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κ-deformed covariant quantum phase spaces as Hopf algebroids

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ABSTRACT

We consider the general $D=4$ $(10+10)$-dimensional κ-deformed quantum phase space as given by Heisenberg double $\mathcal{H}$ of $D=4$ κ-deformed Poincaré–Hopf algebra $\mathbb{H}$. The standard $(4+4)$-dimensional κ-deformed covariant quantum phase space spanned by κ-deformed Minkowski coordinates and commuting momenta generators $(\widehat{x}_\mu, \widehat{p}_\mu)$ is obtained as the subalgebra of $\mathcal{H}$. We study further the property that Heisenberg double defines particular quantum spaces with Hopf algebroid structure. We calculate by using purely algebraic methods the explicit Hopf algebroid structure of standard κ-deformed quantum covariant phase space in Majid–Ruegg bicrossproduct basis. The coproducts for Hopf algebroids are not unique, determined modulo the coproduct gauge freedom. Finally we consider the interpretation of the algebraic description of quantum phase spaces as Hopf algebroids.

1. Introduction

Recently several papers appeared (see e.g. [1–5]) discussing the bialgebroid and Hopf algebroid structures of deformed quantum phase spaces with noncommutative coordinates satisfying the κ-deformed Minkowski space–time algebra [6–8]. In these considerations the covariance under the action of κ-deformed quantum Poincaré symmetries was not properly exposed.[1] However, to obtain κ-deformed quantum phase space with built-in quantum κ-covariance property it is convenient to employ the Heisenberg double construction, which in a given κ-Poincaré algebra basis leads to unique choice of covariant κ-deformed quantum phase space algebra. Further using the property that Heisenberg double algebra defines a Hopf algebroid, we shall point out here some new properties of the quantum phase spaces equipped with coalgebraic sector.

The notion of Hopf algebroid introduces new class of quantum spaces which provide for deformed quantum phase spaces the bialgebraic structure [9–12] with a freedom in coproducts which will be called the coproduct gauge. The mathematical origin of such a freedom is linked with the bialgebroid structure of $\mathcal{H}$ described briefly as follows:

1. Algebraic sector of $\mathcal{H}$ is given by the total algebra H, and its subalgebra A is called the base algebra.
2. There exist two maps: the source map $\alpha : A \longrightarrow H$ which is algebra homomorphism and the target map $\beta : A \longrightarrow H$ which is algebra antihomomorphism. The images of the two maps from A into H commute in H, i.e. for any $a, b \in A$

$$[\alpha(a)\,,\beta(b)] = 0, \qquad (1)$$

what permits an (A, A)-bimodule structure on H, namely $a.h.b = h\beta(a)\alpha(b)$.
3. For bialgebroids the notion of Hopf-algebraic coproducts $\Delta : H \longrightarrow H \otimes H$ ($\otimes$ describes standard tensor product) is replaced by the coproduct map from H into (A, A)-bimodule product $H \otimes_A H$, with respect to the above bimodule structure on H (see [10,12]). It appears that $H \otimes_A H$ as the codomain of the co-algebraic sector of bialgebroids does not inherit the algebra structure from $H \otimes H$.[2] $H \otimes_A H$ can be defined as the quotient of $H \otimes H$ by the left ideal $\mathcal{I}_L$ generated as a left ideal

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[1] Following [7] the covariance of κ-Minkowski algebraic relations under the action of κ-deformed quantum Poincaré algebra is an unseparable part of the definition of κ-deformed noncommutative Minkowski space.

[2] The factorwise algebra multiplication is however well defined in the Takeuchi product [5,12], certain subbimodule $H \times_A H \subset H \otimes_A H$ introduced by Takeuchi [9]. The coproduct Δ is required to take values within $H \times_A H$ and $\Delta : H \to H \times_A H$ must respect the multiplication.

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by the subset of $H \otimes H$ consisting of all elements of the form $\alpha(a) \otimes 1 - 1 \otimes \beta(a)$ [10],[3]

$$\mathcal{I}_L = \langle \alpha(a) \otimes 1 - 1 \otimes \beta(a), \quad a \in A \rangle. \tag{2}$$

If we introduce the canonical choice $\alpha(a) = a$, one gets the left ideal in the form

$$\mathcal{I}_L = \langle a \otimes 1 - 1 \otimes \beta(a), \quad a \in A \rangle, \tag{3}$$

with the target map β (with $\beta(a) \in H$) determining the coalgebra gauge freedom.

If the base algebra is commutative (e.g. for canonical Heisenberg algebra, see [1–4]) and for some other special classes of Hopf algebroids, one can introduce the coproduct gauge as defined by two-sided ideal, namely the bialgebroid coproduct takes values in the standard tensor product $H \otimes H$ divided by a two-sided ideal $\mathcal{I} \subset H \otimes H$.

We recall that in Hopf-algebraic κ-deformation scheme the general covariant κ-deformed phase space is provided by the Heisenberg double[4] $\mathcal{H} = \mathbb{H} \rtimes \widetilde{\mathbb{H}}$ (see e.g. [14]), where $\mathbb{H} = U_\kappa(\widehat{g})$ describe κ-deformed Poincaré–Hopf algebra [15,7] and $\widetilde{\mathbb{H}}$ is the Hopf algebra describing dual κ-deformed quantum Poincaré group. In this paper we employ the general property (see e.g. [10], Sect. 6) that Heisenberg double algebra is equipped with the Hopf algebroid structure. In recent literature (see e.g. [2,3]) the bialgebroid structures of deformed standard quantum phase spaces $(\widehat{x}_\mu, \widehat{p}_\mu)$ with κ-Minkowski space–time sector

$$[\widehat{x}_0, \widehat{x}_i] = -\frac{i}{\kappa}\widehat{x}_i, \qquad [\widehat{x}_i, \widehat{x}_j] = 0, \tag{4}$$

and commuting fourmomenta $\widehat{p}_\mu$ were studied by embedding into canonical quantum phase space algebra (we put $\hbar = 1$)

$$[x_\mu, x_\nu] = [p_\mu, p_\nu] = 0, \qquad [x_\mu, p_\nu] = i\eta_{\mu\nu}. \tag{5}$$

Relation (4) permits the following general class of realizations of quantum phase spaces[5]

$$\widehat{x}_\mu = x_\nu f^\nu_{\ \mu}(p), \qquad \widehat{p}_\mu = p_\mu, \tag{6}$$

where $f^\nu_{\ \mu}(p)$ are chosen in consistency with relations (4), (5) (Jacobi identities) and provide large variety of quantum phase spaces with space–time algebra described by relations (4). The problem with such an approach is the lack of structural indication how to obtain the covariant action of κ-deformed Poincaré–Hopf algebra, which is a part of full definition of quantum κ-deformed Minkowski space (see footnote 1).

In this paper we use the construction of the κ-deformed quantum phase space as the Heisenberg double of $D = 4$ κ-deformed Poincaré–Hopf algebra first presented in [18]. Such construction contains built-in κ-covariance of κ-deformed quantum phase space first observed for κ-Minkowski space–time sector in [7]. In Majid–Ruegg basis [7] we obtain that both κ-Poincaré–Hopf algebra $\mathbb{H}$ and κ-Poincaré group $\widetilde{\mathbb{H}}$ are described by two dual bicrossproduct structures [18,14,19,20], namely[6]

$$\mathbb{H} = U(so(1,3)) \triangleright\!\!\blacktriangleleft_\kappa \mathcal{T}^4 \quad \overset{duality}{\longleftrightarrow} \quad \widetilde{\mathbb{H}} = \widetilde{\mathcal{T}}^4_\kappa \triangleright\!\!\blacktriangleleft_\kappa \mathcal{L}^6, \tag{7}$$

where $\mathcal{L}^6$ describe the functions of Abelian Lorentz parameters $\lambda_{\mu\nu}$ which are dual to $U(so(3,1))$ and $\mathcal{T}^4$ is the fourmomenta sector dual to the algebra $\widetilde{\mathcal{T}}^4_\kappa$ describing noncommutative functions of κ-deformed Minkowski coordinates (see (4)). One can show that the κ-Poincaré covariance of fourmomentum sector $\mathcal{T}^4$ can be derived from the bicrossproduct structure of $\mathbb{H}$ (see (7)).

The κ-deformed Poincaré algebra $\mathbb{H}$ acts on standard κ-deformed quantum phase space $\widetilde{\mathcal{T}}^4_\kappa \otimes \mathcal{T}^4$ in a covariant way. Further, the covariant action of $\mathbb{H}$ on $\mathcal{L}^6$ follows from the duality of $\mathcal{L}^6$ and $U(so(3,1))$ algebras as well as the semidirect product of the coalgebra sectors in $\mathbb{H}$ and $\widetilde{\mathbb{H}}$.

Firstly, in Section 2, we recall the results presented in [18] and provide the $10+10$-dimensional generalized κ-deformed quantum phase space, with standard dual pair of generators $(\widehat{x}_\mu, \widehat{p}_\mu)$ and the dual canonical pair $(\widehat{\lambda}_{\mu\nu}, \widehat{m}_{\mu\nu})$ of Lorentz group parameters and Lorentz algebra generators. In such a way we obtain the κ-deformation of canonical generalized phase space which in undeformed case was used for the geometric description of elementary particles with translational and spin degrees of freedom (see e.g. [21–25]).

In present paper, in order to provide the explicit formulae describing Hopf bialgebroid structure, we shall restrict our considerations to standard κ-deformed quantum phase space for spinless particles, given by Heisenberg double $\mathcal{H}^{(4,4)} \equiv \mathcal{H}_{(p,x)} = \mathbb{H}_p \rtimes \mathbb{H}_x$, where the dual Hopf algebras $\mathbb{H}_p, \mathbb{H}_x$ describe momenta and coordinate sectors

$$\begin{aligned} \mathbb{H}_p: \quad & [\widehat{p}_\mu, \widehat{p}_\nu] = 0, \qquad \Delta(\widehat{p}_i) = \widehat{p}_i \otimes e^{-\frac{\widehat{p}_0}{\kappa}} + 1 \otimes \widehat{p}_i, \\ & \Delta(\widehat{p}_0) = \widehat{p}_0 \otimes 1 + 1 \otimes \widehat{p}_0, \end{aligned} \tag{8}$$

$$\begin{aligned} \mathbb{H}_x: \quad & [\widehat{x}_0, \widehat{x}_i] = -\frac{i}{\kappa}\widehat{x}_i, \qquad [\widehat{x}_i, \widehat{x}_j] = 0, \\ & \Delta(\widehat{x}_\mu) = \widehat{x}_\mu \otimes 1 + 1 \otimes \widehat{x}_\mu. \end{aligned} \tag{9}$$

A Hopf algebroid is defined as a bialgebroid with an antipode. In Section 3, we derive the Hopf algebroid structure of standard κ-deformed quantum phase space $\mathcal{H}_{(p,x)}$. We determine from formula (1) target map (we choose $\alpha(a) = a$) and antipodes; further calculate the coalgebra gauge sector using two alternative ways of determining tensor product $H \otimes_A H$ over noncommutative ring A. In Section 4, we present the interpretation of particular coproduct gauges in the case of nonrelativistic QM phase space and we discuss briefly the dependence of results on the choice of κ-Poincaré algebra basis.

2. Covariant κ-deformed quantum phase spaces as Heisenberg doubles

2.1. Heisenberg double – general remarks

The name "Heisenberg" originates from the simple example of Heisenberg algebra in quantum mechanics, which is the Heisenberg double for dual pair of Abelian Hopf algebras, describing respectively commuting quantum-mechanical momenta and coordinates (see (8), (9) in the limit $\kappa \longrightarrow \infty$). Heisenberg double construction in more general case represents algebraic generalization of the notion of quantum cotangent double for algebraic quantum groups, and provides new models of deformed quantum phase spaces in QM.

A Hopf algebra $\mathbb{H} = (A, m, \Delta, S, \epsilon)$ is a bialgebra (with multiplication $m: A \otimes A \longrightarrow A$, comultiplication $\Delta: A \longrightarrow A \otimes A$ and counit ϵ) supplemented with antipode (coinverse) S. Hopf algebra duality between $\mathbb{H}$ and $\widetilde{\mathbb{H}} = (A^*, m^*, \Delta^*, S^*, \epsilon^*)$ requires

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[3] In fact the bialgebroid structure obtained with the use of left ideal (2) defines a right bialgebroid $\mathcal{H}^R$ (see e.g. [11–13,5]). The choice of the right bialgebroid in this paper is related with further use of coproduct formulae given in [18].

[4] Heisenberg double $\mathcal{H}$ is a special case of smash product algebra $\mathbb{H} \rtimes V$, where V is an $\mathbb{H}$-module algebra, which is in general case not endowed with the Hopf-algebraic structure (see e.g. [16]).

[5] Such realizations can be expressed by differential operators, with p_μ replaced by ($\sqrt{-1}$ times) the partial derivatives ∂_μ [17,1–5].

[6] It follows from formula (7) that the κ-deformation in algebraic sector of $\mathbb{H}$ is present only in cross commutators between fourmomenta and Lorentz generators. From (7), it follows that $\mathbb{H}$ can be described by the action of $U(o(1,3))$ on $\mathcal{T}^4$ as well as the coaction of $\mathcal{T}^4$ on $U(o(1,3))$.

the existence of bilinear pairing $A \otimes A^* \longrightarrow \mathbb{C}$ denoted $< a, a^* >$ ($a \in A, a^* \in A^*$), realizing vector space duality $A \longleftrightarrow A^*$ and relating the multiplication (comultiplication) in $\mathbb{H}$ with comultiplication (multiplication) in $\widetilde{\mathbb{H}}$ [14,17]. For dual Hopf algebras one can introduce the natural action $\mathbb{H} \rhd \widetilde{\mathbb{H}}$

$$a \rhd a^* = a^*_{(1)} < a, a^*_{(2)} >, \tag{10}$$

where we use the coproduct notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$ $(x = a, a^*)$. The action (10) if applied to the product of $a^*, b^* \in \widetilde{\mathbb{H}}$ satisfies the Hopf-algebraic consistency condition [14]

$$\begin{aligned} a \rhd (a^* b^*) &= a^*_{(1)} < a_{(1)}, a^*_{(2)} > b^*_{(1)} < a_{(2)}, b^*_{(2)} > \\ &= (a_{(1)} \rhd a^*)(a_{(2)} \rhd b^*), \end{aligned} \tag{11}$$

i.e. the algebra A^* is an $\mathbb{H}$-module algebra. From relations (10)–(11) one can deduce the cross multiplication rule in the algebra $\mathcal{A} = A \oplus A^*$

$$(a \otimes 1)(1 \otimes a^*) = a^*_{(1)} < a_{(1)}, a^*_{(2)} > a_{(2)}, \tag{12}$$

which completes the multiplication rule in Heisenberg double algebra $\mathcal{A}$. The algebra $\mathcal{A}$ with cross multiplication rule (12) defines the Heisenberg double $\mathcal{H} = \mathbb{H} \rtimes \widetilde{\mathbb{H}}$ with Hopf algebroid structure and provides in noncommutative geometry a class of important examples of quantum spaces with supplemented coalgebra sector.

Following [18], we exhibit below the Heisenberg double $\mathcal{H}^{(10,10)}$ of κ-deformed Poincaré group and its dual κ-deformed Poincaré algebra,[7] in order to get the general κ-deformed quantum phase space containing both the translational and Lorentz sectors. The standard κ-deformed quantum phase space with generators $(\widehat{x}_\mu, \widehat{p}_\mu)$ is a subalgebra $\mathcal{H}^{(4,4)} \subset \mathcal{H}^{(10,10)}$.

2.2. General covariant κ-deformed quantum phase space

2.2.1. κ-Poincaré–Hopf algebra $\mathbb{H}$

The κ-Poincaré–Hopf algebra $\mathbb{H}$ in bicrossproduct basis [7,8] has the following form (with conventions $\mu, \nu, \lambda, \sigma = 0, 1, 2, 3$; $i, j = 1, 2, 3$ and $g_{\mu\nu} = (-1, 1, 1, 1)$)[8]

– *algebra sector*:

$$\begin{aligned} &[\widehat{m}_{\mu\nu}, \widehat{m}_{\lambda\sigma}] = i\left(g_{\mu\sigma}\widehat{m}_{\nu\lambda} + g_{\nu\lambda}\widehat{m}_{\mu\sigma} - g_{\mu\lambda}\widehat{m}_{\nu\sigma} - g_{\nu\sigma}\widehat{m}_{\mu\lambda}\right) \\ &[\widehat{m}_{ij}, \widehat{p}_\mu] = -i\left(g_{i\mu}\widehat{p}_j - g_{j\mu}\widehat{p}_i\right) \\ &[\widehat{m}_{i0}, \widehat{p}_0] = i\widehat{p}_i, \qquad [\widehat{p}_\mu, \widehat{p}_\nu] = 0 \\ &[\widehat{m}_{i0}, \widehat{p}_j] = i\delta_{ij}\left(\kappa \sinh(\frac{\widehat{p}_0}{\kappa})e^{-\frac{\widehat{p}_0}{\kappa}} + \frac{1}{2\kappa}\vec{\widehat{p}}^{\,2}\right) - \frac{i}{\kappa}\widehat{p}_i\widehat{p}_j \end{aligned} \tag{13}$$

– *coalgebra sector*:

$$\begin{aligned} &\Delta(\widehat{m}_{ij}) = \widehat{m}_{ij} \otimes I + I \otimes \widehat{m}_{ij} \\ &\Delta(\widehat{m}_{k0}) = \widehat{m}_{k0} \otimes e^{-\frac{\widehat{p}_0}{\kappa}} + I \otimes \widehat{m}_{k0} + \frac{1}{\kappa}\widehat{m}_{kl} \otimes \widehat{p}_l \\ &\Delta(\widehat{p}_0) = \widehat{p}_0 \otimes I + I \otimes \widehat{p}_0 \\ &\Delta(\widehat{p}_k) = \widehat{p}_k \otimes e^{-\frac{\widehat{p}_0}{\kappa}} + I \otimes \widehat{p}_k \end{aligned} \tag{14}$$

– *counits and antipodes*:

$$\begin{aligned} &S(\widehat{m}_{ij}) = -\widehat{m}_{ij}, \qquad S(\widehat{m}_{i0}) = -\widehat{m}_{i0} + \frac{3i}{2\kappa}\widehat{p}_i \\ &S(\widehat{p}_i) = -e^{\frac{\widehat{p}_0}{\kappa}}\widehat{p}_i, \qquad S(\widehat{p}_0) = -\widehat{p}_0 \\ &\epsilon(\widehat{p}_\mu) = \epsilon(\widehat{m}_{\mu\nu}) = 0. \end{aligned} \tag{15}$$

2.2.2. κ-Poincaré quantum group $\widetilde{\mathbb{H}}$

Using the following canonical form of duality relations

$$<\widehat{x}^\mu, \widehat{p}_\nu> = i\delta^\mu_\nu \qquad <\widehat{\lambda}^\mu{}_\nu, \widehat{m}_{\lambda\rho}> = i(\delta^\mu_\lambda g_{\nu\rho} - \delta^\mu_\rho g_{\nu\lambda}) \tag{16}$$

we obtain the commutation relations defining κ-Poincaré group [6, 19] in the following form

– *algebra sector*:

$$\begin{aligned} &[\widehat{x}^\mu, \widehat{x}^\nu] = \frac{i}{\kappa}(\delta^\mu_0\widehat{x}^\nu - \delta^\nu_0\widehat{x}^\mu), \qquad [\widehat{\lambda}^\mu_\nu, \widehat{\lambda}^\alpha_\beta] = 0 \\ &[\widehat{\lambda}^\mu_\nu, \widehat{x}^\lambda] = -\frac{i}{\kappa}\left((\widehat{\lambda}^\mu_0 - \delta^\mu_0)\widehat{\lambda}^\lambda_\nu - (\widehat{\lambda}^0_\nu - \delta^0_\nu)g^{\mu\lambda}\right) \end{aligned} \tag{17}$$

– *coalgebra sector*:

$$\begin{aligned} &\Delta(\widehat{x}^\mu) = \widehat{\lambda}^\mu{}_\rho \otimes \widehat{x}^\rho + \widehat{x}^\mu \otimes I \\ &\Delta(\widehat{\lambda}^\mu{}_\nu) = \widehat{\lambda}^\mu{}_\rho \otimes \widehat{\lambda}^\rho{}_\nu \end{aligned} \tag{18}$$

– *antipodes and counits*:

$$\begin{aligned} &S(\widehat{\lambda}^\mu{}_\nu) = \widehat{\lambda}_\nu{}^\mu \qquad S(\widehat{x}^\mu) = -\widehat{\lambda}_\nu{}^\mu\widehat{x}^\nu \\ &\epsilon(\widehat{x}^\mu) = 0 \qquad \epsilon(\widehat{\lambda}^\mu{}_\nu) = \delta^\mu{}_\nu \end{aligned} \tag{19}$$

In the Heisenberg double algebra $\mathcal{H}^{(10,10)} = \mathbb{H} \rtimes \widetilde{\mathbb{H}}$ the commutation relations (13) and (17) are supplemented by the following relations obtained from (16), (14) and (18)

– *cross relations*:

$$\begin{aligned} &[\widehat{p}_k, \widehat{x}_l] = -i\delta_{kl} \qquad [\widehat{p}_0, \widehat{x}_0] = i \\ &[\widehat{p}_k, \widehat{x}_0] = -\frac{i}{\kappa}\widehat{p}_k \qquad [\widehat{p}_0, \widehat{x}_l] = 0 \\ &[\widehat{m}_{\lambda\rho}, \widehat{\lambda}^\mu_\nu] = i\left(\delta^\mu_\rho\widehat{\lambda}_{\lambda\nu} - \delta^\mu_\lambda\widehat{\lambda}_{\rho\nu}\right), \qquad [\widehat{p}_\mu, \widehat{\lambda}^\lambda_\rho] = 0 \\ &[\widehat{m}_{\lambda\rho}, \widehat{x}^\mu] = i\left(\delta^\mu_\rho\widehat{x}_\lambda - \delta^\mu_\lambda\widehat{x}_\rho\right) + \frac{i}{\kappa}\left(\delta^0_\rho\widehat{m}_\lambda{}^\mu - \delta^0_\lambda\widehat{m}_\rho{}^\mu\right) \end{aligned} \tag{20}$$

where $\widehat{m}_\lambda{}^\mu = g^{\mu\rho}\widehat{m}_{\lambda\rho}$, $\widehat{m}^\mu{}_\lambda = g^{\mu\rho}\widehat{m}_{\rho\lambda}$.

The generalized covariant κ-deformed phase space is described by sets of commutators (13), (17) and (20). The coproducts (14) and (18) realize the coalgebraic homomorphism of relations (13) and (17), but the relations (20) will be mapped into the coalgebra only in the bialgebroid framework (see Section 3).

2.3. Standard covariant κ-deformed quantum phase space

One can obtain the following distinguished subalgebras of general κ-deformed quantum phase space $\mathcal{H}^{(10,10)}$:

1. By putting consistently in formulae (17)–(20) the value $\widehat{\lambda}^\mu{}_\nu = \delta^\mu{}_\nu$, one obtains the covariant κ-deformed DSR algebra [26,16]. This algebra can be written as the semidirect product $\mathbb{H} \rtimes \widetilde{\mathcal{T}^4_\kappa}$, with the base generators $\widehat{p}_\mu, \widehat{m}_{\mu\nu}, \widehat{x}^\mu$, or due to the duality $\mathcal{T}^4 \longleftrightarrow \widetilde{\mathcal{T}^4_\kappa}$, as the semidirect product $so(3,1) \rtimes (\mathcal{T}^4 \oplus \widetilde{\mathcal{T}^4_\kappa})$. The last formula confirms that the κ-deformed Hopf-algebraic Lorentz sector of $\mathbb{H}$ acts covariantly on the standard κ-deformed quantum phase space $\mathcal{H}^{(4,4)} \equiv \mathcal{H}_{(p,x)} = \mathbb{H}_p \rtimes \widetilde{\mathbb{H}}_x$ defined by relations (8), (9).

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[7] We consider the κ-deformed Poincaré–Hopf algebra in bicrossproduct basis (see e.g. [7]), with classical Lorentz algebra. For some remarks about the dependence of results on the choice of quantum algebra basis see Section 4.

[8] We denote the κ-Poincaré algebra generators by $(\widehat{p}_\mu, \widehat{m}_{\mu\nu})$ and set $\hbar = 1$.

2. If we remove from the general covariant κ-deformed phase space $\mathcal{H}^{(10,10)} = \mathbb{H} \rtimes \widetilde{\mathbb{H}}$ the generators $\widehat{m}_{\mu\nu}$, we obtain in a consistent way the algebra with generators $(\widehat{p}_\mu, \widehat{x}^\mu, \widehat{\lambda}_{\mu\nu})$ which is dual to the considered above κ-deformed DSR algebra.
3. By removing the Lorentz sector from both Hopf algebras $\mathbb{H}$ and $\widetilde{\mathbb{H}}$, one obtains the Heisenberg double $\mathcal{H}^{(4,4)} \equiv \mathcal{H}_{(p,x)}$ with the algebra sector $\mathcal{T}^4 \oplus \widetilde{\mathcal{T}}^4_\kappa$ and the following basic commutators

$$[\widehat{x}^\mu, \widehat{x}^\nu] = \frac{i}{\kappa}(\delta^\mu_0 \widehat{x}^\nu - \delta^\nu_0 \widehat{x}^\mu)$$
$$[p_\mu, p_\nu] = 0$$
$$[\widehat{p}_k, \widehat{x}_l] = -i\delta_{kl} \qquad [\widehat{p}_0, \widehat{x}_0] = i$$
$$[\widehat{p}_k, \widehat{x}_0] = -\frac{i}{\kappa}\widehat{p}_k \qquad [\widehat{p}_0, \widehat{x}_l] = 0. \tag{21}$$

Relations (21) describe the standard κ-deformed quantum phase space. For $\kappa \to \infty$ we get the relativistic quantum phase space described by the canonical Heisenberg commutation relations. The relations (21) cannot be lifted in homomorphic way to the coalgebra sector, i.e. these relations cannot be treated as describing an algebraic sector of a Hopf algebra.[9]

3. Hopf algebroid structure of standard κ-deformed quantum phase space

3.1. The calculation of target map

In this section we shall show that using only relations (1) one can determine for $\mathcal{H}^{(4,4)}$ the ideal (3). We choose the following bialgebroid coproducts in $\mathcal{H}^{(4,4)}$, where base algebra is given by $\widetilde{\mathcal{T}}^4_\kappa$ spanned by generators $\widehat{x}^\mu$

$$\Delta(\widehat{x}^\mu) = 1 \otimes \widehat{x}^\mu,$$
$$\Delta(\widehat{p}_k) = \widehat{p}_k \otimes e^{-\frac{\widehat{p}_0}{\kappa}} + I \otimes \widehat{p}_k, \quad \Delta(\widehat{p}_0) = \widehat{p}_0 \otimes I + I \otimes \widehat{p}_0. \tag{22}$$

The coproducts satisfy the commutation relations (21) in $\mathcal{H}^{(4,4)} \otimes \mathcal{H}^{(4,4)}$; we assume the canonical choice $\alpha(a) = a$. We consider the relations (1) by choosing

1. $a = \widehat{x}_0, b = \widehat{x}_i \longleftrightarrow [\widehat{x}_0, \beta(\widehat{x}_i)] = 0$ (where $i = 1, 2, 3$)
   Choosing $\beta(\widehat{x}_i) = f(p)\widehat{x}_i$ equation (21) implies
$$\beta(\widehat{x}_i) = e^{-\frac{\widehat{p}_0}{\kappa}}\widehat{x}_i. \tag{23}$$
2. $a = \widehat{x}_i, b = \widehat{x}_0 \longleftrightarrow [\widehat{x}_i, \beta(\widehat{x}_0)] = 0$
   Choosing $\beta(\widehat{x}_0) = \widehat{g}(p)\widehat{x}_0 + \widehat{g}_i(p)\widehat{x}_i$ (sum over i understood) we obtain from (21)
$$\beta(\widehat{x}_0) = \widehat{x}_0 - \frac{1}{\kappa}\widehat{p}_i\widehat{x}_i. \tag{24}$$

It is easy to check that for the choice (23)–(24) also the remaining set of eq. (1) $[\widehat{x}_0, \beta(\widehat{x}_0)] = [\widehat{x}_i, \beta(\widehat{x}_j)] = 0$ are valid. Further it can be shown that the construction of the target map as antihomomorphism is consistent with the algebraic relations (4), i.e.

$$[\beta(\widehat{x}_0), \beta(\widehat{x}_i)] = \frac{i}{\kappa}\beta(\widehat{x}_i), \qquad [\beta(\widehat{x}_i), \beta(\widehat{x}_j)] = 0. \tag{25}$$

3.2. The algebraic derivation of coproduct gauge freedom and bialgebraic equivalence classes

In this subsection we shall consider arbitrariness of coproducts satisfying the relations (21) by starting with the formula

$$\widetilde{\Delta}(\widehat{x}_\mu) = \Delta(\widehat{x}_\mu) + \Lambda_\mu(\widehat{x}, \widehat{p}) = \widehat{x}_\rho \otimes \theta^\rho_\mu(\widehat{p}), \tag{26}$$

where $\Delta(\widehat{x}^\mu) = 1 \otimes \widehat{x}^\mu$ and $\theta^\rho_\mu(\widehat{p})$ is the tensor to be determined. The relation (26) describes the homomorphism of deformed quantum phase space algebra (21) if the Λ-tensor operators $\Lambda_\mu \in \mathcal{H}^{(4,4)} \otimes \mathcal{H}^{(4,4)}$ satisfy the relations[10]

$$[\Delta(\widehat{x}_{[\mu}), \Lambda_{\nu]}] + [\Lambda_\mu, \Lambda_\nu] = C^{(\kappa)\rho}_{\mu\nu}\Lambda_\rho, \tag{27}$$
$$[\Delta(\widehat{p}_\mu), \Lambda_\nu] = 0. \tag{28}$$

The relations (27)–(28) are required if the transformation $\Delta(\widehat{x}_\mu) \longrightarrow \widetilde{\Delta}(\widehat{x}_\mu)$ is todescribe the coproduct gauge. Postulating that $(\widetilde{\Delta}(\widehat{x}_\mu), \Delta(\widehat{p}_\mu))$ satisfies the quantum phase space algebra relations (21) one derives algebraically the formulae fixing the tensor $\theta^\rho_\mu(\widehat{p})$

$$\widetilde{\Delta}(\widehat{x}_i) = \widehat{x}_i \otimes e^{\frac{\widehat{p}_0}{\kappa}}, \tag{29}$$
$$\widetilde{\Delta}(\widehat{x}_0) = \widehat{x}_0 \otimes 1 + \frac{1}{\kappa}\widehat{x}_i \otimes e^{\frac{\widehat{p}_0}{\kappa}}\widehat{p}_i. \tag{30}$$

As follows from (26) one gets[11]

$$\Lambda_i = \widehat{x}_i \otimes e^{\frac{\widehat{p}_0}{\kappa}} - 1 \otimes \widehat{x}_i, \tag{31}$$
$$\Lambda_0 = \widehat{x}_0 \otimes 1 - 1 \otimes \widehat{x}_0 + \frac{1}{\kappa}\widehat{x}_i \otimes e^{\frac{\widehat{p}_0}{\kappa}}\widehat{p}_i. \tag{32}$$

One can check subsequently that the relations (27)–(28) are satisfied; in particular in place of eq. (27) we get two equations

$$[\Lambda_\mu, \Lambda_\nu] = C^{(\kappa)\rho}_{\mu\nu}\Lambda_\rho, \qquad [\Delta(\widehat{x}_{[\mu}), \Lambda_{\nu]}] = 0. \tag{33}$$

In the limit $\kappa \longrightarrow \infty$ we obtain that $\widetilde{\Delta}(\widehat{x}_\mu) = \widehat{x}_\mu \otimes 1$.

Further we observe that

1. In relations (26) one can replace $\Lambda_\mu \longrightarrow \alpha\Lambda_\mu$ (α-arbitrary constant) without changing the relations (28) and (33). The resulting coproducts
$$\widetilde{\Delta}_{(\alpha)}(\widehat{x}_i) = (1-\alpha)(1 \otimes \widehat{x}_i) + \alpha\widehat{x}_i \otimes e^{\frac{\widehat{p}_0}{\kappa}}, \tag{34}$$
$$\widetilde{\Delta}_{(\alpha)}(\widehat{x}_0) = (1-\alpha)(1 \otimes \widehat{x}_0) + \alpha(\widehat{x}_0 \otimes 1 + \frac{1}{\kappa}\widehat{x}_i \otimes e^{\frac{\widehat{p}_0}{\kappa}}\widehat{p}_i), \tag{35}$$
   provide parameter-dependent coproduct gauges.
2. Let us replace in (26) ($k \geq 1$)
$$\Lambda_\mu \longrightarrow \Lambda^{(k)}_\mu \equiv A_\mu{}^{\nu_1 \ldots \nu_k}\Lambda_{\nu_1} \ldots \Lambda_{\nu_k}. \tag{36}$$
   In such a way we enlarge the class of possible coproduct gauges to any power of exchange tensor Λ_μ. Because, using short-hand notation

[9] One can add that there were efforts to describe the canonical or deformed Heisenberg algebra in the framework of Hopf algebras (see e.g. [27–29]), but these proposals were in conflict with the basic physical postulate of standard QM that Planck constant $\hbar$ is an universal numerical constant, e.g. the same for all multiparticle states. Such impossibility of providing Hopf-algebraic framework is valid for all Lie algebras centrally extended by numerical central charges.

[10] We denote the algebraic relations described by (9) as $[\widehat{x}_\mu, \widehat{x}_\nu] = C^{(\kappa)\rho}_{\mu\nu}\widehat{x}_\rho$, where $C^{(\kappa)\rho}_{\mu\nu} = \frac{1}{\kappa}(\delta^0_\mu\eta^\rho_\nu - \delta^0_\nu\eta_\mu{}^\rho)$.

[11] Similar tensors $R_\mu = \widehat{x}_\mu \otimes 1 - \tilde{\theta}^\rho_\mu(\widehat{p}) \otimes \widehat{x}_\rho$, where $\tilde{\theta}^\nu_\mu$ is the matrix inverse to θ^ρ_λ (see (26)), have been introduced in [1] for canonical twisted Heisenberg algebra and considered in [3,4] for κ-deformed quantum phase space generated by κ-deformed Poincaré–Hopf algebra with classical Poincaré algebra sector.

$$[\Lambda^{(k)}, \Lambda^{(l)}] \subset \Lambda^{(k+l-1)}, \tag{37}$$

$$[\widetilde{\Delta}(\widehat{x}_\mu), \Lambda^{(k)}] = 0, \tag{38}$$

we see that the homomorphism of coproducts remains valid modulo possible change of coproduct gauge (36), i.e. the algebra (21) is satisfied by coproducts in equivalence class which is defined by the gauge freedom described by the basis (36).

3. Finally we insert in (26) the 2-tensor $\Lambda^{(k,l,m)}$ depending as well on any power of space–time coproducts and powers of fourmomenta coproducts ($k \geq 1, l \geq 0, m \geq 0$)

$$\Lambda_\mu \longrightarrow \Lambda_\mu^{(k,l,m)} \equiv A_\mu{}^{\nu_1 \ldots \nu_k; \rho_1 \ldots \rho_l; \sigma_1 \ldots \sigma_m} \Lambda_{\nu_1} \ldots \Lambda_{\nu_k} \Delta(\widehat{x}_{\rho_1}) \ldots \Delta(\widehat{x}_{\rho_l}) \Delta(\widehat{p}_{\sigma_1}) \ldots \Delta(\widehat{p}_{\sigma_m}). \tag{39}$$

After such substitution and using the properly that $(\Delta(\widehat{x}_\mu), \Delta(\widehat{p}_\mu))$ satisfy the algebra (21) one can show that the coproducts $(\widetilde{\Delta}(\widehat{x}_\mu), \Delta(\widehat{p}_\mu))$ satisfy as well the algebra (21), but modulo coproduct gauges with the basis (39).

4. Further, one can introduce analogous coproduct gauge freedom in the fourmomenta coproducts (8)

$$\widetilde{\Delta}(\widehat{p}_\mu) = \Delta(\widehat{p}_\mu) + \Lambda_\mu^{(k',l',m')}. \tag{40}$$

One can check that the coproducts $\widetilde{\Delta}(\widehat{x}_\mu), \widetilde{\Delta}(\widehat{p}_\mu)$ will satisfy the relations (21) modulo the gauge freedom spanned by the 2-tensors given by (39).

The coproduct gauges (39) define the maximal equivalence class of coproducts in $\mathcal{H}^{(4,4)}$ inside which the algebra (21) of coproducts is satisfied. One can say equivalently that the coproduct gauge-independent description is provided by the equivalence classes of $\mathcal{H}^{(4,4)} \otimes \mathcal{H}^{(4,4)}$ which are obtained if we divide by the ideal with the basis (39). Analogous ideal using alternative methods of calculations[12] has been recently considered in [4].

We can show that the coproduct gauge freedom is within the left ideal $\mathcal{I}_L$ given by the relations (2) by noticing that $\Lambda_\mu(\hat{;}\hat{p}) = (1 \otimes \theta_\mu^\rho)(\alpha(\widehat{x}_\rho) \otimes 1 - 1 \otimes \beta(\widehat{x}_\rho)) \in \mathcal{I}_L$ (see (26)).

3.3. Antipode

In order to obtain the Hopf algebroid one should supplement the bialgebroid $\mathcal{H}$ with an antipode map $\tau : H \longrightarrow H$, which is a linear antiautomorphism of its total algebra H. It is required [10, 12,13] that τ satisfies the properties which, in terms of the right bialgebroid, read

$$\tau\beta = \alpha, \tag{41}$$

$$m(\tau \otimes \mathrm{id})\widetilde{\Delta} = \alpha\epsilon, \tag{42}$$

$$m(\mathrm{id} \otimes \tau)\widetilde{\Delta} = \beta\epsilon\tau. \tag{43}$$

(41) implies that $m(\mathrm{id} \otimes \tau)(\mathcal{I}_L) = 0$ hence the left hand side of (43) does not depend on coproduct gauge. In general, however, $m(\tau \otimes \mathrm{id})(\mathcal{I}_L) \neq 0$, hence the formula (42) can be valid only for the subclass $\widetilde{\Delta}_\gamma$ of coproducts obtained for specific restrictions of coproduct gauges (see Section 3.2). J.H. Lu [10] makes a choice of linear section $\gamma : H \otimes_A H \to H \otimes H$ such that for an abstract coproduct $\Delta : H \to H \otimes_A H$ the map $\Delta_\gamma = \gamma \circ \Delta : H \to H \otimes H$ is a specific gauge for which (42) holds. In [4] a subalgebra $\mathcal{B}$ in $H \otimes H$ is singled out within which all gauge choices satisfy (42).

For our momentum sector $\mathcal{T}^4$, the antipode τ is chosen to agree with the Hopf-algebraic antipode: $\tau(\widehat{p}_0) = S(\widehat{p}_0) = -\widehat{p}_0$ and $\tau(\widehat{p}_i) = S(\widehat{p}_i) = -e^{\frac{\widehat{p}_0}{\kappa}}\widehat{p}_i$ (see (15)). For the coordinate sector, we solve the equations $\tau(\beta(\widehat{x}_\mu)) = \alpha(\widehat{x}_\mu) = \widehat{x}_\mu$ using that τ is an antihomomorphism of algebras and inserting its values on $\widehat{p}_i$. We obtain

$$\tau(\widehat{x}_i) = e^{-\frac{\widehat{p}_0}{\kappa}}\widehat{x}_i = \tau^{-1}(\widehat{x}_i) = \beta(\widehat{x}_i), \tag{44}$$

$$\tau(\widehat{x}_0) = \widehat{x}_0 - \frac{1}{\kappa}\widehat{x}_i\widehat{p}_i = \beta(\widehat{x}_0) - \frac{3i}{\kappa}, \tag{45}$$

$$\tau^2(\widehat{p}_\mu) = \widehat{p}_\mu, \qquad \tau^2(\widehat{x}_i) = \widehat{x}_i, \qquad \tau^2(\widehat{x}_0) = \widehat{x}_0 - \frac{3i}{\kappa}. \tag{46}$$

The counit ϵ satisfies defining equations $h_{(1)}\alpha(\epsilon(h_{(2)})) = h = h_{(2)}\beta(\epsilon(h_{(1)}))$ and on generators is given by

$$\epsilon(\widehat{x}_\mu) = \widehat{x}_\mu, \qquad \epsilon(\widehat{p}_\mu) = 0, \qquad \epsilon(1) = 1. \tag{47}$$

Counit is not an algebra homomorphism but satisfies weaker properties [12,13]

$$\epsilon\left(\alpha(\epsilon(h))h'\right) = \epsilon(hh') = \epsilon\left(\beta(\epsilon(h))h'\right)$$
$$\epsilon(\alpha(\widehat{x}_\mu)) = \epsilon(\beta(\widehat{x}_\mu)) = \widehat{x}_\mu, \tag{48}$$

and $(f, h) \mapsto \epsilon(\alpha(f)h)$ is a right action, where $f \in \widetilde{\mathcal{T}_\kappa^4}$ and $h, h' \in \mathcal{H}^{(4,4)} = \mathcal{T}^4 \oplus \widetilde{\mathcal{T}_\kappa^4}$.

4. Discussion

The noncommutative Hopf algebras are useful as the tool describing quantum symmetry algebras and quantum groups [31–33]. The notion of Hopf algebroids at present is well understood as mathematical structure however its possible physical applications still have to be explored.

An important question in the bialgebroid framework is the physical meaning of the coproduct freedom, which describes the coproduct gauge.

The momentum sector is described in standard Hopf-algebraic way, with momenta coproducts $\Delta(p_\mu)$ describing total 2-particle momentum. For the coordinates we know how to interpret physically only in very special case if we deal with nonrelativistic phase space $(x_i^{(a)}, p_i^{(a)};\ i = 1,2,3;\ a = 1,2)$. For undeformed case $(\kappa \longrightarrow \infty)$ the coproduct gauge freedom (34)–(35) looks as follows

$$\Delta(\widehat{x}_i) = \alpha\widehat{x}_i \otimes 1 + (1 - \alpha)1 \otimes \widehat{x}_i. \tag{49}$$

It can be interpreted as characterizing nonrelativistic center-of-mass coordinate $\widehat{x}_i^{(1+2)}$

$$\widehat{x}_i^{(1+2)} = \frac{m_1}{m_1 + m_2}\widehat{x}_i^{(1)} + \frac{m_2}{m_1 + m_2}\widehat{x}_i^{(2)}, \tag{50}$$

if we put $\alpha = \frac{m_1}{m_1+m_2}$. We see therefore that the coproduct gauge (parameter α) is fixed by dynamical parameter of particles. The values of those parameters is not reflected in the algebraic relations satisfied by total 2-particle momenta and center-of-mass coordinates.

Unfortunately, the center-of-mass coordinate for a pair of relativistic particles (see e.g. [38,39]) leads to more complicated energy-dependent formula for center-of-mass coordinates what does not permit an analogous interpretation.

In this paper we considered the quantum κ-deformed phase space calculated in the Majid–Ruegg basis [7], with the bicrossproduct structure of κ-deformed Poincaré algebra successfully applicable for covariance properties. Such bicrossproduct structure

[12] Basic tool in [4] providing κ-deformed coproducts for κ-Minkowski space–time is a deformed Leibniz formula which describes the action of $\widehat{x}_\mu$ on the product $f(\widehat{x})g(\widehat{x})$ of noncommutative functions on κ-Minkowski space. In their derivation besides the algebra (9) of κ-Minkowski coordinates some additional input was used provided by the cross commutators between the κ-deformed coordinates and fourmomenta (see also [30]).

remains valid as well if we introduce in $\mathcal{T}^4$ (see (7)) an arbitrary fourmomentum basis

$$\widehat{p}_\mu \longrightarrow \widehat{p}'_\mu = F_\mu(\widehat{p}). \tag{51}$$

In particular (see e.g. [34–36]) one can choose the transformation (51) in a way leading to the classical Poincaré basis, and obtain bicrossproduct structure of κ-Poincaré–Hopf algebra with classical algebra basis (see e.g. [37]). The corresponding Heisenberg double will provide different Hopf algebroid formulae for κ-covariant quantum phase space, with classical action of Lorentz generators on fourmomenta but complicated fourmomenta coproducts.

One can address the well-known problem in Hopf-algebraic description of quantum symmetries how to select some privileged algebra bases of the bialgebroid. In the case of standard κ-deformed quantum phase space one can choose the class of κ-Poincaré algebra bases obtained by linear choice $F_\mu(p) = a_\mu^\nu p_\nu$ which via Heisenberg double construction provide class of quantum phase spaces described by κ-deformed centrally extended 8-dimensional Lie algebras. We add in particular that if we choose κ-Poincaré algebra with classical algebra basis, the corresponding quantum κ-deformed phase space algebra is not described by such Lie-algebraic formula (see e.g. [37]).

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