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Diophantine m-tuples for quadratic polynomials

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Abstract

The poster presents one polynomial variant of the problem of Diophantus, described by A. Jurasić [16], and ilustrates that results with some examples from the paper of A. Dujella and A. Jurasić [12]. We proved that there does not exist a set with more than 98 nonzero polynomials in $\mathbb{Z}[X]$, such that the product of any two of them plus a quadratic polynomial n is a square of a polynomial from $\mathbb{Z}[X]$ (we exclude the possibility that all elements of such set are constant multiples of a linear polynomial $p \in \mathbb{Z}[X]$ such that $p^2|n$). Specially, we prove that if such a set contains only polynomials of odd degree, then it has at most 18 elements.

Keywords: Diophantine m-tuples, polynomials.

1 Diophantine m-tuples

Diophantus of Alexandria [2] first studied the problem of finding sets with the property that the product of any two of its distinct elements increased by one

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is a perfect square. Such a set consisting of m elements is therefore called a Diophantine m-tuple. Diophantus found the first Diophantine quadruple of rational numbers $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$, while the first Diophantine quadruple of integers $\{1, 3, 8, 120\}$ was found by Fermat. Many generalizations of this problem were considered since then, for example by adding a fixed integer n instead of 1, looking at kth powers instead of squares, or considering the problem over other domains than $\mathbb{Z}$ or $\mathbb{Q}$.

Definition 1.1 Let n be a nonzero integer. A set of m different positive integers $\{a_1, a_2, ..., a_m\}$ is called a Diophantine m-tuple with the property $D(n)$ or simply $D(n)$-m-tuple if the product $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$.

Diophantus [2] found the first such quadruple $\{1, 33, 68, 105\}$ with the property $D(256)$. The first $D(1)$-quadruple is the above mentioned Fermat's set. The folklore conjecture is that there does not exist a $D(1)$-quintuple. Baker and Davenport [1] proved that Fermat's set cannot be extended to a $D(1)$-quintuple. Dujella [6] proved that there does not exist a $D(1)$-sextuple and there are only finitely many $D(1)$-quintuples. But, for example, the set $\{1, 33, 105, 320, 18240\}$ has the property $D(256)$ [4], and $D(2985984)$ is the property of the set $\{99, 315, 9920, 32768, 44460, 19534284\}$ [13]. The natural question is to find upper bounds for the numbers M_n defined by

$$M_n = \sup\{|S| : S \text{ has the property } D(n)\}$$

where $|S|$ denotes the number of elements in the set S. Dujella [5,3] proved that $M_n \leq 31$ for $|n| \leq 400$, and $M_n < 15.476 \log |n|$ for $|n| > 400$.

The first polynomial variant of the above problem was studied by Jones [15,14] and it was for the case $n = 1$.

Definition 1.2 Let $n \in \mathbb{Z}[X]$ and let $\{a_1, a_2, ..., a_m\}$ be a set of m nonzero polynomials with integer coefficients. We assume that there does not exist a polynomial $p \in \mathbb{Z}[X]$ such that $\frac{a_1}{p}, ..., \frac{a_m}{p}$ and $\frac{n}{p^2}$ are integers. The set $\{a_1, a_2, ..., a_m\}$ is called a polynomial $D(n)$-m-tuple if for all $1 \leq i < j \leq m$ the following holds: $a_i \cdot a_j + n = b_{ij}^2$ where $b_{ij} \in \mathbb{Z}[X]$.

For $n \in \mathbb{Z}$ the assumption concerning the polynomial p means that not all elements of $\{a_1, a_2, ..., a_m\}$ are allowed to be constant. In analogy to the above results we are interested in the size of

$$P_n = \sup\{|S| : S \text{ is a polynomial } D(n)\text{-tuple}\}.$$

Dujella and Fuchs [7] proved that $P_{-1} = 3$ and their result from [8] implies that $P_1 = 4$. Moreover, from [11, Theorem 1] it follows that $P_n \leq 7$ for all $n \in \mathbb{Z}\backslash\{0\}$. It is an improvement of the previous bound $P_n \leq 22$, which follows from [5, Theorem 1]. Dujella and Fuchs, jointly with Tichy [9] and later with Walsh [10], considered the case $n = \mu_1 X + \mu_0$ with integers $\mu_1 \neq 0$ and μ_0. They defined

$$L = \sup\{|S| : S \text{ is a polynomial } D(\mu_1 X + \mu_0)\text{-tuple for some } \mu_1 \neq 0, \mu_0 \in \mathbb{Z}\},$$

and they denoted by L_k the number of polynomials of degree k in a polynomial $D(\mu_1 X + \mu_0)$-tuple S. The results from [10] are sharp bounds $L_0 \leq 1$, $L_1 \leq 4$, $L_k \leq 3$ for all $k \geq 2$, and finally $L \leq 12$.

2 Diophantine m-tuples for quadratic polynomials

We handled the case where n is a quadratic polynomial in $\mathbb{Z}[X]$, which is more complicated than the case with linear n, mostly because quadratic polynomials need not be irreducible. Let us define

$$Q = \sup\{|S| : S \text{ is a polynomial } D(\mu_2 X^2 + \mu_1 X + \mu_0)\text{-tuple} \\ \text{for some } \mu_2 \neq 0, \mu_1, \mu_0 \in \mathbb{Z}\}.$$

Let us also denote by Q_k the number of polynomials of degree k in a polynomial $D(\mu_2 X^2 + \mu_1 X + \mu_0)$-tuple S. We proved the following theorem:

Theorem 2.1 *There are at most* 98 *elements in a polynomial* $D(n)$*-tuple for a quadratic polynomial* n*, i.e.*

$$Q \leq 98.$$

In the proof of Theorem 2.1, we also proved the following statement.

Corollary 2.2 *If a polynomial* $D(n)$*-*m*-tuple for a quadratic* n *contains only polynomials of odd degree, then* $m \leq 18$.

In the proof of Theorem 2.1, we followed the strategy used in [9] and [10] for linear n. First, we estimated the numbers Q_k of polynomials of degree k. We proved:

Proposition 2.3

1.) $Q_0 \leq 2$.

2.) $Q_1 \leq 4$.

Proposition 2.3 completely solves the problem for constant and linear polynomials because, for example, the set $\{3, 5\}$ is a polynomial $D(9X^2+24X+1)$-pair, and the set

$$\{2X, 10X + 20, 4X + 14, 2X + 8\}$$

is a polynomial $D(-4X^2 - 16X + 9)$-quadruple. By further analysis, we got:

Proposition 2.4

1.) $Q_2 \leq 81$.

2.) $Q_3 \leq 5$.

3.) $Q_4 \leq 6$.

4.) $Q_k \leq 3$ *for* $k \geq 5$.

Let us mention that it is not obvious that the number Q_2 is bounded, so the result from Proposition 2.4 1.) is nontrivial. Quadratic polynomials have the major contribution to the bound from Theorem 2.1. The bound from Proposition 2.4 4.) is sharp. For example, the set

$$\{X^{2l-1} + X, X^{2l-1} + 2X^l + 2X, 4X^{2l-1} + 4X^l + 5X\}$$

is a polynomial $D(-X^2)$-triple for any integer $l \geq 2$, and the set

$$\{X^{2l} + X^l, X^{2l} + X^l + 4X, 4X^{2l} + 4X^l + 8X\}$$

is a polynomial $D(4X^2)$-triple for any integer $l \geq 1$.

The poster also presents some other examples of the polynomial $D(n)$-triples and quadruples, where n is a quadratic polynomial. These examples show that several auxiliary results from the proofs of the cases $k = 2, 3, 4$ from Proposition 2.4 are sharp, i.e. in the situations when the existence of Diophantine triples with certain properties cannot be excluded, such triples indeed exist.

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