Hopf algebroid twists in deformation quantization

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1 Hopf algebroids (quantum groupoids)

Hopf algebroids are generalizations of Hopf algebras; while commutative Hopf algebras arise as function algebras on groups, commutative Hopf algebroids come as structure on functions on groupoids. A Hopf algebroid comprises two algebras, the **base algebra** A and the **total algebra** $H = (H, \mu)$ which is A-bimodule and is equipped with coassociative coproduct $\Delta : H \to H \otimes_A H$ with a counit ϵ. (H, Δ, ϵ) is hence a comonoid in the category of A-bimodules (we say that H is an A-**coring**). Base algebra is a generalization of the field of units for H: more precisely, there is a **source map** $\alpha : A \to H$ and a **target map** $\beta : A^{\rm op} \to H$ which are algebra maps with commuting images $[\alpha(a), \beta(a')] = 0$ that is $a, a' \in A$; we then say that H is an $A \otimes A^{\rm op}$-**ring**.

Definition 1 *[4, 1, 3] An $A \otimes A^{\rm op}$-ring (H, μ, α, β) and an A-coring (H, Δ, ϵ) on the same A-bimodule H form a* **left** A**-bialgebroid** $(H, \mu, \alpha, \beta, \Delta, \epsilon)$ *if they satisfy the following compatibility conditions:*

- *(C1) the underlying A-bimodule structure of the A-coring structure is determined by the source and target map (part of the $A \otimes A^{\rm op}$-ring structure): $r.a.r' = \alpha(r)\beta(r')a$.*
- *(C2) formula $\sum_\lambda h_\lambda \otimes f_\lambda \mapsto \epsilon(\sum_\lambda h_\lambda \alpha(f_\lambda))$ defines an action $\blacktriangleright$: $H \otimes A \to A$ which extends the left regular action $A \otimes A \to A$ along the inclusion $A \otimes A \stackrel{\alpha \otimes A}{\longrightarrow} H \otimes A$.*
- *(C3) the linear map $h \otimes (g \otimes k) \mapsto \Delta(h)(g \otimes k)$, $H \otimes (H \otimes H) \to H \otimes H$, induces a well defined action $H \otimes (H \otimes_A H) \to H \otimes_A H$.*

$H \otimes_A H$ is not an algebra by componentwise product in general, hence Δ can not be an algebra map. Indeed the kernel of the projection $H \otimes H \to H \otimes_A H$ of A-bimodules is the right ideal generated by $\beta(a) \otimes 1 - 1 \otimes \alpha(a)$, for $a \in A$. The condition (C3) however ensures that there is a well defined A-subbimodule $H \times_A H \subset H \otimes_A H$, the **Takeuchi product** [1] containing the image of Δ with canonical (componentwise) algebra structure and the corestriction $\Delta| : H \to H \times_A H$ is an algebra map. There is also a right handed version, a right bialgebroid.
A (symmetric) **Hopf algebroid** consists of a left A-algebroid, of a right $A^{\rm op}$-bialgebroid whose total algebras H are identified and an antipode map $\tau : H \to H$ which is an antihomomorphism of algebras with some compatibilities with the two bialgebroid structures, cf. [1, 2].

2 Ping Xu on twists for Hopf algebroids

Ping Xu [11] generalized the Drinfel'd twists to bialgebroids. Unlike in our publications, I here use the convention of Xu about twist (elsewhere we use $\mathcal{F}$ for his $\mathcal{F}^{-1}$).

Definition 2 *[11] $\mathcal{F} \in H \otimes_A H$ is a Drinfeld twist for a left A-bialgebroid $(H, \mu, \alpha, \beta, \Delta, \epsilon)$ if the 2-cocycle condition*

$$(\Delta \otimes_A \mathrm{id})(\mathcal{F})(\mathcal{F} \otimes_A 1) = (\mathrm{id} \otimes_A \Delta)(\mathcal{F})(1 \otimes_A \mathcal{F})$$

and the counitality $(\epsilon \otimes_A \mathrm{id})(\mathcal{F}) = 1_H = (\mathrm{id} \otimes_A \epsilon)(\mathcal{F})$ hold.
In terms of $\mathcal{F}^{-1}$ we can alternatively write the condition

$$(\mathcal{F}^{-1} \otimes_{A_\star} 1)(\Delta \otimes_{A_\star} \mathrm{id})(\mathcal{F}^{-1}) = (1 \otimes_{A_\star} \mathcal{F}^{-1})(\mathrm{id} \otimes_{A_\star} \Delta)(\mathcal{F}^{-1}).$$

Use the Sweedler-like notation for twist $\mathcal{F} = f^1 \otimes f_1$.

Theorem 1 *[11] If H is a left A-bialgebroid then the formula*

$$a \star b = (f^1 \blacktriangleright a)(f_1 \blacktriangleright b)$$

defines an associative algebra $A_\star = (A, \star)$ structure on A with the same unit; the formulas $\alpha_{\mathcal{F}}(a) = \alpha(f^1 \blacktriangleright a) f_1$ and $\beta_{\mathcal{F}}(a) = \beta(f^1 \blacktriangleright a) f_1$ define respectively an algebra homomorphism and antihomomorphism $A_\star \to H$ turning H into a $A_\star$-ring; the formula

$$\Delta_{\mathcal{F}}(h) = \mathcal{F}^{-1}\Delta(h)\mathcal{F}$$

defines a map $\Delta_{\mathcal{F}} : H \to H \otimes_{A_\star} H$ which is coassociative and counital with the same counit. Moreover, $(H, \mu, \alpha_{\mathcal{F}}, \beta_{\mathcal{F}}, \Delta_{\mathcal{F}}, \epsilon)$ is a left $A_\star$-bialgebroid.

Unlike in Xu's work, we can also treat the antipode (if the latter is invertible). The formula for twisting the antipode from Hopf algebra case does not extend straightforwardly, but the less used formula for the inverse of the antipode does!
Basic example: $A = C^\infty(M)$ where M is a smooth manifold. $H = \mathcal{D}$ is the algebra of **differential operators** with smooth coefficients. Define $\Delta(D)(f, g) = D(f \cdot g)$. The base is commutative and $\alpha = \beta$ is the canonical embedding of functions into differential operators; the counit is taking the constant term. (Xu claims there is no antipode, but there is for some manifolds as clarified by N. Kowalzig). Here $\blacktriangleright$ is the usual action of differential operators on functions.
Deformation quantization: Xu extends $C^\infty(M)$ to $C^\infty(M)[[h]]$ where h is a formal variable. Then $\mathcal{D}[[h]]$ is a left A-bialgebroid by extending the scalars; there he implicitly considers the completed tensor product. He proves:

Theorem 2 *If M is Poisson manifold and the formal bidifferential operator $\mathcal{F} \in \mathcal{D}[[h]]$ defines a deformation quantization of M, Then $\mathcal{F}$ is a Drinfeld twist for left $C^\infty(M)[[h]]$-bialgebroid $\mathcal{D}[[h]]$. Consequently, each deformatiom quantization defines also a deformation of that bialgebroid.*

We are interested how to use the Hopf algebroid techniques to find formulas for $\mathcal{F}$ and also to describe the above Hopf algebroid in detail in special cases.

3 Twists for doubles of enveloping algebras

3.1 Phase spaces of Lie type as Hopf algebroids

Throughout, $\mathfrak{g}$ is a fixed Lie algebra over $\mathbb{C}$ with basis $\hat{x}_1, \ldots, \hat{x}_n$, $U(\mathfrak{g})$ is the universal enveloping and $S(\mathfrak{g})$ the symmetric algebra of $\mathfrak{g}$; the generators of $U(\mathfrak{g})$ also denoted $\hat{x}_1, \ldots, \hat{x}_n$ but the corresponding generators of $S(\mathfrak{g})$ are $x_1, \ldots, x_n$. The structure constants $C^\lambda_{\mu\nu}$ are given by

$$[\hat{x}_\mu, \hat{x}_\nu] = C^\lambda_{\mu\nu}\hat{x}_\lambda. \tag{1}$$

Let $\partial^1, \ldots, \partial^n$ be the dual basis of $\mathfrak{g}^*$, which are also (commuting) generators of $S(\mathfrak{g}^*)$. Let $\hat{S}(\mathfrak{g}^*)$ be the formal completion of $S(\mathfrak{g})$. We introduce an auxiliary matrix $\mathcal{C}$ with entries

$$\mathcal{C}^\alpha_\beta := C^\alpha_{\beta\gamma}\partial^\gamma \in S(\mathfrak{g}^*), \tag{2}$$

where we adopted the Einstein convention of understood summation over repeated indices. In this notation introduce the matrices $\mathcal{O} := \exp(\mathcal{C}) \in M_n(\hat{S}(\mathfrak{g}^*))$ and

$$\phi := \frac{-\mathcal{C}}{e^{-\mathcal{C}} - 1} = \sum_{N=0}^\infty \frac{(-1)^N B_N}{N!}\mathcal{C}^N, \qquad \tilde{\phi} := \frac{\mathcal{C}}{e^{\mathcal{C}} - 1} = \sum_{N=0}^\infty \frac{B_N}{N!}\mathcal{C}^N, \tag{3}$$

where B_N are the Bernoulli numbers. By $\hat{A}_n$ denote the completion by the degree of a differential operator of the n-th Weyl algebra A_n with generators $x_1, \ldots, x_n, \partial^1, \ldots, \partial^n$. The underlying vector space of $\hat{A}_n$ is thus a completion of $S(\mathfrak{g}) \otimes S(\mathfrak{g}^*)$.

Now define the elements $\hat{x}^\phi, \hat{y}^\phi \in \hat{A}_n$

$$\hat{x}^\phi_\rho := \sum_\tau x_\tau \phi^\tau_\rho, \quad \hat{y}^\phi_\rho := \sum_\tau x_\tau \tilde{\phi}^\tau_\rho. \tag{4}$$

Then $\hat{x}_\rho \mapsto \hat{x}^\phi_\rho$ extends to a unique algebra map $\alpha : U(\mathfrak{g}) \to \hat{A}_n$ and $\hat{x}_\rho \mapsto \hat{y}^\phi_\rho$ to a unique algebra map $\beta : U(\mathfrak{g})^{\rm op} \to \hat{A}_n$. This *realization* map is related to the symmetrization (PBW) isomorphism $S(\mathfrak{g}) \cong U(\mathfrak{g})$; for other coalgebra isomorphisms we have different choice of ϕ (or different ordering). Our ϕ corresponds to symmetric ordering (Gutt star product). It follows that $\hat{y}^\phi_\alpha = \hat{x}^\phi_\beta \mathcal{O}^\beta_\alpha$ and $[\hat{x}^\phi_\alpha, \hat{y}^\phi_\beta] = 0$.
With appropriate completions implicit [7], $H = \hat{A}_n$ is a Hopf algebroid over $U(\mathfrak{g})$ with coproduct Δ which on $\hat{S}(\mathfrak{g}^*) \cong U(\mathfrak{g})^*$ (identified via PBW map) agrees with the transpose of the multiplication in $U(\mathfrak{g})$ and $\Delta(u) = u \otimes 1$ for $u \in U(\mathfrak{g})$. The source and target map are α and β above!
Alternatively, the map $U(\mathfrak{g}) \to \hat{A}_n$ sends an element in $U(\mathfrak{g})$ to an operator on $\hat{S}(\mathfrak{g})$; this action is a right Hopf action and the total algebra H is the smash product of $U(\mathfrak{g})$ and $\hat{S}(\mathfrak{g}^*)$. This is however isomorphic as an algebra to $\hat{A}_n$. We shall thus identify $\hat{x}_\mu \in U(\mathfrak{g})$ and $\hat{x}^\phi_\mu \in \hat{A}_n$ etc.

3.2 New twist

Theorem 3 *[8] In symmetric ordering, the deformed coproduct Δ on $\hat{S}(\mathfrak{g}^*) \cong U(\mathfrak{g})^*$ is given by*

$$\Delta\partial^\mu = 1 \otimes \partial^\mu + \partial^\alpha \otimes [\partial^\mu, \hat{x}_\alpha] + \frac{1}{2}\partial^\alpha\partial^\beta \otimes [[\partial^\mu, \hat{x}_\alpha], \hat{x}_\beta] + \ldots = \exp(\partial^\alpha \otimes \mathrm{ad}\,(-\hat{x}_\alpha))(1 \otimes \partial^\mu)$$

Hadamard's formula $\mathrm{Ad}(\exp(A))(B) = \exp(\mathrm{ad}\,A)(B)$ then implies $\Delta\partial^\mu = \exp(-\partial^\rho \otimes \hat{x}_\rho)(1 \otimes \partial^\mu)\exp(\partial^\sigma \otimes \hat{x}_\sigma)$ and undeformed case $\Delta_0\partial^\mu = \exp(-\partial^\alpha \otimes x_\alpha)(1 \otimes \partial^\mu)\exp(\partial^\alpha \otimes x_\alpha)$. Comparing the two expressions we obtain $\Delta(\partial^\mu) = \mathcal{F}_L^{-1}\Delta_0(\partial^\mu)\mathcal{F}_L$ where $\mathcal{F}_L$ is the product of the two exponentials:

$$\mathcal{F}_L = \exp(-\partial^\rho \otimes x_\rho)\exp(\partial^\sigma \otimes \hat{x}_\sigma) \tag{5}$$

To show that $\mathcal{F}_L$ is in fact a twist we prove analogous formulas for the rest of generators, say $\Delta(x_\mu) = \mathcal{F}_L^{-1}(x_\mu \otimes 1)\mathcal{F}_L$. Applying "inner" exponentials, we easily get

$$\exp(\partial^\rho \otimes x_\rho)(x_\mu \otimes 1)\exp(-\partial^\sigma \otimes x_\sigma) = x_\mu \otimes 1 + 1 \otimes x_\mu = x_\mu \otimes 1 + 1 \otimes \hat{y}_\tau \mathcal{O}^\tau_\mu.$$

Now we need to apply outer exponentials to each of the two summands. We again use the Hadamard's formula and the formula

$$\mathrm{ad}^k(\partial^\rho \otimes \hat{x}_\rho)(1 \otimes \hat{x}_\mu) = [(-\mathcal{C})^k]^\tau_\mu \otimes \hat{x}_\tau$$

to obtain

$$\exp(-\partial^\sigma \otimes \hat{x}_\sigma)(x_\mu \otimes 1)\exp(\partial^\rho \otimes x_\rho) = x_\mu \otimes 1 - (\tilde{\phi}^{-1})^\tau_\mu \otimes \hat{x}_\tau$$

and

$$\exp(-\partial^\sigma \otimes \hat{x}_\sigma)(1 \otimes \hat{y}_\tau \mathcal{O}^\tau_\mu)\exp(\partial^\rho \otimes x_\rho) = (1 \otimes \hat{y}_\tau)\Delta(\mathcal{O}^\tau_\sigma) = \Delta(x_\mu)$$

where we used known fact that $\Delta(\hat{y}_\tau) = 1 \otimes \hat{y}_\tau$. In sum, we obtained the additional $x_\mu \otimes 1 - (\tilde{\phi}^{-1})^\tau_\mu \otimes \hat{x}_\tau$, but this can be shown to be in the ideal! Indeed, $x_\mu = \hat{x}_\sigma(\phi^{-1})^\sigma_\mu$ and $\tilde{\phi}^{-1} = \mathcal{O}\phi^{-1}$ and the right ideal is generated by $\hat{x}_\rho \otimes 1 - \mathcal{O}^\tau_\rho \otimes \hat{x}_\tau$. It is clear here that for the twist to work it is essential that the base is larger than the field. This freedom needed for twist has also some tentative physical interpretation vaguely similar to gauge freedom [5].
Finally, one proves that the undeformed right ideal generated by $x_\mu \otimes 1 - 1 \otimes x_\mu$ after twist ends in the deformed right ideal.
There is also an alternative twist in terms of y^ϕ_α.

$$\mathcal{F}_R = \exp(-x_\alpha \otimes \partial^\alpha)\exp(\hat{y}^\phi_\beta \otimes \partial^\beta) \tag{6}$$

Theorem 4 *(with S. Meljanac) $\mathcal{F}_L$ and $\mathcal{F}_R$ are Drinfel'd twists for Hopf algebroid on completed Weyl algebra and by twisting they yield the Heisenberg double of the corresponding universal enveloping algebra with its canonical Hopf algebroid structure.*

The formula for the twist (6) can be modified for other orderings (work in progress with Martina Stojić). The key lemma in that direction is that in the symmetric ordering $\exp(\partial^\beta \otimes \hat{x}_\beta) = \sum_I \partial^I \otimes \hat{x}_I$ (I runs over multiindices) where on the right hand side we have an infinite-dimensional version of the *canonical element* (does not depend on ordering). If we act on vacuum with the left hand side in the second tensor factor we obtain $\exp(\partial^\beta \otimes x_\beta)$; to get the same result in other orderings one seeks for function K^{-1} such that $\exp((K^{-1}(\partial))^\beta \otimes \hat{x}_\beta)(1 \otimes |0\rangle) = \exp(\partial^\beta \otimes x_\beta)$. The methods for finding K^{-1} are known from [9].

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