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Twist for Snyder space

Daniel Meljanac[1,a], **Stjepan Meljanac**[2,b], **Salvatore Mignemi**[3,4,c], **Danijel Pikutić**[2,d], **Rina Štrajn**[2,e]

[1] Division of Materials Physics, Ruđer Bošković Institute, Bijenička cesta 54, 10002 Zagreb, Croatia
[2] Division of Theoretical Physics, Ruđer Bošković Institute, Bijenička cesta 54, 10002 Zagreb, Croatia
[3] Dipartimento di Matematica e Informatica, Università di Cagliari, viale Merello 92, 09123 Cagliari, Italy
[4] INFN, Sezione di Cagliari, Cittadella Universitaria, 09042 Monserrato, Italy

Abstract We construct the twist operator for the Snyder space. Our starting point is a non-associative star product related to a Hermitian realisation of the noncommutative coordinates originally introduced by Snyder. The corresponding coproduct of momenta is non-coassociative. The twist is constructed using a general definition of the star product in terms of a bi-differential operator in the Hopf algebroid approach. The result is given by a closed analytical expression. We prove that this twist reproduces the correct coproducts of the momenta and the Lorentz generators. The twisted Poincaré symmetry is described by a non-associative Hopf algebra, while the twisted Lorentz symmetry is described by the undeformed Hopf algebra. This new twist might be important in the construction of different types of field theories on Snyder space.

1 Introduction

Since the beginning, the research of a quantum field theory brought with it the problem of ultraviolet divergences and it was already Heisenberg who proposed the idea of noncommutative spaces as a possible solution [1]. The idea was formulated mathematically for the first time in 1947 by Snyder [2].

In his paper, Snyder has shown that the introduction of a minimal unit of length necessarily leads to a noncommutative algebra of spacetime coordinates, but also that the assumption of Lorentz covariance does not impose a requirement for

the spacetime to be continuous. The noncommutative coordinates can be realised as Lie algebra generators and being represented as operators that encode rotations, their spectrum is discrete, but Lorentz invariance is preserved. The momentum space can be identified with the de Sitter space $\sim SO(4, 1)/SO(3, 1)$.

Because of the success of the renormalisation theory, the idea of noncommutative spaces was forgotten for many years, until it was rediscovered by mathematicians [3] and appeared in the context of string theory [4]. When noncommutative geometry resurfaced as an important field of research [5–8], new models were introduced. These include e.g. the most simple type of a noncommutative space, known as the Moyal plane [6–8], and the κ-Minkowski spacetime [9,10], a Lie algebra-type deformation of ordinary phase space, which is especially interesting because of its connection to doubly special relativity [11]. The κ-Minkowski spacetime is also a noncommutative geometry where the momentum space is the de Sitter space, obtained from the Iwasawa decomposition of $SO(4, 1)$ [12,13] and that has deformed commutation relations between the momenta and the noncommutative coordinates, which leads to a noncommutative addition of momenta. It is a model that has been extensively studied in different directions, e.g., in defining star products [14–18], the differential calculus [19], the conserved charges [20,21], generalising the description of the κ-Poincaré-Hopf algebra [22], or classifying the differential calculi and considering the related field theories [23].

Despite the renewed interest in noncommutative geometry, the emergence of new models and their study using the formalism of Hopf algebras [12,24], less attention was given to Snyder's original proposal. Some of the work done includes the consideration of different field theories on the Snyder spacetime [25] and its study in connection to quantum gravity phenomenology [26]. Its classical and quantum aspects have been studied from a phenomenological point

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[a] e-mail: Daniel.Meljanac@irb.hr

[b] e-mail: meljanac@irb.hr

[c] e-mail: smignemi@unica.it

[d] e-mail: dpikutic@irb.hr

[e] e-mail: Rina.Strajn@irb.hr

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of view, outside of the noncommutative geometry formalism [27–36]. The model was also considered in a series of papers [37–40], and the star product, coproduct and antipodes of its Hopf algebra were calculated.

Using a spacetime picture based on the concept of realisations of a noncommutative geometry, the authors of [38] constructed the co-algebraic sector corresponding to any realisation of the Snyder model, and also defined a non-ambiguous self interacting scalar field theory for the Snyder spacetime, computing explicitly the first order correction terms of the corresponding Lagrangian. In [41,42], the problem was considered from a geometrical point of view, with equivalent results to those of [38]. Two types of scalar field theory were constructed. The first one, based on the natural momenta addition, uncovers a non-associative deformation of the Poincaré symmetries, and the second considers Snyder spacetime as a subspace of a larger noncommutative space. The restriction of the extra-dimensional scalar field theory was also discussed.

More recently, the construction of a free field theory was considered in [43], both for a hermitian realisation of the Snyder spacetime, showing that the free theory is equivalent to the commuatative one, and for a generalisation of the model, up to the first order in the deformation parameter. In [44], a truncated form of the nonassociative and noncommutative Snyder ϕ^4 field theory was defined and quantized using the functional method in momentum space. Different nonassociative star/cross product geometries, as well as the related field theories, have also been considered in [45–50].

In this paper we construct the non-associative star product related to a Hermitian realisation of the noncommutative coordinates originally introduced by Snyder [2]. This construction is performed using the method proposed in [38,51,55]. Using a general definition of the star product, defined by a bi-differential twist operator, we calculate the twist in the Hopf algebroid approach to all orders in the deformation parameter. The result is given by a closed analytical expression. This twist does not satisfy the cocycle condition, but we prove that it reproduces the star product and the coproducts of the momenta and the Lorentz generators. Also, it generates a quasi-bialgebra and quasi-bialgebroid structure related to the Snyder space. This twist might be important in the construction of different types of field theories on the Snyder space.

In Sect. 2, the construction of the star product related to the original Snyder realisation is presented. In Sect. 3, the corresponding twist operator is constructed. It is proved that it reproduces the coproducts of the momenta and the Lorentz generators. In Sect. 4, the outlook and discussion are given. Two appendices, related to Sects. 2 and 3 respectively, which include important details of the proofs are added.

2 From Snyder algebra to star product

In its original formulation [2], the Snyder algebra is generated by noncommutative coordinates $\hat{x}_\mu$, momenta p_μ and Lorentz generators $M_{\mu\nu}$ that satisfy the following commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\beta M_{\mu\nu}, \quad [p_\mu, \hat{x}_\nu]$$
$$= -i(\eta_{\mu\nu} + \beta p_\mu p_\nu), \quad [p_\mu, p_\nu] = 0,$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}),$$
$$[M_{\mu\nu}, \hat{x}_\lambda] = i(\eta_{\mu\lambda}\hat{x}_\nu - \eta_{\nu\lambda}\hat{x}_\mu), \quad [M_{\mu\nu}, p_\lambda]$$
$$= i(\eta_{\mu\lambda} p_\nu - \eta_{\nu\lambda} p_\mu), \tag{1}$$

where β is a constant of the order $1/M_{Pl}^2$, M_{Pl} being the Planck mass and the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$ has signature $(1, n - 1)$. The Lorentz generators satisfy the standard relations and their commutation relations with the noncommutative coordinates and the momenta are also undeformed, i.e. vector like, hence Lorentz invariance is preserved, but the commutation relations of the momenta and the noncommutative coordinates are deformed.

It is possible to find a realisation of the Snyder algebra in terms of x_μ and p_μ, the generators of the undeformed Heisenberg algebra [37,38,51], which satisfy

$$[x_\mu, x_\nu] = 0, \quad [p_\mu, p_\nu] = 0, \quad [p_\mu, x_\nu] = -i\eta_{\mu\nu}. \tag{2}$$

Moreover, we concentrate on a hermitian realisation given by

$$\hat{x}_\mu = x_\mu + \frac{\beta}{2}(Dp_\mu + p_\mu D) = x_\mu + \beta\left(Dp_\mu - \frac{i}{2}p_\mu\right),$$
$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad p_\mu = -i\frac{\partial}{\partial x^\mu}, \tag{3}$$

where we define the dilatation operator D as

$$D = \frac{1}{2}(x \cdot p + p \cdot x) = x \cdot p - \frac{in}{2}, \tag{4}$$

and $x \cdot p$ denotes the scalar product, $x \cdot p = x^\alpha p_\alpha$. Hence

$$\hat{x}_\mu = x_\mu + \beta\left(x \cdot pp_\mu - i\frac{n+1}{2}p_\mu\right). \tag{5}$$

Note that $M_{\mu\nu}^\dagger = M_{\mu\nu}$, $D^\dagger = D$ and $(\hat{x}_\mu)^\dagger = \hat{x}_\mu$.

For the undeformed Heisenberg algebra, $\mathcal{H} = \mathcal{A}\mathcal{T}$, where $\mathcal{A}$ is the enveloping algebra generated by x_μ and $\mathcal{T}$ is generated by p_μ, we define the action $\triangleright$ on $\mathcal{A}$, $\triangleright : \mathcal{H} \otimes \mathcal{A} \to \mathcal{A}$, by

$$x_\mu \triangleright f(x) = x_\mu f(x), \quad p_\mu \triangleright f(x) = -i\frac{\partial f(x)}{\partial x^\mu}, \quad f(x) \in \mathcal{A}. \tag{6}$$

It can be shown [43,52–55] that for realisations of non-commutative coordinates which can be written in the form $\hat{x}_\mu = x_\alpha \varphi^\alpha{}_\mu(p) + \chi_\mu(p)$, one has

$$e^{ik\cdot\hat{x}} \triangleright 1 = e^{iK(k)\cdot x + ig(k)}, \tag{7}$$

and

$$e^{ik\cdot\hat{x}} \triangleright e^{iq\cdot x} = e^{i\mathcal{P}(k,q)\cdot x + i\mathcal{Q}(k,q)}, \tag{8}$$

where Eqs. (7) and (8) are the defining relations for the functions $K(k)$, $g(k)$, $\mathcal{P}(k,q)$ and $\mathcal{Q}(k,q)$. It is easily seen that

$$\mathcal{P}_\mu(k,0) = K_\mu(k), \quad \mathcal{P}_\mu(0,q) = q_\mu. \tag{9}$$

Writing the inverse of Eq. (7),

$$e^{ik\cdot x} = e^{iK^{-1}(k)\cdot\hat{x} - ig(K^{-1}(k))} \triangleright 1, \tag{10}$$

it follows that for any function $f(x) \in \mathcal{A}$ that can be Fourier transformed, we can define a function $\hat{f}(\hat{x}) \in \mathcal{H}$ in the following way

$$f(x) = \int d^n k \, \tilde{f}(k) e^{ik\cdot x},$$
$$\hat{f}(\hat{x}) = \int d^n k \, \tilde{f}(k) \, e^{iK^{-1}(k)\cdot\hat{x} - ig(K^{-1}(k))}, \tag{11}$$

and $\hat{f}$ will have the property

$$\hat{f} \triangleright 1 = f(x). \tag{12}$$

We then define the star product, $\star : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, of two functions $f(x)$, $g(x) \in \mathcal{A}$ by

$$f(x) \star g(x) = \hat{f}\hat{g} \triangleright 1 = \hat{f} \triangleright g(x). \tag{13}$$

From (10), it also follows that the star product of two plane waves is given by

$$e^{ik\cdot x} \star e^{iq\cdot x} = e^{iK^{-1}(k)\cdot\hat{x} - ig(K^{-1}(k))} \triangleright e^{iq\cdot x}$$
$$= e^{i\mathcal{P}(K^{-1}(k),q)\cdot x + i\mathcal{Q}(K^{-1}(k),q) - ig(K^{-1}(k))}. \tag{14}$$

Noting that $g(k) = \mathcal{Q}(k,0)$, and defining the functions $\mathcal{D}_\mu(k,q)$ and $\mathcal{G}(k,q)$ by

$$\mathcal{D}_\mu(k,q) = \mathcal{P}_\mu(K^{-1}(k),q), \tag{15}$$
$$\mathcal{G}(k,q) = \mathcal{Q}(K^{-1}(k),q) - \mathcal{Q}(K^{-1}(k),0), \tag{16}$$

one finds that this star product can be written as

$$e^{ik\cdot x} \star e^{iq\cdot x} = e^{i\mathcal{D}(k,q)\cdot x + i\mathcal{G}(k,q)}. \tag{17}$$

The star product of two functions $f(x)$, $g(x) \in \mathcal{A}$ that can be Fourier transformed is then given by

$$f(x) \star g(x) = \int d^n k d^n q \, \tilde{f} \tilde{g} e^{i\mathcal{D}(k,q)\cdot x + i\mathcal{G}(k,q)}. \tag{18}$$

For the Hermitain realisation given by (3), it can be shown (for more details, see the appendix) that $\mathcal{D}_\mu(k,q)$ and $\mathcal{G}(k,q)$ are given by

$$\mathcal{D}_\mu(k,q) = \frac{1}{1 - \beta k\cdot q}\left[\left(1 - \frac{\beta k\cdot q}{1 + \sqrt{1 + \beta k^2}}\right)k_\mu \right.$$
$$\left. + \sqrt{1 + \beta k^2}q_\mu\right], \tag{19}$$

$$\mathcal{G}(k,q) = i\frac{n+1}{2}\ln(1 - \beta k\cdot q). \tag{20}$$

Furthermore, we define the coproduct, $\Delta : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$ as

$$\Delta p_\mu = \mathcal{D}_\mu(p \otimes 1, 1 \otimes p), \tag{21}$$

from where one obtains the known expression for the coproduct for the so-called Snyder realisation [38]

$$\Delta p_\mu = \frac{1}{1 - \beta p_\alpha \otimes p^\alpha}\left(p_\mu \otimes 1 - \frac{\beta}{1 + \sqrt{1 + \beta p^2}}p_\mu p_\alpha\right.$$
$$\left. \otimes p^\alpha + \sqrt{1 + \beta p^2} \otimes p_\mu\right). \tag{22}$$

The addition of momenta which corresponds to the coproduct (22) can be found in [41,42]. The antipode $S(p)$ is undeformed $S(p_\mu) = -p_\mu$. The star product for the Snyder space (18) is non-associative, while the corresponding coproduct (22) is non-coassociative.

3 Twist from star product

The star product $f \star g$ can also be written using the twist operator $\mathcal{F}$

$$f(x)\star g(x) = m\left(\mathcal{F}^{-1}(\triangleright \otimes \triangleright)(f(x) \otimes g(x))\right), \quad f, g \in \mathcal{A}, \tag{23}$$

where $m : \mathcal{A} \otimes \mathcal{H} \to \mathcal{H}$ is the multiplication map. From (18), within the Hopf algebroid setting, we can extract the form for $\mathcal{F}^{-1}$, the inverse of the twist operator [53,54,56–60]

$$\mathcal{F}^{-1} =: \exp\left[i(1 \otimes x^\alpha)(\Delta - \Delta_0)p_\alpha + i\mathcal{G}(p \otimes 1, 1 \otimes p)\right] :$$
$$=: \exp\left[i(1 \otimes x^\alpha)(\Delta - \Delta_0)p_\alpha\right] : e^{i\mathcal{G}(p\otimes1, 1\otimes p)}, \tag{24}$$

where $\Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu$ and $::$ denotes normal ordering in which the coordinates x_α stand on the left of the momenta p_α. Twist $\mathcal{F}^{-1}$ is determined by (24) up to the right ideal $\mathcal{I}_0$ of $\mathcal{H}$, which has the property $m\left(\mathcal{I}_0(\triangleright \otimes \triangleright)(f \otimes g)\right) = 0$.

Another way for obtaining the twist is using the perturbative approach introduced in [56]. The coproduct (22) is expanded with respect to the deformation parameter β as $\Delta p_\mu = \Sigma_{k=0}^\infty \Delta_k p_\mu$, with $\Delta_k p_\mu \propto \beta^k$. For $\chi(\beta p^2) = 0$ (where the realisation of the noncommutative coordinates is written as $\hat{x}_\mu = x_\alpha \varphi_\mu^\alpha(\beta p^2) + \beta p_\mu \chi(\beta p^2)$, see appendix), the corresponding twist is $\mathcal{F}_0 = e^f$, where $f = \Sigma_{k=1}^\infty f_k$, $f_k \propto \beta^k$, and we choose $f_k \in \mathcal{T} \otimes \mathcal{L}(x)\mathcal{T}$, where $\mathcal{L}(x)$ is linear span of x^μ. After writing down the most general ansatz for the f_k, one finds the unknown coefficients by requiring that the twist reproduces the coproduct (22) order by order, in the following way

$$\Delta p_\mu = \mathcal{F}_0 \Delta_0 p_\mu \mathcal{F}_0^{-1}. \tag{25}$$

After calculating $\mathcal{F}_0$, we find that $\mathcal{F}^{-1}$, corresponding to $\chi(\beta p^2) = -i\frac{n+1}{2}$, is given by

$$\mathcal{F}^{-1} = \mathcal{F}_0^{-1}\left(\frac{1}{1 \otimes 1 - \beta p_\alpha \otimes p^\alpha}\right)^{\frac{n+1}{2}}$$
$$= \exp\left\{i\left(\frac{1}{2}p^2 \otimes x \cdot p + \frac{1}{2}p_\alpha p_\beta \otimes x^\alpha p^\beta + p_\alpha \otimes x \cdot p\, p^\alpha\right)\right.$$
$$\left. \times \left(\frac{\ln(1 + \beta p^2)}{p^2} \otimes 1\right)\right\}\left(\frac{1}{1 \otimes 1 - \beta p_\alpha \otimes p^\alpha}\right)^{\frac{n+1}{2}}. \tag{26}$$

This expression is identical to the one in (24), which can be proved order by order.

Note that, in the Hopf algebroid approach, we have another expression related to the twist operator

$$\mathcal{F}^{-1} = e^{-ip_\alpha \otimes x^\alpha}\, e^{ip_\gamma^W \otimes (x^\gamma + \beta x \cdot p p^\gamma)}\, e^{i\mathcal{G}(p \otimes 1, 1 \otimes p)}, \tag{27}$$

where p_μ^W is the momentum corresponding to the Weyl ordering, which is given by $p_\mu^W = K_\mu^{-1}(p)$. The inverse K_μ^{-1} of the function $K_\mu(p)$, defined in (7), can be calculated from the equation

$$K(K^{-1}(p))_\mu = p_\mu. \tag{28}$$

For the realisation (3), the function $K_\mu(p)$ is given by $K_\mu = p_\mu \frac{\tan\sqrt{\beta p^2}}{\sqrt{\beta p^2}}$ [38], from where it follows that K_μ^{-1} is given by

$$p_\mu^W = K_\mu^{-1}(p) = p_\mu \frac{\arctan\sqrt{\beta p^2}}{\sqrt{\beta p^2}}. \tag{29}$$

Generally, the following identity holds [58]

$$:\, e^{i(1 \otimes x^\alpha)(\Delta - \Delta_0)p_\alpha}\, := e^{-ip_\alpha \otimes x^\alpha}\, e^{iK_\gamma^{-1}(p) \otimes x_\beta \varphi^{\beta\gamma}(p)}. \tag{30}$$

Having the form of the twist operator (26), it is important to check the consistency condition (25). Since $e^{i\mathcal{G}(p \otimes 1, 1 \otimes p)}$ commutes with $\Delta_0 p_\mu$, it is enough to calculate $\mathcal{F}_0 \Delta_0 p_\mu \mathcal{F}_0^{-1}$

$$\mathcal{F}_0 \Delta_0 p_\mu \mathcal{F}_0^{-1} = p_\mu \otimes 1 + \sum_{n=0}^\infty \sum_{k=0}^\infty \beta^{n-k} \frac{(-1)^{k+n}}{k!}$$
$$\times A_{n,k}(p^{2(n-k)} \otimes 1)\, ad_{f_1}^k(1 \otimes p_\mu), \tag{31}$$

where

$$A_{n,k} = \sum_{r_1+\cdots+r_k=n} \frac{1}{r_1 r_2 \ldots r_k} \tag{32}$$

and

$$ad_{f_1}^k(1 \otimes p_\mu) = \beta^k\left(\sum_{l=0}^k c_{k-l,l}(p_\mu(p^2)^{k-l} \otimes 1)(p_\alpha \otimes p^\alpha)^l\right.$$
$$\left. + \sum_{l=0}^k d_{k-l,l}((p^2)^{k-l} \otimes p_\mu)(p_\alpha \otimes p^\alpha)^l\right). \tag{33}$$

The coefficients $c_{k-l,l}$ and $d_{k-l,l}$ satisfy the recursive relations [51]

$$c_{k-l+1,l} = lc_{k-l,l} + c(l-1)c_{k-l+1,l-1} + \frac{1}{2}d_{k-l+1,l-1}, \tag{34}$$

$$d_{k-l+1,l} = \left(l + \frac{1}{2}\right)d_{k-l,l} + ld_{k-l+1,l-1}, \tag{35}$$

with $c_{0,0} = 0$ and $d_{0,0} = 1$. The solutions of the recursive relations are

$$c_{k-l,l} = \sum_{s=0}^{l-1}(-1)^s\binom{l-1}{s}(l-s)^k - d_{k-l+1,l-1}, \tag{36}$$

$$d_{k-l,l} = \sum_{s=0}^{l}(-1)^s\binom{l}{s}\left(l-s+\frac{1}{2}\right)^k. \tag{37}$$

The coefficients $c_{k-l,l}$ and $d_{k-l,l}$ satisfy the following identities, see appendix

$$\sum_{n=0}^\infty \sum_{k=0}^n \frac{(-1)^{n+k}}{k!}A_{n,k}c_{k-l,l}x^{n-l} = \frac{1+x-\sqrt{1+x}}{x}, \tag{38}$$

$$\sum_{n=0}^\infty \sum_{k=0}^n \frac{(-1)^{n+k}}{k!}A_{n,k}d_{k-l,l} = \binom{\frac{1}{2}}{n-l}, \quad \forall n, l. \tag{39}$$

Inserting the expressions for $c_{k-l,l}$ and $d_{k-l,l}$ into (31), and using the identities (38) and (39) leads exactly to (22).

Another check of consistency is to use the twist to calculate $\Delta M_{\mu\nu}$. One gets that

$$\Delta M_{\mu\nu} = \mathcal{F}\Delta_0 M_{\mu\nu}\mathcal{F}^{-1} = \Delta_0 M_{\mu\nu}, \tag{40}$$

where $\Delta_0 M_{\mu\nu} = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}$. The coproduct of the Lorentz generators turns out to be the primitive one, since

$[\mathcal{F}, \Delta_0 M_{\mu\nu}] = 0$. The antipode $S(M_{\mu\nu})$ is also undeformed, $S(M_{\mu\nu}) = -M_{\mu\nu}$.

Because of the non-coassociativity of the coproduct, the twist for the Snyder space does not satisfy the cocycle condition.

Finally, from the star product (23), we define

$$\hat{f}(x, p) = m\left(\mathcal{F}^{-1}(\triangleright \otimes 1)(f(x) \otimes 1)\right), \quad \hat{f}(x, p) \triangleright 1 = f(x),$$
(41)

and $\hat{\mathcal{A}} = \left\{\hat{f}(x, p)\right\} \subset \mathcal{H}$. The space $\hat{\mathcal{A}}$ is not an algebra. The non-associative algebra $\mathcal{A}_\star$, where the product of functions is given by (13), (18), is defined by $\mathcal{A}_\star = \hat{\mathcal{A}} \triangleright 1$. Particularly,

$$\hat{x}_\mu(x, p) = m\left(\mathcal{F}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)\right)$$
$$= x_\mu + \beta\left(x \cdot p p_\mu - i\frac{n+1}{2}p_\mu\right),$$
(42)

which gives back the Snyder realisation from which we have started and this proves the consistency of our approach.

Using the flipped twist operator $\mathcal{F}_{21}$, we define dual non-commutative coordinates $\hat{y}_\mu(x, p)$

$$\hat{y}_\mu(x, p) = m\left(\mathcal{F}_{21}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)\right)$$
$$= \left(x_\mu + x \cdot p p_\mu \frac{\sqrt{1 + \beta p^2} - 1}{p^2}\right)$$
$$\times \sqrt{1 + \beta p^2} - i\frac{n+1}{2}\beta p_\mu.$$
(43)

They satisfy a Snyder type algebra [51]

$$[\hat{y}_\mu, \hat{y}_\nu] = iM_{\mu\nu}\left(\frac{1 - \sqrt{1 + \beta p^2}}{p^2}\right)\sqrt{1 + \beta p^2},$$
(44)

and $\hat{y}_\mu \triangleright f(x) = f(x) \star x_\mu$. Note that $[\hat{x}_\mu, \hat{y}_\nu] \neq 0$, as a consequence of the non-associativity of the star product. The relation between $\hat{y}_\mu$ and $\hat{x}_\mu$ is given by

$$\hat{y}_\mu = \left(\hat{x}^\alpha - \lambda^\alpha(p)\right)O_{\alpha\mu}^{-1}(p),$$
(45)

from which it follows

$$x_\mu \star f(x) = \left(O_{\mu\alpha}(p) \triangleright f(x)\right) \star x^\alpha + \lambda_\mu(p) \triangleright f(x).$$
(46)

4 Outlook and discussion

In this paper we have constructed the non-associative star product related to a Hermitian realisation of noncommutative coordinates, originally introduced by Snyder. Using a general definition of the star product, defined by a bi-differential twist operator, we have calculated the twist in

the Hopf algebroid approach up to all orders in the deformation parameter. The result is given by a closed analytical expression. This twist does not satisfy the cocycle condition because of the non-coassociativity of the corresponding coproduct. However, we have proved that it reproduces the star product and the coproducts of the momenta and the Lorentz generators, as well as the Snyder realisation of the noncommutative coordinates. Even though the corresponding coproduct is non-coassociative, it satisfies the 3-cocycle condition (see Appendix C). Twisted coalgebraic structures related to the Snyder space and Poincaré symmetry are quasi-bialgebroid and quasi-bialgebra respectively. Generalizations which include the antipode are under investigation.

We point out that the associativity of star products generally reflects the associativity of compositions of quantum mechanical operators. However, applications of the deformation quantization procedure to string theory sometimes require deformation in the direction of a quasi-Poisson bracket, which does not satisfy the Jacobi identity and hence leads to non-associativity. It was found that for string endpoints attached to a Dirichlet brane in a non-constant background B-field, the star products describing the correlation functions are both noncommutative and non-associative. Similarly, it was found that closed strings in flat non-geometric R-flux backgrounds M also probe non-associative geometry. Another example of non-associative deformations arising in quantum mechanics and gravity in three dimensions is given by considering the dynamics of electrons in uniform distributions of magnetic charge. All of the above lead to non-associativity becoming an interesting area of research in physics in recent years [45–50,61]; for example, in connection to non-geometric R-flux backgrounds, non-associative star products have been investigated [46] and perturbative non-associative field theories were constructed [47]; the construction of non-associative Weyl star products was undertaken [48]; physical consequences of non-associativity were examined [50], finding that momentum space is quantized and implying a coarse-graining of momentum space with a uniform monopole background. Regarding different directions of the research on non-associativity in quantum physics, the twist we have constructed in this paper could be helpful in studying the properties of the non-associative star product corresponding to the Snyder spacetime. Especially, it could be helpful in investigating the physical consequences of this non-associativity by generalizing approaches presented in [23,62].

An example is given by the investigation of QFT in a Snyder background. The knowledge of the star product allows one to study the quantization using standard methods of non-commutative QFT. In this way, the free scalar field theory in the Hermitian realization (3) has been developed in [43], where it was shown that it is equivalent to the standard com-

mutative theory. The interacting ϕ^4 theory has been investigated in a linear approximation in β in [44] and in the full β dependence in [63]. Of course, the interacting theory is no longer equivalent to its commutative counterpart. In particular, the noncommutativity and nonassociativity of the star product introduce several new features. For example, nonassociativity entails that the ϕ^4 interaction term is not uniquely defined, but three inequivalent possibilities occur. Moreover, the total 4-momentum is not conserved in some loop diagrams. This fact makes the computation of these diagrams particularly difficult. A partial evaluation has been performed in [63], where it has been shown that the Snyder star product can introduce a natural regularization on at least some of the one-loop diagrams of the two-point functions. However, in the limit of vanishing incoming momenta, a variant of the UV/IR mixing of noncommutative field theory [7,8,64] appears. It would be interesting to further investigate this subject. Finally, we notice that, in the context of QFT, the knowledge of the exact form of the twist obtained in the present paper can be useful for the study of the statistics of identical particles in Snyder spacetime [57,65].

A Construction of the star product

Using (8), it can be checked that [51,52,55]

$$e^{-i\lambda k\cdot\hat{x}} p_\mu e^{i\lambda k\cdot\hat{x}} \triangleright e^{iq\cdot x} = \mathcal{P}_\mu(\lambda k, q)e^{iq\cdot x}, \tag{A.1}$$

with λ a real parameter. Differentiating both sides of this equation by λ and noting that the realisation of the noncommutative coordinates can be written in the form $\hat{x}_\mu = x_\alpha \varphi_\mu{}^\alpha(\beta p^2) + \beta p_\mu \chi(\beta p^2)$, it follows that the function $\mathcal{P}_\mu(\lambda k, q)$ satisfies the differential equation

$$\frac{d\mathcal{P}_\mu(\lambda k, q)}{d\lambda} = k_\alpha \varphi_\mu{}^\alpha\left(\mathcal{P}(\lambda k, q)\right). \tag{A.2}$$

Defining $\hat{x}^\mu_{(0)} = x^\alpha \varphi_\mu{}^\alpha$, one obtains that the following equality holds

$$e^{-i\lambda k\cdot\hat{x}_{(0)}} p_\mu e^{i\lambda k\cdot\hat{x}} \triangleright e^{iq\cdot x} = \mathcal{P}_\mu(\lambda k, q)e^{iq\cdot x + i\mathcal{Q}(\lambda k, q)}, \tag{A.3}$$

with λ a real parameter. Differentiating both sides of (A.3) by λ and using (A.2), it follows that $\mathcal{Q}(\lambda k, q)$ satisfies the differential equation

$$\frac{d\mathcal{Q}(\lambda k, q)}{d\lambda} = k_\alpha \chi^\alpha\left(\mathcal{P}(\lambda k, q)\right), \tag{A.4}$$

with $\mathcal{Q}(0, q) = 0$ and $\chi^\alpha \equiv \beta p^\alpha \chi(\beta p^2)$.

Differential equations (A.2) and (A.3) can be used to calculate the functions $\mathcal{P}_\mu(k, q)$ and $\mathcal{Q}(k, q)$ for the hermitian realisation (3), which corresponds to $\varphi_\mu{}^\alpha = \delta_\mu^\alpha + \beta p_\mu p^\alpha$ and $\chi = -i\frac{n+1}{2}$. Having the functions $\mathcal{P}_\mu(k, q)$ and $\mathcal{Q}(k, q)$, one can calculate $\mathcal{D}_\mu(k, q)$ and $\mathcal{G}(k, q)$ to get

$$\mathcal{P}_\mu(k, q) = \frac{q_\mu + \left(\frac{\sin\sqrt{\beta k^2}}{\sqrt{\beta k^2}} + \frac{k\cdot q}{k^2}(\cos\sqrt{\beta k^2} - 1)\right)k_\mu}{\cos\sqrt{\beta k^2} - \frac{k\cdot q}{k^2}\sqrt{\beta k^2}\sin\sqrt{\beta k^2}}, \tag{A.5}$$

$$\mathcal{Q}(k, q) = i\frac{n+1}{2}\ln\left(\cos\sqrt{\beta k^2} - \frac{k\cdot q}{k^2}\sqrt{\beta k^2}\sin\sqrt{\beta k^2}\right). \tag{A.6}$$

From here it follows that

$$\mathcal{D}_\mu(k, q) = \frac{1}{1 - \beta k\cdot q}\left[\left(1 - \frac{\beta k\cdot q}{1 + \sqrt{1 + \beta k^2}}\right)k_\mu + \sqrt{1 + \beta k^2}q_\mu\right], \tag{A.7}$$

$$\mathcal{G}(k, q) = i\frac{n+1}{2}\ln(1 - \beta k\cdot q). \tag{A.8}$$

B Proofs of identities

B.1 General identity

In order to prove the identities (39) and (38), it is helpful to first prove the general identity

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n+k}}{k!}A_{n,k}\alpha^k x^n = (1 + x)^\alpha. \tag{B.1}$$

One starts by noticing that the term $(1 + x)^\alpha$ can be written in the following way

$$
\begin{aligned}
(1 + x)^\alpha &= e^{\alpha\ln(1+x)} = \sum_{k=0}^{\infty}\frac{(\alpha\ln(1+x))^k}{k!} \\
&= \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}\left(\sum_{r=1}^{\infty}\frac{(-1)^{r+1}x^r}{r}\right)^k \\
&= \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}\prod_{i=1}^{k}\sum_{r_i=1}^{\infty}\frac{(-1)^{r_i+1}x^{r_i}}{r_i}
\end{aligned}
\tag{B.2}
$$

Since, for fixed k, the terms with x^n are such that $r_1 + \cdots + r_k = n$ and $k \leq n$, this can further be written as

$$
\begin{aligned}
(1+x)^\alpha &= \sum_{n=0}^\infty \sum_{k=0}^n \frac{\alpha^k}{k!} \left(\sum_{r_1+\cdots r_k=n} \prod_{i=1}^k \frac{(-1)^{r_i+1}}{r_i} \right) x^n \\
&= \sum_{n=0}^\infty \sum_{k=0}^n \frac{\alpha^k}{k!} (-1)^{n+k} \left(\sum_{r_1+\cdots r_k=n} \prod_{i=1}^k \frac{1}{r_i} \right) x^n \\
&= \sum_{n=0}^\infty \sum_{k=0}^n \frac{\alpha^k}{k!} (-1)^{n+k} A_{n,k} x^n.
\end{aligned}
$$
(B.3)

In the second equality, $\sum_{i=1}^k (r_i + 1) = n + k$ is used and in the third equality, the definition of $A_{n,k}$ (32).

B.2 Identity for d-s

To prove (39), the identity satisfied by the coefficients $d_{k-l,l}$, (39) is first multiplied by x^{n-l} and the resulting expression is then summed over n from 0 to ∞ to get

$$
\sum_{n=0}^\infty \sum_{k=0}^n \frac{(-1)^{n+k}}{k!} A_{n,k} d_{k-l,l} x^{n-l} = \sqrt{1+x}.
$$
(B.4)

Using (37), the definition of the coefficients d, and rearranging the left hand side of (B.4), one gets

$$
\begin{aligned}
\sum_{s=0}^l (-1)^s \binom{l}{s} x^{-l} \sum_{n=0}^\infty \sum_{k=0}^n \frac{(-1)^{n+k}}{k!} A_{n,k} \\
\times \left(l - s + \frac{1}{2} \right)^k x^n = \sqrt{1+x},
\end{aligned}
$$
(B.5)

which, using the general identity (B.1), simplifies to

$$
\sum_{s=0}^l (-1)^s \binom{l}{s} x^{-l} (1+x)^{l-s+\frac{1}{2}} = \sqrt{1+x}.
$$
(B.6)

Therefore

$$
\sum_{s=0}^l (-1)^s \binom{l}{s} x^{-l} (1+x)^{l-s} = 1.
$$
(B.7)

Using the binomial formula, the left hand side reduces to

$$
x^{-l} \sum_{s=0}^l \binom{l}{s} (-1)^s (1+x)^{l-s} = x^{-l} [-1 + (1+x)]^l = 1,
$$
(B.8)

concluding the proof.

B.3 Identity for c-s

To prove (38), the identity satisfied by the coefficients c, one proceeds in a similar way as for the coefficients d. The left hand side of (38) is given by

$$
\begin{aligned}
\sum_{n=0}^\infty \sum_{k=0}^n \frac{(-1)^{n+k}}{k!} A_{n,k} \\
\times \left[\sum_{s=0}^{l-1} (-1)^s \binom{l-1}{s} (l-s)^k - d_{k-l+1,l-1} \right] x^{n-l},
\end{aligned}
$$
(B.9)

which, after setting $l = \tilde{l} + 1$, becomes

$$
\begin{aligned}
\frac{1}{x} \sum_{n=0}^\infty \sum_{k=0}^n \frac{(-1)^{n+k}}{k!} A_{n,k} \\
\times \left[\sum_{s=0}^{\tilde{l}} (-1)^s \binom{\tilde{l}}{s} (\tilde{l}-s+1)^k - d_{k-\tilde{l},\tilde{l}} \right] x^{n-\tilde{l}}
\end{aligned}
$$
(B.10)

Using (B.4), this can further be written as

$$
\frac{1}{x} \sum_{s=0}^{\tilde{l}} (-1)^s \binom{\tilde{l}}{s} x^{-\tilde{l}} \sum_{n=0}^\infty \sum_{k=0}^n \frac{(-1)^{n+k}}{k!} A_{n,k} (\tilde{l}-s+1)^k x^n - \frac{\sqrt{1+x}}{x}.
$$
(B.11)

The obtained expression is then simplified using the general identity (B.1)

$$
\begin{aligned}
\frac{1}{x} &\sum_{s=0}^{\tilde{l}} (-1)^s \binom{\tilde{l}}{s} x^{-\tilde{l}} (1+x)^{\tilde{l}-s+1} - \frac{\sqrt{1+x}}{x} \\
&= \frac{1+x}{x} \sum_{s=0}^{\tilde{l}} (-1)^s \binom{\tilde{l}}{s} x^{-\tilde{l}} (1+x)^{\tilde{l}-s} - \frac{\sqrt{1+x}}{x} \\
&= \frac{1+x}{x} x^{-\tilde{l}} \sum_{s=0}^{\tilde{l}} \binom{\tilde{l}}{s} (-1)^s (1+x)^{\tilde{l}-s} - \frac{\sqrt{1+x}}{x} \\
&= \frac{1+x}{x} x^{-\tilde{l}} [-1 + (1+x)]^{\tilde{l}} - \frac{\sqrt{1+x}}{x} \\
&= \frac{1+x-\sqrt{1+x}}{x}.
\end{aligned}
$$
(B.12)

The final expression in (B.12) is precisely the right hand side of (38), thus concluding the proof.

C 3-Cocycle condition

A quasi-bialgebra over $\mathbb{C}$ is defined as an associative algebra A together with the counit $\epsilon : A \to \mathbb{C}$, coproduct $\Delta : A \to A \otimes A$ and an invertible element $\Phi \in A \otimes A \otimes A$, called the co-associator, with the following properties [24]

$$(1 \otimes \Delta)\Delta a = \Phi^{-1}[(\Delta \otimes 1)\Delta a]\Phi, \quad \forall a \in A, \qquad (C.1)$$

$$[(\Delta \otimes 1 \otimes 1)\Phi][(1 \otimes 1 \otimes \Delta)\Phi]$$
$$= (\Phi \otimes 1)[(1 \otimes \Delta \otimes 1)\Phi](1 \otimes \Phi), \qquad (C.2)$$

$$(\epsilon \otimes 1)\Delta a = a = (1 \otimes \epsilon)\Delta a, \quad \forall a \in A, \qquad (C.3)$$

$$(1 \otimes \epsilon \otimes 1)\Phi = 1 \otimes 1. \qquad (C.4)$$

Property (C.2) is called the 3-cocycle condition.

Starting with a bialgebra with coassociative coproduct Δ_0, and counit ϵ, it is possible to construct a quasi-bialgebra using twist $\mathcal{F}$, satisfying normalization condition, but not necessarily satisfying the cocycle condition. The coproducts are related by

$$\Delta a = \mathcal{F}\Delta_0 a \mathcal{F}^{-1}, \quad \forall a \in A. \qquad (C.5)$$

This construction indeed leads to a quasi-bialgebra. Property (C.3) trivially follows from the normalization condition and the co-associator is given by

$$\Phi = (\mathcal{F} \otimes 1)[(\Delta_0 \otimes 1)\mathcal{F}][(1 \otimes \Delta_0)\mathcal{F}^{-1}](1 \otimes \mathcal{F}^{-1}). \quad (C.6)$$

It trivially satisfies the property (C.4), which also follows from the normalization condition. It is also straightforward, although tedious, to show that the remaining property of quasi-bialgebra – the 3-cocycle condion (C.2) – also holds. It follows directly from the construction (C.6) and from the coassociativity of the coproduct Δ_0.

Similarly, one can extend this construction to quasi-bialgebroid.

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