One-scale H-distributions

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Introduction

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$$u_n(x) := e^{2\pi i n x} \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}} 0\,,$$

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It is bounded in $\mathrm{L}^1_{\mathrm{loc}}(\Omega) \hookrightarrow \mathcal{M}(\Omega) = (\mathrm{C}_c(\Omega))'$, so

$$|u_{n'}|^2 \xrightharpoonup{*} \nu\,.$$

ν is called the defect measure.

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If the defect measure is not trivial we need another objects to determine all the properties of the sequence:

- H-measures
- semiclassical measures
- ...

Outline

1. H-measures and semiclassical measures
2. One-scale H-measures
3. One-scale H-distributions
4. Multi-scale problems

Outline

H-measures

$\Omega \subseteq \mathbf{R}^d$ open.

Theorem (Tartar, 1990)

If $u_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$*, then there exist a subsequence* $(u_{n'})$ *and* $\mu_H \in \mathcal{M}(\Omega \times \mathrm{S}^{d-1})$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$

$$\lim_{n'} \int_{\mathbf{R}^d} (\widehat{\varphi_1 u_{n'}})(\boldsymbol{\xi}) \overline{(\widehat{\varphi_2 u_{n'}})(\boldsymbol{\xi})} \psi\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) \, d\boldsymbol{\xi} = \langle \mu_H, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \,.$$

(Unbounded) Radon measure μ_H *we call the H-measure corresponding to the (sub)sequence* (u_n)*.*

Notation:

$\mathbf{x} = (x^1, x^2, \dots, x^d) \in \Omega \,, \quad \boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_d) \in \mathbf{R}^d$

$\hat{u}(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \boldsymbol{\xi} \cdot \mathbf{x}} u(\mathbf{x}) \, d\mathbf{x}$

$\mathsf{a} \cdot \mathsf{b} = \sum_{i=1}^d a^i \bar{b}^i$ ($\mathsf{a}, \mathsf{b} \in \mathbf{C}^r$)

$\langle \cdot, \cdot \rangle$ sesquilinear dual product

$\mathcal{M}(X) = (\mathrm{C}_c(X))' \,, \quad (\varphi \boxtimes \psi)(\mathbf{x}, \boldsymbol{\xi}) = \varphi(\mathbf{x}) \psi(\boldsymbol{\xi})$

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Corollary

$$u_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \mu_H = 0 \, .$$

Semiclassical measures

Theorem (Gérard, 1991)

If $u_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$, $\omega_n \to 0^+$, *then there exist a subsequence* $(u_{n'})$ *and* $\mu_{sc}^{(\omega_{n'})} \in \mathcal{M}(\Omega \times \mathbf{R}^d)$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\Omega)$ *and* $\psi \in \mathcal{S}(\mathbf{R}^d)$

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(Unbounded) measure $\mu_{sc}^{(\omega_{n'})}$ *we call the semiclassical measure with characteristic length* $(\omega_{n'})$ *corresponding to the (sub)sequence* $(u_{n'})$.

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$$u_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \mu_{sc}^{(\omega_n)} = 0 \quad \& \quad (u_n) \text{ is } (\omega_n) - \text{oscillatory} .$$

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Example 1: Oscillations - one characteristic length

$\alpha > 0$, $\mathsf{k} \in \mathbf{Z}^d \setminus \{0\}$,

$$u_n(\mathbf{x}) := e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}} \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}} 0\,,\ n \to \infty\,,$$

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$$\begin{aligned} \nu &= \lambda \\ \mu_H &= \lambda \boxtimes \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}} \\ \mu_{sc}^{(\omega_n)} &= \lambda \boxtimes \begin{cases} \delta_0 & , \quad \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty\rangle \\ 0 & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases} \end{aligned}$$

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Theorem

If $u_n \longrightarrow u$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ *is* (ω_n)*-oscillatory and* $\mu_{sc}^{(\omega_n)}(\Omega \times \{0\}) = 0$, *then* $u = 0$ *and*

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If $u_n \longrightarrow u$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ is (ω_n)-oscillatory and $\mu_{sc}^{(\omega_n)}(\Omega \times \{0\}) = 0$, then $u = 0$ and

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(ω_n)-concentrating property

Definition

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$(\forall\, \varphi \in \mathrm{C}_c^\infty(\Omega)) \quad \lim_{R\to\infty} \limsup_n \int_{|\boldsymbol{\xi}|\geqslant \frac{R}{\omega_n}} |\widehat{\varphi u_n}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} = 0\,.$

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$$(u_n)\ \omega_n\text{-concentrating} \iff \mu_{sc}^{(\omega_n)}(\Omega \times \{0\}) = 0\,.$$

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Theorem

If $u_n \rightharpoonup u$ *in* $\mathrm{L}^2_{\rm loc}(\Omega)$ *is* (ω_n)*-oscillatory and* (ω_n)*-concentrating, then* $u=0$ *and*

$$\langle \mu_H, \varphi \boxtimes \psi\rangle = \left\langle \mu_{sc}^{(\omega_n)}, \varphi \boxtimes \psi\Big(\frac{\cdot}{|\cdot|}\Big)\right\rangle\,.$$

For an arbitrary bounded sequence (u_n) in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ is there a characteristic length $\omega_n \to 0^+$ such that (u_n) is

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- M.E., M. Lazar: *Characteristic scales of bounded L^2 sequences*, Asymptotic Analysis **109**(3-4) (2018) 171–192.

Example 2: Oscillations - two characteristic length

$0 < \alpha < \beta$, $\mathsf{k}, \mathsf{s} \in \mathbf{Z}^d \setminus \{0\}$,

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μ_H ($\mu_{sc}^{(\omega_n)}$) is H-measure (semiclassical measure with characteristic length (ω_n), $\omega_n \to 0^+$) associated to $(u_n + v_n)$.

$$\mu_H = \lambda \boxtimes \left(\delta_{\frac{\mathsf{k}}{|\mathsf{k}|}} + \delta_{\frac{\mathsf{s}}{|\mathsf{s}|}}\right)$$

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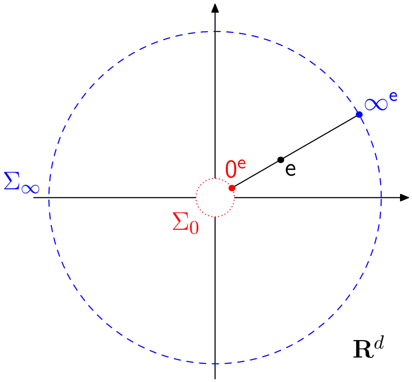
Outline

$K_{0,\infty}(\mathbf{R}^d)$

$K_{0,\infty}(\mathbf{R}^d)$ is a compactification of $\mathbf{R}^d_* := \mathbf{R}^d \setminus \{0\}$ homeomorphic to a spherical layer (i.e. an annulus in $\mathbf{R}^2$):



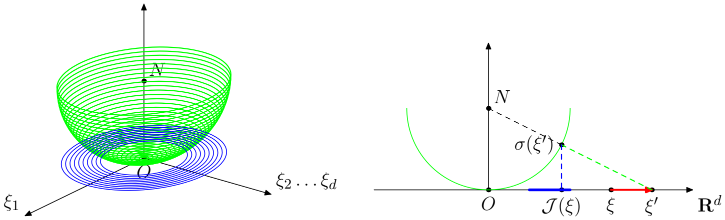
Precise description of $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$ 1/2

For fixed $r_0 > 0$ let us define $r_1 = \frac{r_0}{\sqrt{r_0^2+1}}$, and denote by

$$A[0, r_1, 1] := \left\{ \boldsymbol{\zeta} \in \mathbf{R}^d : r_1 \leqslant |\boldsymbol{\zeta}| \leqslant 1 \right\}$$

closed d-dimensional spherical layer equipped with the standard topology (inherited from $\mathbf{R}^d$).

We want to construct a homeomorphism $\mathcal{J} : \mathbf{R}^d_* \longrightarrow \mathrm{Int}\, A[0, r_1, 1]$.



Precise description of $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$ 2/2

From the previous construction we get that $\mathcal{J} : \mathbf{R}^d_* \longrightarrow \mathrm{Int}\, A[0, r_1, 1]$ is given by

$$\mathcal{J}(\boldsymbol{\xi}) = \frac{\boldsymbol{\xi}}{\sqrt{|\boldsymbol{\xi}|^2 + \left(\frac{|\boldsymbol{\xi}|}{|\boldsymbol{\xi}|+r_0}\right)^2}}\,,$$

and $(A[0, r_1, 1], \mathcal{J})$ is a compactification of $\mathbf{R}^d_*$.

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and $(A[0, r_1, 1], \mathcal{J})$ is a compactification of $\mathbf{R}^d_*$.

Lemma

For $\psi \in \mathrm{C}(\mathbf{R}^d_*)$ we have $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ iff there exist $\psi_0, \psi_\infty \in \mathrm{C}(\mathrm{S}^{d-1})$ such that

$$\psi(\boldsymbol{\xi}) - \psi_0\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) \to 0\,, \quad |\boldsymbol{\xi}| \to 0\,,$$

$$\psi(\boldsymbol{\xi}) - \psi_\infty\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) \to 0\,, \quad |\boldsymbol{\xi}| \to \infty\,.$$

Lemma

i) $\mathcal{S}(\mathbf{R}^d) \hookrightarrow \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, *and*

ii) $\big\{\psi\big(\frac{\cdot}{|\cdot|}\big) \;:\; \psi \in \mathrm{C}(\mathrm{S}^{d-1})\big\} \hookrightarrow \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

One-scale H-measures

Theorem (Tartar, 2009)

If $u_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$, $\omega_n \to 0^+$, *then there exist a subsequence* $(u_{n'})$ *and* $\mu_{osH}^{(\omega_{n'})} \in \mathcal{M}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$

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(Unbounded) Radon measure $\mu_{osH}^{(\omega_{n'})}$ *we call the one-scale H-measure with characteristic length* $(\omega_{n'})$ *corresponding to the (sub)sequence* $(u_{n'})$.

Generalisation of both H-measures and semiclassical measures.

- L. Tartar: *The general theory of homogenization: A personalized introduction*, Springer (2009).

Some properties of μ_{osH}

- $\mu_{osH}^{(\omega_n)} = 0 \iff \mu_H = 0$
- $\mu_{osH}^{(\omega_n)} = 0 \quad \implies \quad \mu_{sc}^{(\omega_n)} = 0$
- $\mu_{osH}^{(\omega_n)}(\Omega \times \Sigma_\infty) = 0 \iff (u_n)$ is $(\omega_n)-$oscillatory
- $\mu_{osH}^{(\omega_n)}(\Omega \times \Sigma_0) = 0 \iff (u_n)$ is $(\omega_n)-$concentrating
- $u_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \mu_{osH}^{(\omega_n)} = 0$

Example 1 revisited

$u_n(\mathbf{x}) = e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}}$,

$$\mu_H = \lambda \boxtimes \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}$$

$$\mu_{sc}^{(\omega_n)} = \lambda \boxtimes \begin{cases} \delta_{\mathsf{0}} & , \quad \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty\rangle \\ 0 & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

$$\mu_{osH}^{(\omega_n)} = \lambda \boxtimes \begin{cases} \delta_{\mathsf{0}^{\frac{\mathsf{k}}{|\mathsf{k}|}}} & , \quad \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty\rangle \\ \delta_{\infty^{\frac{\mathsf{k}}{|\mathsf{k}|}}} & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

Example 2 - revisited

$u_n(\mathbf{x}) = e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}}$, $v_n(\mathbf{x}) = e^{2\pi i n^\beta \mathsf{s}\cdot\mathbf{x}}$,
associated objects to $(u_n + v_n)$:

$$\mu_H = \lambda \boxtimes \left(\delta_{\frac{\mathsf{k}}{|\mathsf{k}|}} + \delta_{\frac{\mathsf{s}}{|\mathsf{s}|}}\right)$$

$$\mu_{sc}^{(\omega_n)} = \lambda \boxtimes \begin{cases} 2\delta_0 & , \quad \lim_n n^\beta \omega_n = 0 \\ (\delta_0 + \delta_{c\mathsf{s}}) & , \quad \lim_n n^\beta \omega_n = c \in \langle 0, \infty\rangle \\ \delta_0 & , \quad \lim_n n^\beta \omega_n = \infty \ \& \ \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty\rangle \\ 0 & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

$$\mu_{osH}^{(\omega_n)} = \lambda \boxtimes \begin{cases} (\delta_{0^{\frac{\mathsf{k}}{|\mathsf{k}|}}} + \delta_{0^{\frac{\mathsf{s}}{|\mathsf{s}|}}}) & , \quad \lim_n n^\beta \omega_n = 0 \\ (\delta_{0^{\frac{\mathsf{k}}{|\mathsf{k}|}}} + \delta_{c\mathsf{s}}) & , \quad \lim_n n^\beta \omega_n = c \in \langle 0, \infty\rangle \\ (\delta_{0^{\frac{\mathsf{k}}{|\mathsf{k}|}}} + \delta_{\infty^{\frac{\mathsf{s}}{|\mathsf{s}|}}}) & , \quad \lim_n n^\beta \omega_n = \infty \ \& \ \lim_n n^\alpha \omega_n = 0 \\ (\delta_{c\mathsf{k}} + \delta_{\infty^{\frac{\mathsf{s}}{|\mathsf{s}|}}}) & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty\rangle \\ (\delta_{\infty^{\frac{\mathsf{k}}{|\mathsf{k}|}}} + \delta_{\infty^{\frac{\mathsf{s}}{|\mathsf{s}|}}}) & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

Alternative proof (Antonić, E., Lazar)

- Cantor diagonal procedure (separability)
- commutation lemma
- a variant of the kernel lemma

Lemma

Let X and Y be two Hausdorff second countable topological manifolds (with or without a boundary), and let B be a non-negative continuous bilinear form on $\mathrm{C}_c(X) \times \mathrm{C}_c(Y)$. Then there exists a Radon measure $\mu \in \mathcal{M}(X \times Y)$ such that

$$(\forall\, f \in \mathrm{C}_c(X))(\forall\, g \in \mathrm{C}_c(Y)) \quad B(f, g) = \langle \mu, f \boxtimes g \rangle \ .$$

Furthermore, the above remains valid if we replace C_c by C_0, and $\mathcal{M}$ by $\mathcal{M}_b$ (the space of bounded Radon measures, i.e. the dual of Banach space C_0).

- N. Antonić, M.E., M. Lazar: *Localisation principle for one-scale H-measures*, Journal of Functional Analysis **272** (2017) 3410–3454.

Localisation principle

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\sum_{0 \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha}} (\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

where

- $\varepsilon_n \to 0^+$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $(\forall\, \varphi \in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1+\sum_{s=0}^m |\varepsilon_n \boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r) \qquad (*)$

Theorem (Antonić, E., Lazar, 2017)

Under previous assumptions, $\boldsymbol{\mu}_{1sH}^{(\varepsilon_n)}$ *associated to* (u_n) *satisfies*

$$\mathbf{p}_1 \boldsymbol{\mu}_{1sH}^\top = \mathbf{0}\,,$$

where

$$\mathbf{p}_1(\mathbf{x}, \boldsymbol{\xi}) := \sum_{0 \leqslant |\boldsymbol{\alpha}| \leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{1 + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

Example 3: equations with characteristic length (1/2)

Let $\Omega \subseteq \mathbf{R}^2$ be open, and let $\mathsf{u}_n := (u_n^1, u_n^2) \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

$$\begin{cases} u_n^1 + \varepsilon_n \partial_{x_1}(a_1 u_n^1) = f_n^1 \\ u_n^2 + \varepsilon_n \partial_{x_2}(a_2 u_n^2) = f_n^2 \end{cases},$$

where $\varepsilon_n \to 0^+$, $\mathsf{f}_n := (f_n^1,\ f_n^2)$ satisfies $(*)$, and $a_1, a_2 \in \mathrm{C}(\Omega; \mathbf{R})$, $a_1, a_2 \neq 0$ everywhere.

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By the localisation principle for one-scale H-measure $\boldsymbol{\mu}_{1sH}$ with characteristic length (ε_n) associated to (u_n) we get the relation

$$\left(\frac{1}{1+|\boldsymbol{\xi}|}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{2\pi i\xi_1}{1+|\boldsymbol{\xi}|}\begin{bmatrix}a_1(\mathbf{x}) & 0\\ 0 & 0\end{bmatrix} + \frac{2\pi i\xi_2}{1+|\boldsymbol{\xi}|}\begin{bmatrix}0 & 0\\ 0 & a_2(\mathbf{x})\end{bmatrix}\right)\boldsymbol{\mu}_{1sH}^{\top} = \mathbf{0}\,,$$

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whose $(1,1)$ component reads

$$\left(\frac{1}{1+|\boldsymbol{\xi}|} + i \frac{2\pi \xi_1}{1+|\boldsymbol{\xi}|} a_1(\mathbf{x}) \right) \mu_{osH}^{11} = 0\,,$$

hence

$$\frac{1}{1+|\boldsymbol{\xi}|} \mu_{osH}^{11} = 0\,, \quad \frac{\xi_1}{1+|\boldsymbol{\xi}|} \mu_{osH}^{11} = 0\,.$$

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$$\operatorname{supp} \mu_{osH}^{11} \subseteq \Omega \times \Sigma_\infty\,, \quad \frac{\xi_1}{1+|\boldsymbol{\xi}|}\mu_{osH}^{11} = 0\,.$$

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$$\operatorname{supp} \mu_{osH}^{11} \subseteq \Omega \times \{\infty^{(0,-1)}, \infty^{(0,1)}\}\,.$$

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Example 3: equations with characteristic length (2/2)

Analogously, from the $(2,2)$ component we get

$$\operatorname{supp}\mu_{osH}^{22} \subseteq \Omega \times \{\infty^{(-1,0)}, \infty^{(1,0)}\},$$

hence $\operatorname{supp}\mu_{osH}^{11} \cap \operatorname{supp}\mu_{osH}^{22} = \emptyset$ which implies $\mu_{osH}^{12} = \mu_{osH}^{21} = 0$.

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The very definition of one-scale H-measures gives $u_n^1 \bar{u}_n^2 \xrightarrow{*} 0$.

This approach can be systematically generalised by introducing a variant of compensated compactness suitable for problems with characteristic length.

Outline

One-scale H-measures

$\Omega \subseteq \mathbf{R}^d$ open

Theorem

If $u_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\rm loc}(\Omega)$, $v_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\rm loc}(\Omega)$ *and* $\omega_n \to 0^+$, *then there exist* $(u_{n'})$, $(v_{n'})$ *and* $\mu_{osH}^{(\omega_{n'})} \in \mathcal{M}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}}(\boldsymbol{\xi}) \overline{\widehat{\varphi_2 v_{n'}}(\boldsymbol{\xi})} \psi(\omega_{n'}\boldsymbol{\xi}) \, d\boldsymbol{\xi} = \langle \mu_{osH}^{(\omega_{n'})}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \ .$$

The measure $\mu_{osH}^{(\omega_{n'})}$ *is called the one-scale H-measure with characteristic length* $(\omega_{n'})$ *associated to the (sub)sequences* $(u_{n'})$ *and* $(v_{n'})$.

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$\Omega \subseteq \mathbf{R}^d$ open

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$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_n}(\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})}\, d\mathbf{x} = \langle \mu_{osH}^{(\omega_{n'})}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \ .$$

The measure $\mu_{osH}^{(\omega_{n'})}$ *is called the one-scale H-measure with characteristic length* $(\omega_{n'})$ *associated to the (sub)sequences* $(u_{n'})$ *and* $(v_{n'})$.

$\mathcal{A}_\psi(u) = (\psi \hat{u})^\vee$, $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$

One-scale H-distributions

$\Omega \subseteq \mathbf{R}^d$ open

Theorem

If $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$, $v_n \rightharpoonup 0$ *in* $\mathrm{L}^{p'}_{\mathrm{loc}}(\Omega)$ *and* $\omega_n \to 0^+$, *then there exist* $(u_{n'})$, $(v_{n'})$ *and* $\nu^{(\omega_{n'})}_{1sH} \in \mathcal{D}'(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\Omega)$ *and* $\psi \in E$

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The distribution $\nu^{(\omega_{n'})}_{1sH}$ *is called the one-scale H-distribution with characteristic length* $(\omega_{n'})$ *associated to the (sub)sequences* $(u_{n'})$ *and* $(v_{n'})$.

$\mathcal{A}_\psi(u) = (\psi \hat{u})^\vee$, $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$

One-scale H-distributions

$\Omega \subseteq \mathbf{R}^d$ open, $p \in \langle 1, \infty \rangle$, $\frac{1}{p} + \frac{1}{p'} = 1$

Theorem

If $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p_{\rm loc}(\Omega)$, $v_n \rightharpoonup 0$ *in* $\mathrm{L}^{p'}_{\rm loc}(\Omega)$ *and* $\omega_n \to 0^+$*, then there exist* $(u_{n'})$, $(v_{n'})$ *and* $\nu^{(\omega_{n'})}_{1sH} \in \mathcal{D}'(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\Omega)$ *and* $\psi \in E$

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Determine E such that

- $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ is continuous
- The First commutation lemma is valid

Differential structure on $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$

For $\kappa \in \mathbf{N}_0 \cup \{\infty\}$ let us define

$$\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) := \Big\{\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) : \psi^* := \psi \circ \mathcal{J}^{-1} \in \mathrm{C}^\kappa(A[0, r_1, 1])\Big\}\,.$$

It is not hard to check that $\mathrm{C}^0(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ and $\mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ coincide.

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It is not hard to check that $\mathrm{C}^0(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ and $\mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ coincide.

For $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ we define $\|\psi\|_{\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))} := \|\psi^*\|_{\mathrm{C}^\kappa(A[0,r_1,1])}$.

$$\mathrm{C}^\kappa(A[0, r_1, 1]) \quad \text{Banach algebra} \quad \Longrightarrow \quad \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) \quad \text{Banach algebra}$$

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Differential structure on $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$

For $\kappa \in \mathbf{N}_0 \cup \{\infty\}$ let us define

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Is $\mathcal{A}_\psi = (\psi\hat{\cdot})^\vee : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ continuous?

Theorem (Hörmander-Mihlin)

If for $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ there exists $C > 0$ such that

$$(\forall\, \boldsymbol{\xi} \in \mathbf{R}^d_*)(\forall\, \boldsymbol{\alpha} \in \mathbf{N}^d_0,\, |\boldsymbol{\alpha}| \leqslant \kappa) \qquad |\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})| \leqslant \frac{C}{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|}}\,,$$

where $\kappa = \lfloor \frac{d}{2} \rfloor + 1$, then ψ is a Fourier multiplier. Moreover, we have

$$\|\mathcal{A}_\psi\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))} \leqslant C_d \max\Big\{p, \frac{1}{p-1}\Big\} C\,.$$

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We shall use *Faá di Bruno formula*: for sufficiently smooth functions $\mathrm{g} : \mathbf{R}^d \longrightarrow \mathbf{R}^r$ and $f : \mathbf{R}^r \longrightarrow \mathbf{R}$ we have

$$\partial^{\boldsymbol{\alpha}}(f \circ \mathrm{g})(\boldsymbol{\xi}) = |\boldsymbol{\alpha}|! \sum_{1 \leqslant |\boldsymbol{\beta}| \leqslant |\boldsymbol{\alpha}|\,,\ \boldsymbol{\beta} \in \mathbf{N}^r_0} C(\boldsymbol{\beta}, \boldsymbol{\alpha})\,,$$

where

$$C(\boldsymbol{\beta}, \boldsymbol{\alpha}) = \frac{(\partial^{\boldsymbol{\beta}} f)(\mathrm{g}(\boldsymbol{\xi}))}{\boldsymbol{\beta}!} \sum_{\substack{\sum_{i=1}^r \boldsymbol{\alpha}_i = \boldsymbol{\alpha}\,, \\ \boldsymbol{\alpha}_i \in \mathbf{N}^d_0}} \prod_{j=1}^{r} \sum_{\substack{\sum_{i=1}^{\beta_j} \boldsymbol{\gamma}_i = \boldsymbol{\alpha}_j\,, \\ \boldsymbol{\gamma}_i \in \mathbf{N}^d_0 \setminus \{0\}}} \prod_{s=1}^{\beta_j} \frac{\partial^{\boldsymbol{\gamma}_s} g_j(\boldsymbol{\xi})}{\boldsymbol{\gamma}_s!}\,.$$

Lemma

For every $j \in 1..d$ *and* $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ *we have*

$$|\partial^{\boldsymbol{\alpha}}(\mathcal{J}_j)(\boldsymbol{\xi})| \leqslant \frac{C_{\boldsymbol{\alpha},d}}{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|}}\,, \quad \boldsymbol{\xi} \in \mathbf{R}^d_*\,.$$

Theorem

Let $\kappa \in \mathbf{N}_0$. *For every* $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *and* $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ *such that* $|\boldsymbol{\alpha}| \leqslant \kappa$ *we have*

$$|\partial^{\boldsymbol{\alpha}}\psi(\boldsymbol{\xi})| \leqslant C_{\kappa,d}\frac{\|\psi\|_{\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))}}{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|}}\,, \quad \boldsymbol{\xi} \in \mathbf{R}^d_*\,.$$

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Therefore, for $\kappa \geqslant \lfloor \frac{d}{2} \rfloor + 1$ and $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ we have

$$\|\mathcal{A}_\psi\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))} \leqslant C_{d,p}\|\psi\|_{\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))}\,.$$

One-scale H-distributions

Theorem

If $u_n \longrightarrow 0$ *in* $\mathrm{L}^p_{\rm loc}(\Omega)$ *and* (v_n) *is bounded in* $\mathrm{L}^q_{\rm loc}(\Omega)$*, for some* $p \in \langle 1, \infty\rangle$ *and* $q \geqslant p'$*, and* $\omega_n \to 0^+$*, then there exist subsequences* $(u_{n'})$*,* $(v_{n'})$ *and a complex (supported) distribution of finite order* $\nu_{1sH}^{(\omega_{n'})} \in \mathcal{D}'(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$*, where* $\kappa = \lfloor \frac{d}{2} \rfloor + 1$*, we have*

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\overline{\varphi_2 v_{n'}}\, d\mathbf{x} = \lim_{n'} \left\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\right\rangle = \left\langle \nu_{1sH}^{(\omega_{n'})}, \varphi_1\bar{\varphi}_2 \boxtimes \psi \right\rangle ,$$

where $\psi_n := \psi(\omega_n \cdot)$*. The distribution* $\nu_{1sH}^{(\omega_{n'})}$ *we call one-scale H-distribution (with characteristic length* $(\omega_{n'})$*) associated to (sub)sequences* $(u_{n'})$ *and* $(v_{n'})$*.*

The existence of one-scale H-distributions: proof 1/2

For $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ and $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ such that $\operatorname{supp}\varphi_1, \operatorname{supp}\varphi_2 \subseteq K_m$, we have

$$|\,\langle \varphi_2 v_n, \mathcal{A}_{\psi_n}(\varphi_1 u_n)\rangle\,| \leqslant C_{m,d}\|\varphi_1\|_{\mathrm{L}^\infty(K_m)}\|\varphi_2\|_{\mathrm{L}^\infty(K_m)}\|\psi\|_{\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))}\,,$$

where K_m are compacts such that $K_m \subseteq \operatorname{Int} K_{m+1}$ and $\bigcup_m K_m = \Omega$.

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where K_m are compacts such that $K_m \subseteq \operatorname{Int} K_{m+1}$ and $\bigcup_m K_m = \Omega$.

By the Cantor diagonal procedure (we have separability) ... we get trilinear form L:

$$L(\varphi_1, \varphi_2, \psi) = \lim_{n'} \left\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'}) \right\rangle \,.$$

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Commutation lemma $\quad\Longrightarrow\quad L(\varphi_1, \varphi_2, \psi) = L(\varphi_1\bar\varphi_2, \zeta_1\zeta_2, \psi)\,.$

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Commutation lemma $\implies$ $L(\varphi_1, \varphi_2, \psi) = L(\varphi_1 \bar{\varphi}_2, \zeta_1 \zeta_2, \psi)\,.$

For $\varphi \in \mathrm{C}_c(\Omega)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ we define

$$\mathcal{L}(\varphi, \psi) := L(\varphi, \zeta, \psi)\,,$$

where $\zeta \equiv 1$ on $\operatorname{supp}\varphi$.
$\mathcal{L}$ is continuous bilinear form on $\mathrm{C}_c(\Omega) \times \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

The existence of one-scale H-distributions: proof 2/2

Theorem

Let $\Omega \subseteq \mathbf{R}^d$ be open, and let B be a continuous bilinear form on $\mathrm{C}_c^\infty(\Omega) \times \mathrm{C}^\infty(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$. Then there exists a unique (supported) distribution $\nu \in \mathcal{D}'(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that

$$(\forall\, f \in \mathrm{C}_c^\infty(\Omega))(\forall\, g \in \mathrm{C}^\infty(\mathrm{K}_{0,\infty}(\mathbf{R}^d))) \quad B(f,g) = \langle \nu, f \boxtimes g \rangle \,.$$

Moreover, if B is continuous on $\mathrm{C}_c^k(\Omega) \times \mathrm{C}^l(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ for some $k, l \in \mathbf{N}_0$, ν is of a finite order $q \leqslant k + l + 2d + 1$.

The existence of one-scale H-distributions: proof 2/2

Theorem

Let $\Omega \subseteq \mathbf{R}^d$ *be open, and let* B *be a continuous bilinear form on* $\mathrm{C}^\infty_c(\Omega) \times \mathrm{C}^\infty(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$*. Then there exists a unique (supported) distribution* $\nu \in \mathcal{D}'(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *such that*

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Moreover, if B *is continuous on* $\mathrm{C}^k_c(\Omega) \times \mathrm{C}^l(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *for some* $k, l \in \mathbf{N}_0$*,* ν *is of a finite order* $q \leqslant k + l + 2d + 1$*.*

Therefore, we have that there exists $\nu^{(\omega_{n'})}_{1sH} \in \mathcal{D}'_{\kappa+2d+1}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that

$$\begin{aligned}\left\langle \nu^{(\omega_{n'})}_{1sH}, \varphi_1\bar{\varphi}_2 \boxtimes \psi \right\rangle =& \mathcal{L}(\varphi_1\bar{\varphi}_2, \psi)\\ =& L(\varphi_1\bar{\varphi}_2, \zeta_1\zeta_2, \psi)\\ =& L(\varphi_1, \varphi_2, \psi) = \lim_{n'} \left\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'}) \right\rangle\end{aligned}$$

Outline

Example 4: oscillations - two characteristic length

$0 < \alpha < \beta$, $\mathsf{k}, \mathsf{s} \in \mathbf{Z}^d \setminus \{\mathsf{0}\}$,

$$u_n(\mathbf{x}) := e^{2\pi i (n^\alpha \mathsf{s} + n^\beta \mathsf{k})\cdot \mathbf{x}} \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0\,,\ n \to \infty$$

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$$\mu_{osH}^{(\omega_n)} = \lambda \boxtimes \begin{cases} \delta_{0^{\frac{\mathsf{k}}{|\mathsf{k}|}}} & , \quad \lim_n n^\beta \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\beta \omega_n = c \in \langle 0, \infty \rangle \\ \delta_{\infty^{\frac{\mathsf{k}}{|\mathsf{k}|}}} & , \quad \lim_n n^\beta \omega_n = \infty \end{cases}$$

Lower order term n^α and corresponding direction of oscillations s we cannot resemble in any case.
Therefore, we need some new methods and/or tools.

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- L. Tartar: *Multi-scale H-measures*, Discrete and Continuous Dynamical Systems - Series S **8** (2015) 77–90.

Still no satisfactory results.

References & The End :) (thank you all)

H-measures

- L. Tartar: *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations,* Proceedings of the Royal Society of Edinburgh, **115A** (1990) 193–230.
- P. Gérard: *Microlocal defect measures,* Comm. Partial Diff. Eq. **16** (1991) 1761–1794.

Semiclassical (Wigner) measures

- P. Gérard: *Mesures semi-classiques et ondes de Bloch,* Sem. EDP 1990–91 (exp. 16), (1991).
- P. L. Lions, T. Paul: *Sur les measures de Wigner,* Revista Mat. Iberoamericana **9,** (1993) 553–618.
- M.E., M. Lazar: *Characteristic scales of bounded L^2 sequences*, Asymptotic Anal. **109**(3-4) (2018) 171–192.

References & The End :) (thank you all)

One-scale H-measures

- L. Tartar: *The general theory of homogenization: A personalized introduction,* Springer (2009), Chapter 32.
- L. Tartar: *Multi-scale H-measures,* Discrete Continuous Dyn. Syst. Ser. S **8** (2015) 77–90.
- N. Antonić, M.E., M. Lazar: *Localisation principle for one-scale H-measures*, J. Funct. Anal. **272** (2017) 3410–3454.

H-distributions and one-scale H-distributions

- N. Antonić, D. Mitrović: *H-distributions – an extension of H-measures to an $L^p - L^q$ setting*, Abstr. Appl. Anal. (2011), Article ID 901084, 12 pages.
- M.E.: *One-scale H-measures and variants* (in Croatian), Ph.D. thesis, University of Zagreb, Zagreb (2016).