On Kac parameters and spectral decomposition of a matrix of specialized roots of Lie algebra $\mathfrak{sl}_n$

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Abstract

This paper presents interesting spectral properties of a particular integer skew-symmetric matrix, used to encode information on $\mathbb{Z}$-gradation of type $\tilde{\mathbf{s}}$ for classical affine Lie algebra $\widetilde{\mathfrak{sl}}_n$. It is shown that the hidden Kac parameters can be revealed using an explicitly computed eigenvector in a Gram-Schmidt orthogonalization process.

Keywords: $\mathbb{Z}$-gradation for classical (affine) Lie algebra; Kac parameters of finite order automorphisms of simple Lie algebras; Matrix of specialized roots; Integer skew-symmetric matrix; Spectral decomposition.
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1 Introduction

The connection between the $\mathbb{Z}$-gradation of type $\tilde{\mathbf{s}}$ of an affine Lie algebra $\tilde{\mathfrak{g}}$ and the finite order automorphisms of the simple Lie algebra $\mathfrak{g}$ is among the earliest important achievements of the theory of affine Lie algebras (see [7], [8] or [5]). In 1968., V.G. Kac ([7]) showed that all N-th order automorphisms of $\mathfrak{g}$ are, in fact, parameterized (in such a way) by a sequence of nonnegative relatively prime integers $(s_0, s_1, \ldots, s_\ell)$ for which $N = k\sum_{i=0}^{\ell} a_i s_i$. Here the a_i's are positive integer labels of the corresponding Dynkin Diagram $\mathcal{X}$ (see [8]). The sequence $(s_0, s_1, ..., s_\ell)$ is in [14] designated by the term *Kac parameters* and denoted by $\tilde{\mathbf{s}}^{Kac}$. Further, [14] provides an algorithm to determine $\tilde{\mathbf{s}}^{Kac}$ for every $\mathbb{Z}$-gradation of the type $\tilde{\mathbf{s}}$ for the classical affine Lie algebras, i.e., Lie algebras with Dynkin diagrams of the type $\mathcal{X} \in \left\{\mathcal{A}_\ell^{(1)}, \mathcal{B}_\ell^{(1)}, \mathcal{C}_\ell^{(1)}, \mathcal{D}_\ell^{(1)}\right\}$.

An increased interest for researching various gradations, corresponding specializations of Weyl-Kac character formula for affine Lie algebras and for certain realizations of modules (i.e. representations) by vertex operators is triggered in the early 1980's, let us just mention [12], [9], [10]. Many of the vertex operator constructions of integrable highest weight representations and the corresponding $\mathbb{Z}$-gradations and specializations are not described in terms of Kac parameters, see e.g. [10], [11], [13].

The realization of the classical simple Lie algebra by traceless matrices (see [3]) enables us to find the connection between their $\mathbb{Z}$-gradations of type $\mathbf{s} = (s_1, ..., s_{n-1})$ and certain $n \times n$ integer skew-symmetric matrix, called the *matrix of specialized roots*, and denoted by $\mathrm{MSR}_{\mathcal{X}}(\mathbf{s})$. In fact, this matrix is a convenient encoding of the corresponding $\mathbb{Z}$-gradation of the type $\mathbf{s}$ of the classical simple Lie algebra (see Definition 2.1). Using some basic properties of the matrix of specialized roots, the algorithm [14] extracts the proper element from the *Weyl group* which points to hidden Kac parameters. In this process of finding the Kac parameters, the algorithm rearranges all elements of $\mathrm{MSR}_{\mathcal{X}}(\mathbf{s})$ in an interesting way. Especially, in the case of Lie algebra with $\mathcal{X} = \mathcal{A}_\ell^{(1)}$, the upper triangle becomes nonnegative, and the lower triangle becomes nonpositive. This interesting phenomenon motivated the authors of this paper to study the structure of the matrix of specialized roots $A = \mathrm{MSR}_{\mathcal{A}_\ell^{(1)}}(\mathbf{s})$.

Although integer, skew-symmetric and of rank only two, A possesses very intriguing structure. For instance, an eigenvector corresponding to a nonzero eigenvalue determines all elements of the Weyl group that point to the hidden Kac parameters – simple sorting of the components of the imaginary part of this eigenvector reveals the key permutation(s). Furthermore, such an eigenvector of a nonzero eigenvalue can be computed by an explicit formula – its real part is constant and the imaginary part is obtained by the Gram-Schmidt orthogonalization of the vector of partial sums of $\mathbf{s}= (s_1, ..., s_{n-1})$ against the vector of ones. Thus, using pure linear algebra, we have devised a simple and efficient algorithm for computing the key elements of the Weyl group of $\mathfrak{sl}_n$. As an illustration of its power, we easily reproduce the results of the algorithm in [14], see Corollary 3.3.

The rest of the paper is organized as follows. In §2, we set the stage, define the matrix of specialized roots and recall the results from [14]. In §3 we first observe that the matrix of specialized roots is of rank two, and give explicit formulas for its eigenvalues and eigenvectors. We show that the permutation that sorts the imaginary part of an eigenvectors reveals the underlying structure. Section 4 presents an application of our results to bosonic and fermionic realization [13], which is connected with some $\mathbb{Z}$-gradations of type $\mathbf{s}$.

2 The matrix of specialized roots of $\mathfrak{sl}_n$

It is well known that traceless $n \times n$ matrices form the Lie subalgebra $\mathfrak{g} = \mathfrak{sl}_n$ of the general linear (Lie) algebra $\mathfrak{gl}_n$. Clearly, $\dim(\mathfrak{sl}_n) = n^2 - 1$, with the standard basis

$$\mathcal{B} = \left\{E_{ij} \equiv e_i e_j^T \mid i \neq j\right\} \cup \left\{H_i \equiv E_{ii} - E_{i+1,i+1} \mid i = 1, \ldots, n-1\right\} .$$

(Here e_i denotes the i-th column of the $n \times n$ identity matrix I, and $E_{ii} = e_i e_i^T$.)
Now it is very important to notice that the set $\{H_i \mid i = 1, \ldots, n-1\}$ spans one maximal toral subalgebra $\mathfrak{h}$ (consisting of semisimple elements). In the case when $\mathfrak{g}$ is (semi)simple, the Lie subalgebra $\mathfrak{h}$ is Cartan subalgebra. Since $\mathfrak{h}$ is abelian, the set

$$ad_{\mathfrak{g}}\mathfrak{h} = \{ad_{\mathfrak{g}}h | h \in \mathfrak{h}\}$$

is simultaneously diagonalizable, where $ad_{\mathfrak{g}}h(x) = [h, x]$, $x \in \mathfrak{g}$. Hence, $\mathfrak{g}$ is the direct sum of the subspaces $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid ad(h)(x) = \alpha(h)x \ , \ h \in \mathfrak{h}\}$, where α is a linear functional ($\alpha \in \mathfrak{h}^\star$). The set of all nonzero $\alpha \in \mathfrak{h}^\star$ for which $\mathfrak{g}_\alpha \neq 0$ is usually denoted by $\mathcal{R}$ and called the root system. The elements from $\mathcal{R}$ are called the *roots* (of $\mathfrak{g}$ relative to $\mathfrak{h}$). Following this notation, we have

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha, \tag{2.1}$$

where $\mathfrak{g}_0 = \mathfrak{h}$. The decomposition (2.1) is called a *Cartan or root space decomposition.*
In the case of $\mathfrak{sl}_n$, the root system $\mathcal{R}$ corresponds to the Dynkin diagram $\mathcal{A}_{n-1}$ and

$$\Delta = (\alpha_1, ..., \alpha_{n-1})$$

denotes the standard basis of the root system $\mathcal{R}$, i.e. each root $\alpha \in \mathcal{R}$ can be written uniquely as

$$\alpha = \sum_{i=1}^{n-1} k_i \alpha_i,$$

with integral coefficients k_i all nonegative or all nonpositive.
Let $\{\hat{\varepsilon}_i, \ i = 1, \ldots, n\}$ be a dual basis of the basis $\{E_{ii}|, \ i = 1, ..., n\}$ of the diagonal matrices from $\mathfrak{gl}_n$. Let ε_i be the restriction of $\hat{\varepsilon}_i$ on the Cartan subalgebra $\mathfrak{h}$. It is well known (see [2] and [3]) that

$$\begin{aligned} &\mathcal{R} = \{\pm(\varepsilon_i - \varepsilon_j), \ 1 \leq i < j \leq n\} \\ &\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, ..., \alpha_{n-2} = \varepsilon_{n-2} - \varepsilon_{n-1}, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n \ . \end{aligned} \tag{2.2}$$

Let now $\mathbf{s} = (s_1, ..., s_{n-1})$ be an $(n-1)$-tuple of integers and $\alpha = \sum_{i=1}^{n-1} k_i \alpha_i$ an arbitrary root. Define the mapping $\deg : \mathcal{R} \to \mathbb{Z}$ by

$$\deg(\alpha) = \sum_{i=1}^{n-1} k_i s_i. \tag{2.3}$$

For the roots from the basis Δ, it holds that

$$\deg(\alpha_i) = s_i \qquad i = 1, \dots, n-1. \tag{2.4}$$

By (2.2) we can interpret (2.4) as a system of linear equations with variables $\deg(\varepsilon_i)$

$$\deg(\varepsilon_i) - \deg(\varepsilon_{i+1}) = s_i \qquad i = 1, \dots, n-1. \tag{2.5}$$

We recall the following notion of *the matrix of specialized roots* [14].

Definition 2.1. *Let $\Delta = (\alpha_1, ..., \alpha_{n-1})$ be a basis of the simple Lie algebra root system $\mathcal{R}$ (for simple Lie algebra $\mathfrak{sl}_n$). Let $\mathbf{s} = (s_1, ..., s_{n-1})$ be an arbitrary $(n-1)$-tuple of integers, and let the mapping $\deg : \mathcal{R} \to \mathbb{Z}$ for the basis Δ be given by (2.3). The matrix $[a_{ij}]_{i,j} \in \mathbb{Z}^{n\times n}$, defined by*

$$a_{ij} = \deg(\varepsilon_i) - \deg(\varepsilon_j) \qquad i, j \in \{1, \cdots, n\} \tag{2.6}$$

is called the matrix of specialized roots and denoted by $\mathrm{MSR}_{\mathcal{A}^{(1)}_{n-1}}(\mathbf{s})$.

Remark 2.1. Note that the values of variables $\deg(\varepsilon_i)$ are not uniquely determined by (2.5), but the above definition does not depend on the choice of solutions $\deg(\varepsilon_i)$. Indeed the functional $\varepsilon_i - \varepsilon_j$ for $i \neq j$ is always a root and we have following equations

$$a_{i,j} \overset{\text{def}}{=} \deg(\varepsilon_i) - \deg(\varepsilon_j) = \deg(\varepsilon_i - \varepsilon_j) = \deg(\alpha) = \deg(\sum_{i=1}^{n-1} k_i \alpha_i),$$

where $\alpha = \varepsilon_i - \varepsilon_j$. For $i = j$ it is evidently $a_{i,i} = \deg(\varepsilon_i) - \deg(\varepsilon_i) = 0$.

Remark 2.2. In [14], the term matrix of specialized roots is simultaneously defined for all classical simple Lie algebras, i.e. for Lie algebras with Dynkin diagrams of the type $\mathcal{X} \in \{\mathcal{A}^{(1)}_\ell, \mathcal{B}^{(1)}_\ell, \mathcal{C}^{(1)}_\ell, \mathcal{D}^{(1)}_\ell\}$, and the corresponding notation is $\mathrm{MSR}_{\mathcal{X}}(\mathbf{s})$. For the sake of brevity, in this paper we focus only on the case $\mathfrak{sl}_n$ (i.e. Dynkin diagram $\mathcal{A}^{(1)}_{n-1}$) and we simplify the notation by writing $\mathrm{MSR}(\mathbf{s}) \equiv \mathrm{MSR}_{\mathcal{A}^{(1)}_{n-1}}(\mathbf{s})$. Our preliminary results indicate that similar development is possible for other Dynkin diagrams.

Example 2.1. *Let $\mathfrak{g} = \mathfrak{sl}_5$ and $\mathbf{s} = (s_1, s_2, s_3, s_4) = (3, 11, 5, -2)$. The root system is*

$$\mathcal{R} = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_4, \pm(\alpha_1+\alpha_2), \pm(\alpha_2+\alpha_3), \pm(\alpha_3+\alpha_4),$$

$$\pm(\alpha_1+\alpha_2+\alpha_3), \pm(\alpha_2+\alpha_3+\alpha_4), \pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\}.$$

The matrix of specialized roots reads

$$\mathrm{MSR}(\mathbf{s}) = \begin{pmatrix} 0 & \mathbf{3} & 14 & 19 & 17 \\ -3 & 0 & \mathbf{11} & 16 & 14 \\ -14 & -11 & 0 & \mathbf{5} & 3 \\ -19 & -16 & -5 & 0 & -\mathbf{2} \\ -17 & -14 & -3 & 2 & 0 \end{pmatrix} .$$

It is important to notice that every deg(α) for $\alpha \in \mathcal{R}$ is placed on (i,j)-position in the matrix which related to corresponding root vector. For instance, deg($\alpha_2 + \alpha_3$)($= 16$) is placed on the $(2,4)$-position. At the same time the matrix $E_{2,4}$ is root vector for the root $\alpha_2 + \alpha_3 \in \mathfrak{h}^\star$, i.e.

$$[h, E_{2,4}] = (\alpha_2 + \alpha_3)(h) \cdot E_{2,4}, \quad h \in \mathfrak{h} \ .$$

2.1 MSR(s) and the Weyl group

As we emphasized in the introduction, the matrix of specialized roots $\mathrm{MSR}_{\mathcal{X}}(\mathbf{s})$ has one of the main roles in the algorithm for finding Kac parameters of $\mathbb{Z}$-gradation of type $\mathbf{s}$ for classical affine Lie algebras. In fact, the matrix $\mathrm{MSR}(\mathbf{s})$ will serve to detect another base of the root system $\mathcal{R}$ that will guarantee the positivity of the Kac parameters (see [14]). Before we show our proposition about the connection between matrix $\mathrm{MSR}(\mathbf{s})$ and the hidden base we introduce an additional notation related to the Weyl group.

Let $\mathcal{E}$ be the ambient Euclidean space for the root system $\mathcal{R}$. The subgroup of $\mathrm{GL}(\mathcal{E})$ generated by the reflections

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \ , \quad \alpha \in \mathcal{R}$$

is the *Weyl group* $\mathcal{W}$. Since any reflection σ_α leaves the roots system $\mathcal{R}$ invariant, the Weyl group in fact permutes the set $\mathcal{R}$. Moreover $\mathcal{W}$ acts simply transitively on bases, i.e.

i) if Δ' is another base of $\mathcal{R}$, then $\Delta' = \sigma(\Delta)$ for some $\sigma \in \mathcal{W}$;

ii) if $\Delta' = \sigma(\Delta')$ then $\sigma = id$.

Since Weyl group of $\mathfrak{sl}_n$ is isomorphic to *symmetric group* S_n, its cardinality rapidly grows with increasing n, $|\mathcal{W}| = n!$. Hence, to determine an appropriate base of the root system $\mathcal{R}$, the naive search is not feasible.

The following proposition gives us the properties of $\mathrm{MSR}(\mathbf{s})$, which will enable us to find an appropriate base via corresponding element from Weyl group.

Proposition 2.1. *Let* MSR($\mathbf{s}$) *be the matrix of specialized roots for an arbitrary* $(n-1)$*-tuple of integers* $\mathbf{s} = (s_1, ..., s_{n-1})$*. Then*

1. *The* MSR($\mathbf{s}$) *is an skew-symmetric matrix.*
2. *The columns* Ae_i*,* $i = 1, \ldots, n$*, of* A *can be ordered so that*
$$Ae_{j_1} \preceq Ae_{j_2} \preceq \cdots \preceq Ae_{j_{n-1}} \preceq Ae_{j_n},$$
where the inequality $\preceq$ *between two vectors is understood entry-wise.*
3. *The matrix* MSR($\mathbf{s}$) *contains at least one row composed of nonnegative integers.*
4. *The permutation* $\pi = \left(\begin{smallmatrix} 1 & 2 & \cdots & n-1 & n \\ j_1 & j_2 & \cdots & j_{n-1} & j_n \end{smallmatrix}\right) \in S_n (\simeq \mathcal{W})$ *yields the basis*
$$\Delta' = (\pi(\alpha_1), ..., \pi(\alpha_{n-1}))$$
of the root system $\mathcal{R}$*, which satisfies the required condition*
$$\deg(\pi(\alpha_i)) = a_{j_i, j_{i+1}} \in \mathbb{Z}_+, \ \ i = 1, \ldots, n-1 \ .$$
5. *For each cyclic permutation* $(i_1, i_2, \cdots, i_{r-1}, i_r) \in S_n$*, it holds that*
$$a_{i_1,i_2} + a_{i_2,i_3} + \cdots + a_{i_{r-1},i_r} + a_{i_r,i_1} = 0.$$

Proof. (1) From the definition (2.6) it follows that MSR($\mathbf{s}$) is skew-symmetric.
(2) Let i, j, k be arbitrary indices from $\{1, 2, \cdots, n\}$. Using (2.6), we have

$$a_{ij} - a_{ik} = deg(\varepsilon_i) - deg(\varepsilon_j) - deg(\varepsilon_i) + deg(\varepsilon_k) = deg(\varepsilon_k) - deg(\varepsilon_j) = a_{kj} \ . \quad (2.7)$$

Hence, the difference between two elements does not depend on the choice of the row index. In fact, we can simultaneously order elements for all rows in the following way

$$a_{i,j_1} \leq a_{i,j_2} \leq \cdots \leq a_{i,j_{n-1}} \leq a_{i,j_n}, \quad i = 1, \ldots, n \ . \quad (2.8)$$

This implies that the claim *(2)* is true.
(3) Since $a_{i,i} = 0$, (2.8) implies that the j_1^{th} row consist of positive elements.
(4) It is well known that permutation π is an element of the Weyl group of Lie algebra $\mathfrak{sl}_n$ and the Weyl group $\mathcal{W}$ of any simple Lie algebra acts transitively on bases (see for instance [6]). Since the mapping between the roots $\alpha_i \longmapsto \pi(\alpha_i)$ is determined by mapping between the functionals $\varepsilon_k \longmapsto \varepsilon_{j_k}$, $k = 1, 2, \ldots, n$, we can write

$$deg(\pi(\alpha_i)) = deg(\pi(\varepsilon_i - \varepsilon_{i+1})) = deg(\varepsilon_{j_i} - \varepsilon_{j_{i+1}})) = a_{j_i, j_{i+1}}$$

for every $i = 1, 2, \dots, n-1$. Using (2.7) and (2.8), we immediately conclude that

$$deg(\pi(\alpha_i)) = a_{j_i,j_{i+1}} = a_{i,j_{i+1}} - a_{i,j_i} \geq 0 \ .$$

(5) Let $(i_1, i_2, \dots, i_r)$ be an arbitrary cyclic permutation. By definition of MSR($\mathbf{s}$) (2.6)

$$\begin{aligned} a_{i_1,i_2} + \cdots + a_{i_{r-1},i_r} + a_{i_r,i_1} &= deg(\varepsilon_{i_1} - \varepsilon_{i_2}) + \cdots + deg(\varepsilon_{i_{r-1}} - \varepsilon_{i_r}) + deg(\varepsilon_{i_r} - \varepsilon_{i_1}) \\ &= deg[(\varepsilon_{i_1} - \varepsilon_{i_2}) + \cdots + (\varepsilon_{i_{r-1}} - \varepsilon_{i_r}) + (\varepsilon_{i_r} - \varepsilon_{i_1})] \\ &= deg(0) = 0 \ . \end{aligned}$$

□

Remark 2.3. From Proposition 2.1 (*5.*) we know that for any closed circuit of any length $k \geq 2$, $a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_{k-1} i_k} + a_{i_k i_1} = 0$. Thus, the tropical spectral radius of A is

$$\rho_{\text{trop}}(A) = \max_{i_1,\dots,i_k} \frac{a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_{k-1} i_k} + a_{i_k i_1}}{k} = 0.$$

2.2 An example

Example 2.2. Take $n = 12$ and $\mathbf{s} = \begin{pmatrix} 2 & -3 & 4 & -1 & -5 & 6 & -3 & -2 & -5 & 10 & 11 \end{pmatrix}$. The matrix MSR($\mathbf{s}$) is given by

$$\text{MSR}(\mathbf{s}) = \begin{pmatrix} 0 & \mathbf{2} & -1 & 3 & 2 & -3 & 3 & 0 & -2 & -7 & 3 & 14 \\ -2 & 0 & \mathbf{-3} & 1 & 0 & -5 & 1 & -2 & -4 & -9 & 1 & 12 \\ 1 & 3 & 0 & \mathbf{4} & 3 & -2 & 4 & 1 & -1 & -6 & 4 & 15 \\ -3 & -1 & -4 & 0 & \mathbf{-1} & -6 & 0 & -3 & -5 & -10 & 0 & 11 \\ -2 & 0 & -3 & 1 & 0 & \mathbf{-5} & 1 & -2 & -4 & -9 & 1 & 12 \\ 3 & 5 & 2 & 6 & 5 & 0 & \mathbf{6} & 3 & 1 & -4 & 6 & 17 \\ -3 & -1 & -4 & 0 & -1 & -6 & 0 & \mathbf{-3} & -5 & -10 & 0 & 11 \\ 0 & 2 & -1 & 3 & 2 & -3 & 3 & 0 & \mathbf{-2} & -7 & 3 & 14 \\ 2 & 4 & 1 & 5 & 4 & -1 & 5 & 2 & 0 & \mathbf{-5} & 5 & 16 \\ \boxed{7} & \boxed{9} & \boxed{6} & \boxed{10} & \boxed{9} & \boxed{4} & \boxed{10} & \boxed{7} & \boxed{5} & \boxed{0} & \boxed{\mathbf{10}} & \boxed{21} \\ -3 & -1 & -4 & 0 & -1 & -6 & 0 & -3 & -5 & -10 & 0 & \mathbf{11} \\ -14 & -12 & -15 & -11 & -12 & -17 & -11 & -14 & -16 & -21 & -11 & 0 \end{pmatrix} .$$

The 10^{th} row of MSR($\mathbf{s}$) contains only nonnegative integers

$$[a_{10,j}] = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 7 & 9 & 6 & 10 & 9 & 4 & 10 & 7 & 5 & 0 & 10 & 21 \\ \hline \end{array} \ .$$

We can order the selected row

$a_{10,10}$	$a_{10,6}$	$a_{10,9}$	$a_{10,3}$	$a_{10,1}$	$a_{10,8}$	$a_{10,2}$	$a_{10,5}$	$a_{10,4}$	$a_{10,7}$	$a_{10,11}$	$a_{10,12}$
0	4	5	6	7	7	9	9	10	10	10	21

,

This determines the proper element of the Weyl group

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 10 & 6 & 9 & 3 & 1 & 8 & 2 & 5 & 4 & 7 & 11 & 12 \end{pmatrix},$$

which points to the right base Δ', and to the corresponding $(n-1)$-tuple of nonnegative integers

$$\mathbf{s}' = (a_{10,6}, a_{6,9}, a_{9,3}, a_{3,1}, a_{1,8}a_{8,2}, a_{2,5}, a_{5,4}, a_{4,7}, a_{7,11}, a_{11,12}) =$$

$$= (4, 1, 1, 1, 0, 2, 0, 1, 0, 0, 11).$$

Then the functionals

$$\begin{array}{ll}
\alpha'_1 = \varepsilon_{10} - \varepsilon_6 = -(\alpha_6 + \alpha_7 + \alpha_8 + \alpha_9) & \alpha'_2 = \varepsilon_6 - \varepsilon_9 = \alpha_6 + \alpha_7 + \alpha_8 \\
\alpha'_3 = \varepsilon_9 - \varepsilon_3 = -(\alpha_3 + \alpha_4 + \cdots + \alpha_8) & \alpha'_4 = \varepsilon_3 - \varepsilon_1 = -(\alpha_1 + \alpha_2 + \alpha_3) \\
\alpha'_5 = \varepsilon_1 - \varepsilon_8 = \alpha_1 + \alpha_2 + \cdots + \alpha_7 & \alpha'_6 = \varepsilon_8 - \varepsilon_2 = -(\alpha_2 + \alpha_3 + \cdots + \alpha_7) \\
\alpha'_7 = \varepsilon_2 - \varepsilon_5 = \alpha_2 + \alpha_3 + \alpha_4 & \alpha'_8 = \varepsilon_5 - \varepsilon_4 = -\alpha_4 \\
\alpha'_9 = \varepsilon_4 - \varepsilon_7 = \alpha_4 + \alpha_5 + \alpha_6 & \alpha'_{10} = \varepsilon_7 - \varepsilon_{11} = \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} \\
\alpha'_{11} = \varepsilon_{11} - \varepsilon_{12} = \alpha_{11} &
\end{array}$$

form the basis $\Delta' = (\alpha'_1, ..., \alpha'_{n-1})$. The matrix of specialized roots MSR($\mathbf{s}'$) is

$$\mathrm{MSR}(\mathbf{s}') = \begin{pmatrix}
\mathbf{0} & 4 & 5 & 6 & 7 & 7 & 9 & 9 & 10 & 10 & 10 & 21 \\
-4 & \mathbf{0} & 1 & 2 & 3 & 3 & 5 & 5 & 6 & 6 & 6 & 17 \\
-5 & -1 & \mathbf{0} & 1 & 2 & 2 & 4 & 4 & 5 & 5 & 5 & 16 \\
-6 & -2 & -1 & \mathbf{0} & 1 & 1 & 3 & 3 & 4 & 4 & 4 & 15 \\
-7 & -3 & -2 & -1 & \mathbf{0} & \mathbf{0} & 2 & 2 & 3 & 3 & 3 & 14 \\
-7 & -3 & -2 & -1 & \mathbf{0} & \mathbf{0} & 2 & 2 & 3 & 3 & 3 & 14 \\
-9 & -5 & -4 & -3 & -2 & -2 & \mathbf{0} & \mathbf{0} & 1 & 1 & 1 & 12 \\
-9 & -5 & -4 & -3 & -2 & -2 & \mathbf{0} & \mathbf{0} & 1 & 1 & 1 & 12 \\
-10 & -6 & -5 & -4 & -3 & -3 & -1 & -1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & 11 \\
-10 & -6 & -5 & -4 & -3 & -3 & -1 & -1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & 11 \\
-10 & -6 & -5 & -4 & -3 & -3 & -1 & -1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & 11 \\
-21 & -17 & -16 & -15 & -14 & -14 & -12 & -12 & -11 & -11 & -11 & \mathbf{0}
\end{pmatrix}.$$

Note that

- The permutation π redistributes the signs so that in the resulting skew–symmetric matrix the upper triangle is positive, and the lower negative. The entries in each column and in each row are monotonically ordered.
- The difference of any two rows (or any two columns) of MSR($\mathbf{s}'$) is a vector with all entries equal. The same property holds for MSR($\mathbf{s}$). (See Proposition 2.1.)

3 Spectral decomposition of $\mathrm{MSR}(s)$

The matrix of specialized roots can be written in compact form as a difference of two dyads (rank-one matrices), parametrized by the integers s_i. This leads to a different and more natural parametrization and allows detailed spectral analysis of $\mathrm{MSR}(\mathbf{s})$ as a function of $\mathbf{s}$, without using the matrix entries (2.6).

3.1 A dyadic representation of $\mathrm{MSR}(s)$

Our key observation is the following simple proposition.

Proposition 3.1. *Let $\mathbf{s} = (s_1, \ldots, s_{n-1})$ be an arbitrary $(n-1)$–tuple of integers. The matrix of specialized roots $\mathrm{MSR}(\mathbf{s})$ can be represented as*

$$\mathrm{MSR}(\mathbf{s}) \equiv A = EST - T^T S^T E^T, \tag{3.1}$$

where $E = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^T$, $S = \begin{pmatrix} 0 & s_1 & \cdots & s_{n-1} \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$. *The vector $S = (0\ \mathbf{s})$ is uniquely determined by A: $S = e_1^T T^{-T} A T^{-1}$, $\mathbf{s} = e_1^T T^{-T} A T^{-1} (e_2\ \ldots\ e_n)$. Further, it holds $E^T A E = S A S^T = 0$.*

Proof. From the Definition 2.1, it immediately follows $a_{i,i+1} = s_i,\ \ i = 1, \ldots, n-1$ (cf. (2.5)), and from [14] we know that $a_{i,j} = a_{i,i+1} + a_{i+1,i+2} + \cdots + a_{j-2,j-1} + a_{j-1,j}$. Hence, for $i < j$, $a_{ij} = s_i + \cdots + s_{j-1}$. Since A is skew symmetric by definition, $a_{ji} = -a_{ij}$. On the other hand, it is straightforward to check that the righthand side of (3.1) gives precisely the same elements of A. The remaining claims are easily proved; the only small technical details is to note that the inverse of T is an upper bidiagonal matrix with unit diagonal and the value of minus one at the first superdiagonal. $\square$

Remark 3.1. *Note that we have actually defined a linear mapping $\mathcal{M}(\cdot) : \mathbb{R}^{n-1} \longrightarrow Skew_n$, $\mathcal{M}(\mathbf{s}) = EST - T^T S^T E^T$ into the vector space $Skew_n$ of skew–symmetric matrices. It is easy to show that $\mathcal{M}(s) = \mathbf{0}_{n\times n}$ if and only if $\mathbf{s} = \mathbf{0}_{n-1}$. It is also clear that the matrices $\mathrm{MSR}(\mathbf{s})$ are integer vectors in an $(n-1)$–dimensional linear subspace of $Skew_n$.*

3.2 Spectral decomposition

The Schur form of A is diagonal (see e.g. [1]), and we want explicit formulas for its eigenvalues and eigenvectors. The special structure of A certainly raises expectations regarding the structure of its spectral decomposition. Our goal is not to merely compute the eigenvalues and the eigenvectors, but to discover more of the intrinsic structure of A that may, in turn, impart new knowledge in the Lie algebra setting.

In the sequel, the Euclidean inner product is denoted by $<\cdot,\cdot>$, and $\|\cdot\| = \sqrt{<\cdot,\cdot>}$ is the corresponding induced norm. The imaginary unit is $\mathfrak{i} = \sqrt{-1}$.

Theorem 3.1. *Let* $\mathbf{s} = (s_1, \ldots, s_{n-1})$ *be an arbitrary* $(n-1)$*–tuple of real scalars,*[1] *and let* $A = EST - T^T S^T E^T$*, where* E, S, T *are as in Proposition 3.1. Further, set*

$$P = (ST)^T = \begin{pmatrix} 0, & s_1, & s_1 + s_2, & \ldots, & s_1 + \cdots + s_{n-1} \end{pmatrix}^T,$$

and let

$$\breve{P} = P - \frac{1}{E^T E} E E^T P \tag{3.2}$$

be the result of the Gram–Schmidt orthogonalization of P *to* E*. Then* $A \equiv EP^T - PE^T$ *has spectral decomposition* $A = W\Lambda W^*$*, where* $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$*,* $W^*W = I_n$*, and*

$$\begin{aligned} \lambda_1 &= \mathfrak{i}\sqrt{<P,P><E,E> - <P,E>^2}, \\ \lambda_2 &= -\mathfrak{i}\sqrt{<P,P><E,E> - <P,E>^2}, \\ \lambda_3 &= \lambda_4 = \ldots = \lambda_n = 0. \end{aligned}$$

The corresponding eigenvectors (the columns of W*) are*

$$\begin{aligned} w_1 &= \frac{\tilde{w}_1}{\|\tilde{w}_1\|}, \ \textit{where}\ \tilde{w}_1 = \frac{\|\breve{P}\|}{\|E\|}E + \mathfrak{i}\breve{P}, \\ w_2 &= \frac{\tilde{w}_2}{\|\tilde{w}_2\|}, \ \textit{where}\ \tilde{w}_2 = \frac{\|\breve{P}\|}{\|E\|}E - \mathfrak{i}\breve{P}, \\ w_3, \ldots, w_n &= \textit{any orthonormal basis of } span(w_1, w_2)^{\perp}. \end{aligned}$$

Proof. We first note that A is of rank two. There are many simple ways to show this. For instance, the congruence $B^T AB$ with $B = T^{-1}$ yields $B^T AB = \begin{pmatrix} 0 & s \\ -s^T & \mathbf{0}_{n-1,n-1} \end{pmatrix}$, which is of rank two for any $s \neq \mathbf{0}$. (Note here that B is upper bidiagonal matrix with unit diagonal and with the value -1 on the first super–diagonal. In fact, B is the Cholesky factor of Cartan matrix.) The null–space of A is easily calculated as $Ker(A) = span(E,P)^{\perp}$. Hence, $Im(A) = span(E,P)$. If we orthogonalize P against E to compute $\breve{P} = P - \frac{1}{n}EE^TP$, then $span(E,P) = span(E,\breve{P})$, $E^T\breve{P} = 0$. Hence, A has only two non–zero eigenvalues of the form (due to skew–symmetry) $\lambda_1 = \mathfrak{i}\omega$, $\lambda_2 = -\mathfrak{i}\omega$, where $\omega \in \mathbb{R}_+$. The remaining eigenvalues are $\lambda_3 = \cdots = \lambda_n = 0$.

It is easily checked that the negative semidefinite $A^2 = -A^T A$ has double nonzero eigenvalue with eigenvectors E and P:

$$\begin{array}{l} A^2 E = \zeta E \\ A^2 P = \zeta P \end{array} \ \text{with}\ \zeta = \left(\sum_{k=1}^{n-1} s_k (n-k) \right)^2 - n \sum_{k=1}^{n-1} \left(\sum_{j=1}^{k} s_j \right)^2 < 0.$$

[1]At this point, we are only interested in real s_i's; going over to complex s follows *mutatis mutandis*.

Note that $\zeta = (E^T P)^2 - nP^T P$, that is, $\zeta = <P, E>^2 - <P, P><E, E>$. Since P and E are not collinear, $\zeta < 0$ by the Cauchy–Schwarz inequality. The number ζ is double eigenvalue of A^2, and the corresponding eigenspace is $span(E, P) = span(E, \breve{P})$. We immediately conclude that $\omega = \sqrt{-\zeta}$.

Let $\tilde{w}_1 = x + \mathrm{i}y$ be an (essentially unique) eigenvector belonging to λ_1. Then $\tilde{w}_2 = x - \mathrm{i}y$ is an eigenvector belonging to λ_2. Further, we equivalently write

$$Ax = -\omega y, \quad Ay = \omega x, \text{ i.e. } A \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}.$$

From $x^T Ax = -\omega x^T y$ it follows $x^T y = 0$, i.e. x and y are perpendicular in the Euclidean inner product. Comparing $y^T Ax$ and $x^T Ay$ we conclude that also $x^T x = y^T y$, i.e. x and y are of the same Euclidean length. If we take $\|x\|_2 = \|y\|_2 = 1$, then $A = \omega(xy^T - yx^T)$. It remains to find x and y.

Note that, due to skew–symmetry,

$$\begin{aligned} A\breve{P} &= \eta_1 E, \quad \eta_1 = \frac{E^T A\breve{P}}{E^T E} = \frac{E^T AP}{E^T E}, \quad \text{and} \\ AE &= \eta_2 \breve{P}, \quad \eta_2 = -\frac{E^T AP}{\breve{P}^T \breve{P}}. \end{aligned}$$

Putting the two relations together and re–scaling yields

$$A \begin{pmatrix} \frac{\|\breve{P}\|}{\|E\|} E & \breve{P} \end{pmatrix} = \begin{pmatrix} \frac{\|\breve{P}\|}{\|E\|} E & \breve{P} \end{pmatrix} \begin{pmatrix} 0 & \frac{E^T AP}{\|E\|\|\breve{P}\|} \\ -\frac{E^T AP}{\|E\|\|\breve{P}\|} & 0 \end{pmatrix}.$$

Hence, we can write

$$\omega = \frac{E^T AP}{\|E\|\|\breve{P}\|}, \quad \lambda_1 = \mathrm{i}\omega, \quad \tilde{w}_1 = \frac{\|\breve{P}\|}{\|E\|} E + \mathrm{i}\breve{P}.$$

$\square$

Remark 3.2. Note that the proof contains the identity

$$-\zeta = \omega^2 = -\eta_1\eta_2 = <P, P><E, E> - <P, E>^2 = \frac{<AP, E>^2}{<E, E><\breve{P}, \breve{P}>} \in n\mathbb{Z}_+.$$

Note that we can say that the spectrum of A is in $\pm\mathrm{i}\sqrt{n\mathbb{Z}_+}$. (Since $A\breve{P} = \eta_1 E$, it is certainly $\eta_1 \in \mathbb{Z}$. If we compare the first entries in the relation $AE = \eta_2\breve{P}$, using the representation $A = EP^T - PE^T$, we immediately get $\eta_2 = -n$. Also note that $A\breve{P} = \eta_1 E$ yields $\eta_1 = P^T\breve{P}$.)

Remark 3.3. An eigenvector $\tilde{w}_1 = x + \mathrm{i}y$ is determined up to multiplication with non–zero complex scalar $\alpha + \mathrm{i}\beta$. Take an arbitrary $\alpha + \mathrm{i}\beta \neq 0$ and consider $(\alpha + \mathrm{i}\beta)(x + \mathrm{i}y) = \tilde{x} + \mathrm{i}\tilde{y}$. Note that

$$\begin{pmatrix} \tilde{x} & \tilde{y} \end{pmatrix} = \rho \cdot \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \alpha/\rho & \beta/\rho \\ -\beta/\rho & \alpha/\rho \end{pmatrix}, \quad \rho = |\alpha + \mathrm{i}\beta|. \tag{3.3}$$

If the scaling factor is such that $|\alpha + \mathrm{i}\beta| = 1$, then this is just a change of orthogonal basis in the image of A.

Corollary 3.1. *For an arbitrary choice of the eigenvector $\tilde{w} = x + \mathrm{i}y$ of the eigenvalue λ_1 (or λ_2), the n points $\mathfrak{P}_j = (x_j, y_j)$, $j = 1, \ldots, n$, are collinear.*

Proof. The claim obviously holds true for the particularly chosen eigenvectors w_1 and w_2 from the Theorem 3.1. Using Remark 3.3 and the fact that transformation (3.3) preserves collinearity[2] of the points (given by two coordinates in each row of $\begin{pmatrix} x & y \end{pmatrix}$) the claim remains true for any other eigenvector of λ_1 (λ_2). □

Corollary 3.2. *Let π be a permutation that sorts the components of the vector $\breve{P}$ in a nonincreasing sequence, and let $\tilde{w} = x + \mathrm{i}y$ be an arbitrary eigenvector belonging to λ_1 or λ_2. Then π is a sorting permutation for both x and y.*

Proof. Again, the claim obviously holds true for the particularly chosen eigenvectors w_1 and w_2 from the Theorem 3.1. Next note that, due to collinearity, a permutation that sorts the components of x, also sorts the components of y (but not necessarily in the same direction, e.g. y may get sorted in nondecreasing sequence, while x is sorted in non–increasing sequence). And finally, since any transformation from Remark 3.3, with $\rho = 1$, just rotates the line with the points as a rigid body, the order is preserved, up to the direction. An additional scaling with $\rho \neq 1$ preserves the order. □

Corollary 3.3. *Let Π be the matrix representation of the permutation π from Corollary 3.2. Then π is also a sorting permutation for P. Further, all entries in the upper triangle of $\Pi^T A \Pi$ are non–negative; by skew–symmetry all entries in the lower triangle are non–positive. In each row of $\Pi^T A \Pi$ the entries are monotonically increasing; in each of its columns, the entries are monotonically decreasing. For each cluster of k equal entries in $\Pi^T \breve{P}$, the matrix $\Pi^T A \Pi$ has a $k \times k$ zero block on the main diagonal. All columns (rows) with indices in such a zero block are mutually equal.*

Proof. We use the formulas from the proof of Theorem 3.1 to write A as

$$A \equiv EP^T - PE^T = \omega \frac{\|\breve{P}\|_2}{\|E\|_2} (E\breve{P}^T - \breve{P}E^T).$$

[2] Use the determinant criterion for any three points.

The conclusion about the distribution of the signs in $\Pi^T A\Pi$ follows from the following simple fact: if the entries of a vector v are non–decreasing, $v_1 \le \cdots \le v_n$, then $(Ev^T - vE^T)_{ij} \ge 0$ for $1 \le i \le j \le n$, and in each row (column) of $Ev^T - vE^T$ the entries are monotonically increasing (decreasing). □

Corollary 3.4. *The exponential of A can be explicitly written as the rotation*

$$\mathbf{e}^A = I + \frac{\sin\omega}{\omega}A + \frac{1-\cos\omega}{\omega^2}A^2, \quad \omega = \|A\|_2.$$

Proof. It is quite easy to setup the quadratic Lagrange polynomial that resembles the exponential function on the spectrum of A: $L_2(\mathrm{i}\omega) = \mathbf{e}^{\mathrm{i}\omega}$, $L_2(-\mathrm{i}\omega) = \mathbf{e}^{-\mathrm{i}\omega}$, $L_2(0) = 1$. Hence, $L_2(z) = 1 + (\sin\omega/\omega)z + ((1-\cos\omega)/\omega^2)z^2$, and $\mathbf{e}^A = L_2(A)$. Note that this is an instance of the Rodrigues formula for 3×3 rotation, and it also follows as a special case from the generalized Rodrigues formula [4], since it holds $A^3 = -\omega^2 A$. □

Example 3.1. *(Example 2.2 continued.) The vector $\breve{P}$ reads*

$$\breve{P} = \frac{1}{6}\begin{pmatrix} -7 & 5 & -13 & 11 & 5 & -25 & 11 & -7 & -19 & -49 & 11 & 77 \end{pmatrix}.$$

Its sorting permutation is the same π as in Example 2.2, and the permuted matrix is $\Pi^T A\Pi = \mathrm{MSR}(\mathbf{s}')$, as in Example 2.2. Altogether $3!\cdot 2!\cdot 2!$ permutations leave the structure of $\Pi^T A\Pi$ intact. This ambiguity of the sorting permutation is easily seen in the repeated entries of the vector $\breve{P}$.

4 Spectral decomposition of $\mathrm{MSR}(\mathbf{s})$ in the case of bosonic and fermionic realization

As we pointed out in §1, many of the vertex operator constructions of integrable highest weight representations and the corresponding gradations and specializations are not described in terms of Kac parameters. For instance, in [13] the bosonic and fermionic realization of the affine algebra $\mathfrak{gl}_n$ for the conjugacy classes of Heisenberg subalgebra are parametrized by partitions of the integer n

$$\underline{n} = \{n_1, n_2, ..., n_r\}; \qquad n_1 \le n_2 \le \cdots \le n_r\ .$$

The associated $\mathbb{Z}$-gradations of the constructed modules are determined by relatively prime integers $s_0, \ldots, s_{n-1}$ defined by

$$\begin{aligned}\begin{pmatrix} s_0 & s_1 & s_2 & \ldots & s_{n-1}\end{pmatrix} = N\Bigg(&\frac{n_1+n_r}{2n_1n_r}, \underbrace{\frac{1}{n_1},\ldots,\frac{1}{n_1}}_{n_1-1}, \frac{n_1+n_2}{2n_1n_2}-1, \underbrace{\frac{1}{n_2},\ldots,\frac{1}{n_2}}_{n_2-1},\\ &\frac{n_2+n_3}{2n_2n_3}-1,\ldots,\underbrace{\frac{1}{n_{r-1}},\ldots,\frac{1}{n_{r-1}}}_{n_{r-1}-1},\frac{n_{r-1}+n_r}{2n_{r-1}n_r}-1,\underbrace{\frac{1}{n_r},\ldots,\frac{1}{n_r}}_{n_r-1}\Bigg).\end{aligned} \tag{4.1}$$

In this particular construction, the positive integer N is computed as

$$N = \begin{cases} N', & \text{if } N'(\frac{1}{n_i} + \frac{1}{n_j}) \in 2\mathbb{Z} \quad \text{for all } i,j \in \{1,\dots,r\}; \\ 2N' & \text{if } N'(\frac{1}{n_i} + \frac{1}{n_j}) \notin 2\mathbb{Z} \quad \text{for some } (i,j), \end{cases} \tag{4.2}$$

where N' is the least common multiple of $\{n_1, n_2, ..., n_r\}$. Since $N(\frac{n_i+n_{i+1}}{2n_in_{i+1}} - 1)$ are negative integers, the above $\mathbb{Z}$-gradations are not parametrized by Kac parameters. The corresponding $(n-1)$-tuple, derived from (4.1) reads

$$\begin{aligned} \mathbf{s} &= \begin{pmatrix} s_1 & s_2 & \dots & s_{n-1} \end{pmatrix} \qquad (4.3) \\ &= (\underbrace{\frac{N}{n_1},\dots,\frac{N}{n_1}}_{n_1-1}, N\frac{n_1+n_2}{2n_1n_2} - N, \underbrace{\frac{N}{n_2},\dots,\frac{N}{n_2}}_{n_2-1},\dots, N\frac{n_{r-1}+n_r}{2n_{r-1}n_r} - N, \underbrace{\frac{N}{n_r},\dots,\frac{N}{n_r}}_{n_r-1}). \end{aligned}$$

Example 4.1. *(See [14, Example 3.8]) Take* $n = 12$ *and* $\{n_1, n_2, n_3\} = \{3, 4, 5\}$*. For a given partition we have* $N = 120$ *and*

$$\mathbf{s} = \begin{pmatrix} 40 & 40 & -85 & 30 & 30 & 30 & -93 & 24 & 24 & 24 & 24 \end{pmatrix}.$$

The corresponding matrix $\mathrm{MSR}(\mathbf{s})$ *has the form*

$$\mathrm{MSR}(\mathbf{s}) = \begin{pmatrix} 0 & \mathbf{40} & 80 & -5 & 25 & 55 & 85 & -8 & 16 & 40 & 64 & 88 \\ -40 & 0 & \mathbf{40} & -45 & -15 & 15 & 45 & -48 & -24 & 0 & 24 & 48 \\ -80 & -40 & 0 & \mathbf{-85} & -55 & -25 & 5 & -88 & -64 & -40 & -16 & 8 \\ 5 & 45 & 85 & 0 & \mathbf{30} & 60 & 90 & -3 & 21 & 45 & 69 & 93 \\ -25 & 15 & 55 & -30 & 0 & \mathbf{30} & 60 & -33 & -9 & 15 & 39 & 63 \\ -55 & -15 & 25 & -60 & -30 & 0 & \mathbf{30} & -63 & -39 & -15 & 9 & 33 \\ -85 & -45 & -5 & -90 & -60 & -30 & 0 & \mathbf{-93} & -69 & -45 & -21 & 3 \\ 8 & 48 & 88 & 3 & 33 & 63 & 93 & 0 & \mathbf{24} & 48 & 72 & 96 \\ -16 & 24 & 64 & -21 & 9 & 39 & 69 & -24 & 0 & \mathbf{24} & 48 & 72 \\ -40 & 0 & 40 & -45 & -15 & 15 & 45 & -48 & -24 & 0 & \mathbf{24} & 48 \\ -64 & -24 & 16 & -69 & -39 & -9 & 21 & -72 & -48 & -24 & 0 & \mathbf{24} \\ -88 & -48 & -8 & -93 & -63 & -33 & -3 & -96 & -72 & -48 & -24 & 0 \end{pmatrix}.$$

Following the algorithm from [14] we have the 8^{th} *row of nonnegative integers*

$$[a_{8,j}] = \boxed{8 \mid 48 \mid 88 \mid 3 \mid 33 \mid 63 \mid 93 \mid 0 \mid 24 \mid 48 \mid 72 \mid 96}\ .$$

We can order the selected row and determine the element of the Weyl group

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 8 & 4 & 1 & 9 & 5 & 2 & 10 & 6 & 11 & 3 & 7 & 12 \end{pmatrix},$$

which points to $(n-1)$*-tuple of nonnegative integers*

$$\begin{aligned}\mathbf{s}' &= (a_{8,4}, a_{4,1}, a_{1,9}, a_{9,5}, a_{5,2}, a_{2,10}, a_{10,6}, a_{6,11}, a_{11,3}, a_{3,7}, a_{7,12}) \\ &= (3, 5, 16, 9, 15, 0, 15, 9, 16, 5, 3)\ .\end{aligned}$$

Since $N = 120$*, the associated* $\mathbb{Z}$*-gradation is determined by relatively prime integers* $\begin{pmatrix} s_0 & s_1 & s_2 & \dots & s_{n-1}\end{pmatrix} = \begin{pmatrix} 32 & 40 & 40 & -85 & 30 & 30 & 30 & -93 & 24 & 24 & 24 & 24 \end{pmatrix}$. *The hidden Kac parameters for the mentioned bosonic and fermionic realization are*

$$(s_0^{Kac}, \dots, s_{11}^{Kac}) = (24, 3, 5, 16, 9, 15, 0, 15, 9, 16, 5, 3),$$

where $s_0^{Kac} = a_{12,8} + N = -96 + 120 = 24$.

Of course, thanks to Theorem *3.1 (or more precisely* Corollary *3.3) the corresponding permutation* Π *redistributes the signs so that in the resulting skew-symmetric matrix the upper triangle is positive, and the lower negative*

$$\Pi^T A \Pi = \begin{pmatrix}
0 & \mathbf{3} & 8 & 24 & 33 & 48 & 48 & 63 & 72 & 88 & 93 & 96 \\
-3 & 0 & \mathbf{5} & 21 & 30 & 45 & 45 & 60 & 69 & 85 & 90 & 93 \\
-8 & -5 & 0 & \mathbf{16} & 25 & 40 & 40 & 55 & 64 & 80 & 85 & 88 \\
-24 & -21 & -16 & 0 & \mathbf{9} & 24 & 24 & 39 & 48 & 64 & 69 & 72 \\
-33 & -30 & -25 & -9 & 0 & \mathbf{15} & 15 & 30 & 39 & 55 & 60 & 63 \\
-48 & -45 & -40 & -24 & -15 & 0 & \mathbf{0} & 15 & 24 & 40 & 45 & 48 \\
-48 & -45 & -40 & -24 & -15 & 0 & 0 & \mathbf{15} & 24 & 40 & 45 & 48 \\
-63 & -60 & -55 & -39 & -30 & -15 & -15 & 0 & \mathbf{9} & 25 & 30 & 33 \\
-72 & -69 & -64 & -48 & -39 & -24 & -24 & -9 & 0 & \mathbf{16} & 21 & 24 \\
-88 & -85 & -80 & -64 & -55 & -40 & -40 & -25 & -16 & 0 & \mathbf{5} & 8 \\
-93 & -90 & -85 & -69 & -60 & -45 & -45 & -30 & -21 & -5 & 0 & \mathbf{3} \\
-96 & -93 & -88 & -72 & -63 & -48 & -48 & -33 & -24 & -8 & -3 & 0
\end{pmatrix}.$$

Note that the Kac parameters are placed on the first superdiagonal in the matrix $\Pi^T A \Pi$*.*

The special structure (4.3) of $\mathbf{s}$ implies additional structure of the key vector $\breve{P}$ defined in (3.2).

Theorem 4.1. *Let* $\underline{n} = \{n_1, \dots, n_r\}$ *be a partition of* n*. Let* $\mathbf{s}$ *be given by (4.3). Then* $\breve{P} \in \frac{1}{2}\mathbb{Z}^n$ *and its nonzero entries come in* $\pm$ *pairs (hence, for even (odd)* n *an even (odd) number of entries of* $\breve{P}$ *can be zero). More precisely, the sign structure of* $\breve{P}$ *can be described as follows: First, partition* $\breve{P}$ *as*

$$\breve{P} = \begin{pmatrix} \breve{P}^{[1]} \\ \vdots \\ \breve{P}^{[r]} \end{pmatrix}, \quad \breve{P}^{[i]} \in \frac{1}{2}\mathbb{Z}^{n_i},\ i = 1, \dots, r. \tag{4.4}$$

Then the vectors $\breve{P}^{[i]}$ *have the following sign structure:*

$$\textit{if } n_i \textit{ is even: } \breve{P}^{[i]}_j = -\breve{P}^{[i]}_{n_i-j+1} < 0,;\ j = 1, \dots, \frac{n_i}{2} \tag{4.5}$$

$$\textit{if } n_i \textit{ is odd: } \begin{cases} \breve{P}^{[i]}_j = -\breve{P}^{[i]}_{n_i-j+1} < 0,;\ j = 1, \dots, \frac{n_i-1}{2} \\ \breve{P}^{[i]}_{\frac{n_i-1}{2}+1} = 0 \end{cases} \tag{4.6}$$

Proof. We first show that $\breve{P} \in \frac{1}{2}\mathbb{Z}^n$. Since P is integer vector and $E^T E = n$, it suffices to show that $\frac{1}{n}E^T P \in \frac{1}{2}\mathbb{Z}$, i.e. $\frac{1}{n}EE^T P \in \frac{1}{2}\mathbb{Z}^n$.
Note that

$$\frac{1}{n}E^T P = \frac{s_1 + (s_1 + s_2) + \dots + (s_1 + s_2 + \dots + s_{n-1})}{n} = (1/n)\sum_{k=1}^{n-1} s_k(n-k). \tag{4.7}$$

Set $M = n_1 \cdot \ldots \cdot n_r$. By induction with respect to the index r, it can be shown that

$$M\sum_{k=1}^{n-1} \dot{s}_k(n-k) = \frac{1}{2} \cdot (n_1 - 1)n_2 \cdots n_r \cdot n, \quad \text{where } \dot{s}_k = \frac{s_k}{N}. \tag{4.8}$$

Then we can write (4.7) as

$$\begin{aligned} \frac{1}{n}E^T P &= (1/n)\frac{N}{M}\sum_{k=1}^{n-1} M\dot{s}_k(n-k) = (1/n)\frac{N}{M} \cdot \frac{1}{2} \cdot (n_1 - 1)n_2 \cdots n_r \cdot n \\ &= \frac{N}{M} \cdot \frac{1}{2} \cdot (n_1 - 1)n_2 \cdots n_r = \frac{N}{2n_1} \cdot (n_1 - 1). \end{aligned} \tag{4.9}$$

From $\frac{N}{n_1} \in \mathbb{Z}$ and $\frac{(n_1-1)}{2} \in \frac{1}{2}\mathbb{Z}$ we can conclude that $\frac{1}{n}E^T P \in \frac{1}{2}\mathbb{Z}$, i.e. $\breve{P} \in \frac{1}{2}\mathbb{Z}^n$. Since $\breve{P}$ and E are orthogonal, it must hold $\sum_{i=1}^n \breve{P}_i = 0$; so we know that $\breve{P}$ must have entries of both signs. Since the numbers s_i are particularly structured, some additional structure of $\breve{P}$ is expected. For instance, the first block in the partition (4.4) reads

$$\breve{P}^{[1]} = \begin{pmatrix} 0 \\ s_1 \\ s_1+s_2 \\ \vdots \\ s_1+s_2+\dots+s_{n_1-1} \end{pmatrix} - \begin{pmatrix} N(n_1-1)/(2n_1) \\ N(n_1-1)/(2n_1) \\ N(n_1-1)/(2n_1) \\ \vdots \\ N(n_1-1)/(2n_1) \end{pmatrix}, \quad \text{where } s_1 = \dots = s_{n_1-1} = \frac{N}{n_1},$$

and one can easily check that the entries of $\breve{P}^{[1]}$ are monotonically increasing and that

$$\breve{P}^{[1]}_j + \breve{P}^{[1]}_{n_1-j+1} = (j-1)\frac{N}{n_1} - \frac{N(n_1-1)}{2n_1} + (n_1 - j)\frac{N}{n_1} - \frac{N(n_1-1)}{2n_1} = 0.$$

The structure of the second block is slightly more complicated and we have

$$\begin{aligned}
\breve{P}^{[2]}_j + \breve{P}^{[2]}_{n_2-j+1} &= s_1 + \cdots + s_{n_1} + (j-1)\frac{N}{n_2} - \frac{N(n_1-1)}{2n_1} + s_1 + \cdots + s_{n_1} \\
&\quad +(n_2-j)\frac{N}{n_2} - \frac{N(n_1-1)}{2n_1} = 0, \text{ where we have used that} \\
s_1 + \cdots + s_{n_1} &= N(\frac{n_1-1}{n_1} + \frac{n_1+n_2}{2n_1n_2} - 1), \;\; s_{n_1+1} = \cdots = s_{n_1+n_2-1} = \frac{N}{n_2}.
\end{aligned}$$

The general case is analogous but slightly more technical, and we skip the details for the sake o brevity. □

Corollary 4.1. *Let the assumptions of Theorem 4.1 hold true. If, in addition, either* $N = 2N'$ *or* $\underline{n} = \{n_1, \ldots, n_r\}$ *is a partition of* n *with relatively prime numbers* n_i*, then* $\breve{P}$ *is an integer vector.*

Proof. Using the calculation (4.9) we have the following discussion. Suppose that $N = 2N'$. From $\frac{N}{2n_1} = \frac{N'}{n_1}$ it is obvious that $\frac{1}{n}E^T P \in \mathbb{Z}$, i.e. $\breve{P}$ is an integer vector. Now let $\underline{n} = \{n_1, \ldots, n_r\}$ be a relatively prime partition. If n_1 is odd, then $(n_1 - 1)$ is even and the right hand side in (4.9) is an integer, hence $\breve{P} \in \mathbb{Z}^n$.
If n_1 is even, one of $\{n_j \;:\; j = 2, ..., r\}$ must be odd (since $n_1, n_2, \ldots, n_r$ are relatively prime). If n_j is odd, then (by (4.2)) $N'(\frac{1}{n_1} + \frac{1}{n_j}) \in 2\mathbb{Z}+1$, which implies that $N = 2N'$, and the right hand side in (4.9) is again an integer. Hence, $\breve{P}$ is an integer vector. □

Corollary 4.2. *Let* Π *be a permutation matrix such that* $\hat{P} \equiv \Pi^T \breve{P}$ *has entries in non–decreasing order. Then the matrix* $\hat{A} \equiv \Pi^T A \Pi$ *is symmetric with respect to the anti–diagonal* $(i, n-i+1)$*,* $i = 1, \ldots, n$*.*

Proof. We already know that $\hat{A} = \eta_1(E\hat{P}^T - \hat{P}E^T)$. (Since $A\breve{P} = \eta_1 E$, it is certainly $\eta_1 \in \mathbb{Z}$, but that is immaterial at this point.) Because of the structure of $\breve{P}$ (Theorem 4.1), we conclude that $\hat{P}_i = -\hat{P}_{n-i+1}$, $i = 1, \ldots, n$. Hence, the entries $\hat{a}_{ij}$ of $\hat{A}$ satisfy $\hat{a}_{ij} = \eta_1(\hat{P}_j - \hat{P}_i) = \eta_1(\hat{P}_{n-i+1} - \hat{P}_{n-j+1}) = \hat{a}_{n-j+1,n-i+1}$. □

Example 4.2. *For instance, in the case of* $A = \mathrm{MSR}(\mathbf{s})$ *for* $\mathbf{s}$ *such as in Example 4.1, we clearly see the symmetry respect to the anti–diagonal which is based of the symmetry of the corresponding tuple*

$$(s_1^{Kac}, s_2^{Kac}, \cdots, s_{11}^{Kac}) = (3, 5, 16, 9, 15, 0, 15, 9, 16, 5, 3)\;.$$

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