Cent. Eur. J. Math. • 11(2) • 2013 • 197-225
DOI: 10.2478/s11533-012-0111-x

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Central European Journal of **Mathematics**

Combinatorial bases of modules for affine Lie algebra $B_2^{(1)}$

Research Article

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Received 2 December 2011; accepted 23 March 2012

Abstract: We construct bases of standard (i.e. integrable highest weight) modules $L(\Lambda)$ for affine Lie algebra of type $B_2^{(1)}$ consisting of semi-infinite monomials. The main technical ingredient is a construction of monomial bases for Feigin–Stoyanovsky type subspaces $W(\Lambda)$ of $L(\Lambda)$ by using simple currents and intertwining operators in vertex operator algebra theory. By coincidence $W(k\Lambda_0)$ for $B_2^{(1)}$ and the integrable highest weight module $L(k\Lambda_0)$ for $A_1^{(1)}$ have the same parametrization of combinatorial bases and the same presentation $\mathcal{P}/\mathcal{I}$.

MSC: 17B67, 17B69, 05A19

Keywords: Affine Lie algebras • Vertex operator algebras • Combinatorial bases

1. Introduction

B.L. Feigin and A.V. Stoyanovsky gave in [29] a construction of bases of standard (i.e. integrable highest weight) modules $L(\Lambda)$ for affine Lie algebra $\hat{\mathfrak{g}}$ of type $A_1^{(1)}$ consisting of semi-infinite monomials. In [26] such a construction is extended to all standard modules for affine Lie algebras of type $A_n^{(1)}$. The construction starts with choosing a particular $\mathbb{Z}$-grading of the corresponding simple Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 \tag{1}$$

and a particular group element e which normalizes the subalgebra $\hat{\mathfrak{g}}_1 = \mathfrak{g}_1 \otimes \mathbb{C}[t, t^{-1}]$. Then

$$L(\Lambda) = \bigcup_{m=0}^{\infty} e^{-m} U(\hat{\mathfrak{g}}_1) v_\Lambda, \qquad e^{-m-1} U(\hat{\mathfrak{g}}_1) v_\Lambda \supset e^{-m} U(\hat{\mathfrak{g}}_1) v_\Lambda,$$

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and semi-infinite monomials appear by "taking a limit"

$$\lim_{m\to\infty} e^{-m} U(\widehat{\mathfrak{g}}_1) v_\Lambda.$$

On the other side, for any classical simple Lie algebra $\mathfrak{g}$ and any possible $\mathbb{Z}$-grading (1) such construction is given in [27] for the basic $\widehat{\mathfrak{g}}$-module $L(\Lambda_0)$. In each of these cases a weight basis of $\mathfrak{g}_1$ is interpreted as a perfect crystal for the quantum group $U_q(\widehat{\mathfrak{g}}_0)$ and in a proof of linear independence a crystal base character formula [20] is used, but it was not clear "why" this proof works and how such an approach could be extended to higher level standard modules. A new understanding came from the works of G. Georgiev [17] and S. Capparelli, J. Lepowsky and A. Milas [5, 6] based on a general idea of J. Lepowsky to use intertwining vertex operators to build bases of standard modules and obtain Rogers–Ramanujan type recursions for their graded dimensions. Their way of using intertwining operators inspired a simpler proof of linear independence for $A_n^{(1)}$ in [28], and new constructions for $D_4^{(1)}$ by I. Baranović in [2] and for $A_n^{(1)}$ by G. Trupčević in [30] for all possible $\mathbb{Z}$-gradings (1).

In this paper we use Capparelli–Lepowsky–Milas' approach to extend the construction of [27] to all standard modules $L(\Lambda)$ for an affine Lie algebra $\widehat{\mathfrak{g}}$ of type $B_2^{(1)}$. In this case we neither have a lattice construction of level 1 modules nor Dong–Lepowsky's intertwining operators [7], but we manage to construct intertwining operators we need in a proof of linear independence by using vertex operator algebra theory and results of C. Dong, H. Li and G. Mason [8] on simple currents. Along the way we also obtain a presentation theorem for Feigin–Stoyanovsky type subspaces. The underlying structure of Feigin–Stoyanovsky type subspaces is parallel to the structure of principal subspaces studied, for example, in [1, 3, 4, 17].

Since the list of all possible $\mathbb{Z}$-gradings (1) coincides with the list of all possible level 1 simple currents constructed in [8], the results and methods used in [2, 30] and this paper give hope that the construction in [27] might be extended to all standard modules of all classical affine Lie algebras by using intertwining operators. In return, one should expect a rich and interesting combinatorial structure behind this construction, on one side extending combinatorics of infinite paths used in [20], and on the other side extending $(k, n+1)$-admissible configurations – combinatorial objects introduced and studied in a series of papers [9, 10]. Moreover, it might be that the reason "why" the proof in [27] works can be explained by some connection of tensor multiplication of vertex operator algebra modules with simple currents, cf. [8, 16, 18], on one side and tensor multiplication of affine crystals with perfect crystals, cf. [20], on the other.

Let $\mathfrak{g}$ be a simple complex Lie algebra of type B_2, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$$

a $\mathbb{Z}$-grading of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{g}_0$. We fix a basis of $\mathfrak{g}_1$ consisting of root vectors denoted as

$$x_{\underline{2}}, \qquad x_0, \qquad x_2.$$

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c + \mathbb{C}d$ be the associated affine Lie algebra with the canonical central element c. For $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$ we write $x(n) = x \otimes t^n$. Then for integral dominant weight

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$$

of level $k = k_0 + k_1 + k_2$ a basis of standard module $L(\Lambda)$ can be parametrized by semi-infinite monomials

$$\prod_{j\in\mathbb{Z}} x_{\underline{2}}(-j)^{c_j} x_0(-j)^{b_j} x_2(-j)^{a_j}, \qquad c_j = b_j = a_j = 0 \quad \text{for} \quad -j \ll 0, \tag{2}$$

with quasi-periodic tail with the period of length 6,

$$(\dots, c_{-2n}, b_{-2n}, a_{-2n}, c_{-2n-1}, b_{-2n-1}, a_{-2n-1}, \dots) = (\dots, k_1, k_2, k_1, k_0, k_2, k_0, \dots)$$

for $n \gg 0$, satisfying for all $j \in \mathbb{Z}$ the so-called difference conditions

$$c_{j+1} + b_{j+1} + c_j \le k, \qquad b_{j+1} + a_{j+1} + c_j \le k, \qquad a_{j+1} + c_j + b_j \le k, \qquad a_{j+1} + b_j + a_j \le k. \tag{3}$$

This is Corollary 9.5 of Theorem 9.3. The main technical ingredient in the proof is a construction of monomial bases for Feigin–Stoyanovsky type subspaces defined as

$$W(\Lambda) = U(\widehat{\mathfrak{g}}_1)v_\Lambda \subset L(\Lambda),$$

where $\widehat{\mathfrak{g}}_1 = \mathfrak{g}_1 \otimes \mathbb{C}[t, t^{-1}]$ and v_Λ is a highest weight vector in $L(\Lambda)$. By Theorem 3.1 the constructed basis for level k subspace $W(\Lambda)$ consists of finite monomials of the form (2) with $-j \le -1$, satisfying difference conditions (3) and the so-called initial conditions

$$a_1 \le k_0, \qquad b_1 + a_1 \le k_0 + k_2, \qquad c_1 + b_1 \le k_0 + k_2.$$

Another consequence of this result is Theorem 10.1 which gives a presentation

$$W(\Lambda) \cong \mathcal{P}/\mathcal{I}_\Lambda,$$

where $\mathcal{P}$ is a polynomial algebra $\mathbb{C}[x_{\underline{2}}(j), x_0(j), x_2(j) : j \le -1]$ and $\mathcal{I}_\Lambda$ is the ideal generated by the set of polynomials

$$\bigcup_{n \le -k-1} U(\mathfrak{g}_0) \cdot \left(\sum_{\substack{j_1, \dots, j_{k+1} \le -1 \\ j_1 + \dots + j_{k+1} = n}} x_2(j_1) \dots x_2(j_{k+1}) \right) \cup \{x_2(-1)^{k_0+1}\} \cup U(\mathfrak{g}_0) \cdot x_2(-1)^{k_0+k_2+1}.$$

By coincidence, $W(k\Lambda_0)$ for $B_2^{(1)}$ and the integrable highest weight module $L(k\Lambda_0)$ for $A_1^{(1)}$ have the same parametrization of combinatorial bases and the same presentation $\mathcal{P}/\mathcal{I}$. Due to this coincidence, E. Feigin's fermionic formula [12] for $A_1^{(1)}$-module $L(k\Lambda_0)$ is also a character formula of Feigin–Stoyanovsky type subspace $W(k\Lambda_0)$ for $B_2^{(1)}$.

As it was already said, in our construction we use simple currents and intertwining operators for vertex operator algebra $L(\Lambda_0)$ associated with the affine Lie algebra $\widehat{\mathfrak{g}}$ at level 1. To be more precise, we use results from [8, 23] to see the existence of level 1 "simple current operators"

$$L(\Lambda_0) \xrightarrow{[\omega]} L(\Lambda_1) \xrightarrow{[\omega]} L(\Lambda_0), \qquad L(\Lambda_2) \xrightarrow{[\omega]} L(\Lambda_2)$$

which are linear bijections with the crucial property

$$x(n)[\omega] = [\omega]x(n+1) \qquad \text{for all} \quad x(n) \in \widehat{\mathfrak{g}}_1. \tag{4}$$

From [22] we have fusion rules

$$\dim I \begin{pmatrix} L(\Lambda_2) \\ L(\Lambda_2) \; L(\Lambda_0) \end{pmatrix} = 1 \qquad \text{and} \qquad \dim I \begin{pmatrix} L(\Lambda_1) \\ L(\Lambda_2) \; L(\Lambda_2) \end{pmatrix} = 1,$$

from which we deduce that there are coefficients $[\omega_2]$ and $[\omega_{\underline{2}}]$ of intertwining operators

$$\begin{aligned}
&L(\Lambda_0) \xrightarrow{[\omega_2]} L(\Lambda_2) \xrightarrow{[\omega_2]} L(\Lambda_1), \qquad v_{\Lambda_0} \xrightarrow{[\omega_2]} v_{\Lambda_2} \xrightarrow{[\omega_2]} v_{\Lambda_1}, \qquad [\omega_2]w_{\underline{2}} = 0, \\
&L(\Lambda_0) \xrightarrow{[\omega_{\underline{2}}]} L(\Lambda_2) \xrightarrow{[\omega_{\underline{2}}]} L(\Lambda_1), \qquad v_{\Lambda_0} \xrightarrow{[\omega_{\underline{2}}]} w_{\underline{2}} \xrightarrow{[\omega_{\underline{2}}]} v_{\Lambda_1}, \qquad [\omega_{\underline{2}}]v_{\Lambda_2} = 0
\end{aligned}$$

which commute with the action of $\widehat{\mathfrak{g}}_1$. We consider higher level standard modules as submodules of tensor products of level 1 modules

$$L(\Lambda) \subset L(\Lambda_0)^{\otimes k_0} \otimes L(\Lambda_1)^{\otimes k_1} \otimes L(\Lambda_2)^{\otimes k_2}.$$

Behind all combinatorial properties of our construction seems to be relation (13) for $[\omega]v_\Lambda$, written in terms of tensor products of level 1 highest weight vectors as

$$\begin{aligned}[\omega]\big(v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_1}^{\otimes k_1} \otimes v_{\Lambda_2}^{\otimes k_2}\big) &= \big([\omega]v_{\Lambda_0}\big)^{\otimes k_0} \otimes \big([\omega]v_{\Lambda_1}\big)^{\otimes k_1} \otimes \big([\omega]v_{\Lambda_2}\big)^{\otimes k_2} \\ &= C\, x_{\underline{2}}(-1)^{k_1} x_0(-1)^{k_2} x_2(-1)^{k_1} \big(v_{\Lambda_1}^{\otimes k_0} \otimes v_{\Lambda_0}^{\otimes k_1} \otimes v_{\Lambda_2}^{\otimes k_2}\big).\end{aligned} \tag{5}$$

In particular, it is this relation that for level 1 modules makes the use of crystal base character formula [20] in [27] possible.

Very roughly speaking, we prove linear independence by induction on degree of basis elements in two steps: for monomial vectors $x(\pi)v_\Lambda$, which appear with nontrivial coefficients $c_\pi \neq 0$ in a linear combination $\sum c_\pi x(\pi)v_\Lambda = 0$, we first use intertwining operators $x(\pi)v_\Lambda \to x(\pi)v_{\Lambda'}$ to be able to apply formula (5) to vectors $x(\pi)v_{\Lambda'}$ and get a combination of monomial vectors of the form $\sum c_\pi x(\pi')[\omega]v_{\Lambda''}$. Then, as a second step, we commute $[\omega]$ to the left and, by using (4) and induction hypothesis, we get that c_π equals zero — a contradiction. Of course, the actual argument is a bit more complicated and, as in [2], we have to use two basis elements of 4-dimensional spinor $\mathfrak{g}$-module on the top of $L(\Lambda_2)$ and the corresponding coefficients $[\omega_2]$ and $[\omega_{\underline{2}}]$ of intertwining operators.

A part of this paper was written while the author was a member of the Erwin Schrödinger Institute in Vienna in February of 2009. He would like to thank J. Schwermer for his hospitality.

2. Affine Lie algebra of type $B_2^{(1)}$

Let $\mathfrak{g}$ be a complex simple Lie algebra of type B_2 and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{h} + \sum \mathfrak{g}_\alpha$ be a root space decomposition of $\mathfrak{g}$. The corresponding root system R may be realized in $\mathbb{R}^2$ with the canonical basis $\varepsilon_1, \varepsilon_2$ as

$$R = \big\{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 + \varepsilon_2)\big\} \cup \{\pm\varepsilon_1, \pm\varepsilon_2\}.$$

We fix simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2$ and denote by $\omega_1 = \varepsilon_1$ and $\omega_2 = (\varepsilon_1 + \varepsilon_2)/2$ the corresponding fundamental weights. Note that $\theta = \varepsilon_1 + \varepsilon_2$ is the maximal root. Set

$$\Gamma = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1, \varepsilon_1 + \varepsilon_2\}.$$

Denote by $\langle \cdot, \cdot \rangle$ the normalized Killing form such that $\langle \theta, \theta \rangle = 2$. We identify $\mathfrak{h} \cong \mathfrak{h}^*$ via $\langle \cdot, \cdot \rangle$. We fix $\omega = \omega_1 = \varepsilon_1$. Then we have $\alpha(\omega) = \langle \alpha, \omega \rangle$ and

$$\Gamma = \{\alpha \in R : \alpha(\omega) = 1\}.$$

Obviously we have a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ for

$$\mathfrak{g}_0 = \mathfrak{h} + \sum_{\alpha(\omega)=0} \mathfrak{g}_\alpha = \mathfrak{h} + \mathbb{C}x_{\varepsilon_2} + \mathbb{C}x_{-\varepsilon_2}, \qquad \mathfrak{g}_{\pm 1} = \sum_{\alpha \in \pm\Gamma} \mathfrak{g}_\alpha.$$

Clearly $\mathfrak{g}_1$ is an irreducible 3-dimensional $\mathfrak{g}_0$-module. We shall briefly write

$$\underline{2} = \varepsilon_1 - \varepsilon_2, \qquad 0 = \varepsilon_1, \qquad 2 = \varepsilon_1 + \varepsilon_2,$$

so that $\Gamma = \{\underline{2}, 0, 2\}$, a notation as in [2, 27]. For each root α fix a root vector x_α. For $\alpha = \underline{2}, 0, 2$ we shall write x_α respectively as

$$x_{\underline{2}}, \qquad x_0, \qquad x_2.$$

These vectors form a basis of $\mathfrak{g}_0$-module $\mathfrak{g}_1$.

Denote by $\widehat{\mathfrak{g}}$ the affine Lie algebra of type $B_2^{(1)}$ associated to $\mathfrak{g}$,

$$\widehat{\mathfrak{g}} = \sum_{n\in\mathbb{Z}} \mathfrak{g}\otimes t^n + \mathbb{C}c + \mathbb{C}d,$$

with the canonical central element c and the degree element d such that $[d, x\otimes t^n] = nx\otimes t^n$. Set

$$\widehat{\mathfrak{g}}_{<0} = \sum_{n<0} \mathfrak{g}\otimes t^n, \qquad \widehat{\mathfrak{g}}_{\leq 0} = \sum_{n\leq 0} \mathfrak{g}\otimes t^n + \mathbb{C}c + \mathbb{C}d.$$

Let α_0, α_1 and α_2 be simple roots of $\widehat{\mathfrak{g}}$ with the root subspaces $\mathfrak{g}_{-\theta}\otimes t^1$, $\mathfrak{g}_{\alpha_1}\otimes t^0$ and $\mathfrak{g}_{\alpha_2}\otimes t^0$ respectively, and let $\Lambda_0, \Lambda_1, \Lambda_2$ be the corresponding fundamental weights of $\widehat{\mathfrak{g}}$, cf. [19]. We write

$$x(n) = x\otimes t^n$$

for $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$ and denote by $x(z) = \sum_{n\in\mathbb{Z}} x(n)z^{-n-1}$ a formal Laurent series in formal variable z. For

$$\widehat{\mathfrak{g}}_0 = \sum_{n\in\mathbb{Z}} \mathfrak{g}_0\otimes t^n + \mathbb{C}c + \mathbb{C}d, \qquad \widehat{\mathfrak{g}}_{\pm 1} = \sum_{n\in\mathbb{Z}} \mathfrak{g}_{\pm 1}\otimes t^n$$

we have $\mathbb{Z}$-grading $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_{-1} + \widehat{\mathfrak{g}}_0 + \widehat{\mathfrak{g}}_1$. In particular, $\widehat{\mathfrak{g}}_1$ is a commutative Lie subalgebra of $\widehat{\mathfrak{g}}$ with a basis

$$\widehat{\Gamma} = \{x_{\underline{2}}(n), x_0(n), x_2(n) : n \in \mathbb{Z}\} = \{x_\gamma(n) : \gamma \in \Gamma, n \in \mathbb{Z}\}.$$

On $\widehat{\Gamma}$ we use the linear order

$$\ldots \prec x_2(n-1) \prec x_{\underline{2}}(n) \prec x_0(n) \prec x_2(n) \prec x_{\underline{2}}(n+1) \prec \ldots$$

3. Feigin–Stoyanovsky type subspaces $W(\Lambda)$

Denote by $L(\Lambda)$ a standard (i.e. integrable highest weight) $\widehat{\mathfrak{g}}$-module with a dominant integral highest weight

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2,$$

$k_0, k_1, k_2 \in \mathbb{Z}_+$. Throughout the paper we denote by $k = \Lambda(c)$ the level of $\widehat{\mathfrak{g}}$-module $L(\Lambda)$,

$$k = k_0 + k_1 + k_2,$$

cf. [19]. For each fundamental $\widehat{\mathfrak{g}}$-module $L(\Lambda_i)$ fix a highest weight vector v_{Λ_i}. By complete reducibility of tensor products of standard modules, for level $k > 1$ we have

$$L(\Lambda) \subset L(\Lambda_0)^{\otimes k_0}\otimes L(\Lambda_1)^{\otimes k_1}\otimes L(\Lambda_2)^{\otimes k_2}$$

with a highest weight vector

$$v_\Lambda = v_{\Lambda_0}^{\otimes k_0}\otimes v_{\Lambda_1}^{\otimes k_1}\otimes v_{\Lambda_2}^{\otimes k_2}.$$

Later on we shall also realize $L(\Lambda)$ in a symmetric algebra

$$L(\Lambda) \subset S^k\big(L(\Lambda_0) \oplus L(\Lambda_1) \oplus L(\Lambda_2)\big), \qquad v_\Lambda = v_{\Lambda_0}^{k_0} v_{\Lambda_1}^{k_1} v_{\Lambda_2}^{k_2}.$$

We set $d v_\Lambda = 0$. Then $L(\Lambda)$ is $\mathbb{Z}$-graded by the degree operator d,

$$L(\Lambda) = L(\Lambda)_0 + L(\Lambda)_{-1} + L(\Lambda)_{-2} + \dots,$$

and we say that $\mathfrak{g}$-module $L(\Lambda)_0 = U(\mathfrak{g}) v_\Lambda$ is the "top" of $L(\Lambda)$. The top of $L(\Lambda_0)$ is trivial $\mathfrak{g}$-module $\mathbb{C}v_{\Lambda_0}$, the top of $L(\Lambda_1)$ is a 5-dimensional vector representation $L(\omega_1)$ and the top of $L(\Lambda_2)$ is a 4-dimensional spinor $\mathfrak{g}$-module $L(\omega_2)$.

For each integral dominant Λ we have a Feigin–Stoyanovsky type subspace

$$W(\Lambda) = U(\widehat{\mathfrak{g}}_1) v_\Lambda \subset L(\Lambda).$$

Denote by $\pi\colon \{x_\gamma(-j) : \gamma \in \Gamma,\ j \geq 1\} \to \mathbb{Z}_+$ a "colored partition" for which a finite number of "parts" $x_\gamma(-j)$ (of degree j and color γ) appear $\pi(x_\gamma(-j))$ times, and denote by

$$x(\pi) = \prod x_\gamma(-j)^{\pi(x_\gamma(-j))} \in U(\widehat{\mathfrak{g}}_1) = S(\widehat{\mathfrak{g}}_1)$$

the corresponding monomials. We can identify π with a sequence

$$a_1, b_1, c_1, a_2, b_2, c_2, \dots$$

with finitely many non-zero terms $a_j = \pi(x_2(-j))$, $b_j = \pi(x_0(-j))$, $c_j = \pi(x_{\underline{2}}(-j))$ and

$$x(\pi) = \cdots x_{\underline{2}}(-j)^{c_j} x_0(-j)^{b_j} x_2(-j)^{a_j} \cdots x_{\underline{2}}(-1)^{c_1} x_0(-1)^{b_1} x_2(-1)^{a_1}.$$

For a monomial $x(\pi)$ we say that $x(\pi)v_\Lambda \in W(\Lambda)$ is a monomial vector. The main result of this paper is the following:

Theorem 3.1.
The set of monomial vectors $x(\pi)v_\Lambda$ satisfying difference conditions

$$c_{j+1} + b_{j+1} + c_j \leq k, \qquad b_{j+1} + a_{j+1} + c_j \leq k, \qquad a_{j+1} + c_j + b_j \leq k, \qquad a_{j+1} + b_j + a_j \leq k \tag{6}$$

for all $j \geq 1$, and initial conditions

$$a_1 \leq k_0, \qquad b_1 + a_1 \leq k_0 + k_2, \qquad c_1 + b_1 \leq k_0 + k_2, \tag{7}$$

is a basis of level k Feigin–Stoyanovsky type subspace $W(\Lambda)$.

4. Difference conditions and initial conditions

By Poincaré–Birkhoff–Witt theorem we have a spanning set of monomial vectors $x(\pi)v_\Lambda$ in Feigin–Stoyanovsky type level k subspace $W(\Lambda)$. To reduce this spanning set to a basis described in Theorem 3.1 we use vertex operator algebra relations

$$x_\theta(z)^{k+1} = \sum_{n\in\mathbb{Z}} \left(\sum_{j_1+\cdots+j_{k+1}=n} x_\theta(j_1)\cdots x_\theta(j_{k+1}) \right) z^{-n-k-1} = 0 \qquad \text{on} \quad L(\Lambda)$$

and their consequences $U(\mathfrak{g}_0)\cdot x_\theta(z)^{k+1} = 0$, where $\cdot$ denotes the adjoint action of $\mathfrak{g}_0$ on $\widehat{\mathfrak{g}}_1$. That is, the adjoint action of $\mathfrak{g}_0$ on coefficients of formal Laurent series $x_\theta(z)^{k+1}$ gives relations

$$\sum c_\mu x(\mu) = 0 \qquad \text{on} \quad W(\Lambda),$$

where for each relation the sum is over an infinite set of colored partitions μ. By choosing a proper order on the set of monomials, for each sum we can determine the smallest term $x(\rho)$, the so-called leading term, which can be replaced on $W(\Lambda)$ by a sum of higher (bigger) terms. The list of leading terms is

$$\begin{aligned}
x_{\underline{2}}(-j-1)^{c_{j+1}} x_0(-j-1)^{b_{j+1}} x_{\underline{2}}(-j)^{c_j}, &\qquad c_{j+1}+b_{j+1}+c_j = k+1,\\
x_0(-j-1)^{b_{j+1}} x_2(-j-1)^{a_{j+1}} x_{\underline{2}}(-j)^{c_j}, &\qquad b_{j+1}+a_{j+1}+c_j = k+1,\\
x_2(-j-1)^{a_{j+1}} x_{\underline{2}}(-j)^{c_j} x_0(-j)^{b_j}, &\qquad a_{j+1}+c_j+b_j = k+1,\\
x_2(-j-1)^{a_{j+1}} x_0(-j)^{b_j} x_2(-j)^{a_j}, &\qquad a_{j+1}+b_j+a_j = k+1,
\end{aligned} \tag{8}$$

for all $j \geq 1$. So by induction we see that $W(\Lambda)$ is spanned by monomial vectors $x(\pi)v_\Lambda$ which do not have factors of the form (8), i.e., by monomial vectors which satisfy difference conditions (6) (for details of this argument see [21, 25, 27] or [11]).

Lemma 4.1.
$x_2(-1)v_{\Lambda_1} = x_0(-1)v_{\Lambda_1} = x_{\underline{2}}(-1)v_{\Lambda_1} = 0.$

Proof. For $\alpha \in R$ denote by $\mathfrak{sl}_2(\alpha) \subset \mathfrak{g}$ a Lie subalgebra generated with x_α and $x_{-\alpha}$ and by

$$\widehat{\mathfrak{sl}}_2(\alpha) = \sum_{n\in\mathbb{Z}} \mathfrak{sl}_2(\alpha)\otimes t^n + \mathbb{C}c + \mathbb{C}d \subset \widehat{\mathfrak{g}} \tag{9}$$

denote the corresponding affine Lie algebra of type $A_1^{(1)}$. Note that for a level one $\widehat{\mathfrak{g}}$-module V the restriction to $\widehat{\mathfrak{sl}}_2(\alpha)$ is a level one representation if α is a long root, and it is a level two representation if α is a short root. Also note that $U(\widehat{\mathfrak{sl}}_2(\alpha))v_{\Lambda_1}$ is a standard $A_1^{(1)}$-module and that its $\mathfrak{sl}_2(\alpha)$-submodule on the top is a submodule of 5-dimensional vector representation for B_2.

In the case $\alpha = \varepsilon_1 - \varepsilon_2 = \underline{2}$ we have level one representation on $U(\widehat{\mathfrak{sl}}_2(\alpha))v_{\Lambda_1}$ with 2-dimensional $\mathfrak{sl}_2(\alpha)$-module on the top, so it must be the standard $A_1^{(1)}$-module $L(\Lambda_1)$. Hence $x_{\varepsilon_1-\varepsilon_2}(-1)v_{\Lambda_1} = 0$. Similarly $x_\alpha(-1)v_{\Lambda_1} = 0$ for $\alpha = \varepsilon_1+\varepsilon_2$. On the other hand, in the case $\alpha = \varepsilon_1$ we have level two representation on $U(\widehat{\mathfrak{sl}}_2(\alpha))v_{\Lambda_1}$ with 3-dimensional $\mathfrak{sl}_2(\alpha)$-module on the top, so it must be the standard $A_1^{(1)}$-module $L(2\Lambda_1)$. Hence again $x_\alpha(-1)v_{\Lambda_1} = 0$. □

Lemma 4.2.
We have

(i) $x_2(-1)x_2(-1)v_{\Lambda_0} = x_{\underline{2}}(-1)x_{\underline{2}}(-1)v_{\Lambda_0} = 0,$

(ii) $x_0(-1)x_2(-1)v_{\Lambda_0} = x_{\underline{2}}(-1)x_0(-1)v_{\Lambda_0} = 0,$

(iii) $x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0} = Cx_0(-1)x_0(-1)v_{\Lambda_0}$ *for some* $C \neq 0$,

(iv) $x_0(-1)^3 v_{\Lambda_0} = 0$.

Proof. Note that $x_\gamma(j)v_{\Lambda_0} = 0$ for all $j \geq 0$, so the relation $x_\gamma(z)^2 = 0$ on $L(\Lambda_0)$ for a long root γ implies

$$x_\gamma(-1)^2 v_{\Lambda_0} = \left(x_\gamma(-1)x_\gamma(-1) + 2x_\gamma(-2)x_\gamma(0) + 2x_\gamma(-3)x_\gamma(1) + \cdots\right) v_{\Lambda_0} = 0$$

and (i) follows. Since $x_{\varepsilon_2}(0)v_{\Lambda_0} = x_{-\varepsilon_2}(0)v_{\Lambda_0} = 0$, the action of $x_{\varepsilon_2}(0)$ or $x_{-\varepsilon_2}(0)$ on (i) gives relations (ii) and (iii). Since for level one $\widehat{\mathfrak{g}}$-module V the restriction to $\widehat{\mathfrak{sl}}_2(\alpha)$ is level two representation if α is a short root, on $L(\Lambda_0)$ we have $x_0(z)^3 = 0$ and (iv) follows. □

We fix vectors $w_2 = v_{\Lambda_2}$ and $w_{\underline{2}}$ with weights

$$\omega_2 = \frac{\varepsilon_1 + \varepsilon_2}{2} \qquad \text{and} \qquad \omega_{\underline{2}} = \frac{\varepsilon_1 - \varepsilon_2}{2}$$

in the 4-dimensional spinor $\mathfrak{g}$-module on the top of $L(\Lambda_2)$. By using arguments as above we obtain the following:

Lemma 4.3.

We have

(i) $x_2(-1)v_{\Lambda_2} = 0$ *and* $x_{\underline{2}}(-1)w_{\underline{2}} = 0$,

(ii) $x_{\underline{2}}(-1)x_{\underline{2}}(-1)v_{\Lambda_2} = x_{\underline{2}}(-1)x_0(-1)v_{\Lambda_2} = x_0(-1)x_0(-1)v_{\Lambda_2} = 0$.

Lemma 4.4.

The set of monomial vectors $x(\pi)v_\Lambda$ satisfying difference conditions (6) and initial conditions (7) spans $W(\Lambda)$.

Proof. We have already mentioned how the relation $x_\theta(z)^{k+1} = 0$ on level k standard module $L(\Lambda)$ leads to a spanning set of monomial vectors satisfying difference conditions (6). Following an idea from [30] we reduce the problem of initial conditions (7) for level k Feigin–Stoyanovsky type subspace to a problem of difference conditions for level $k' < k$ Feigin–Stoyanovsky type subspace: we shall consider $\widehat{\mathfrak{g}}$-submodules in tensor products

$$L(\Lambda_0)^{\otimes k_0} \otimes L(\Lambda_2)^{\otimes k_2} \qquad \text{and} \qquad L(\Lambda_1)^{\otimes k_1} \otimes \left(L(\Lambda_0)^{\otimes k_0} \otimes L(\Lambda_2)^{\otimes k_2}\right)$$

of levels $k' = k_0 + k_2$ and $k = k_0 + k_1 + k_2$ generated by highest weight vectors

$$v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2} \qquad \text{and} \qquad v_{\Lambda_1}^{\otimes k_1} \otimes \left(v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2}\right).$$

Assume that for

$$x(\pi) = x_{\underline{2}}(-1)^{c_1} x_0(-1)^{b_1} x_2(-1)^{a_1}$$

the monomial vector $x(\pi)v_\Lambda$ does not satisfy initial conditions (7),

$$a_1 \leq k_0, \qquad b_1 + a_1 \leq k_0 + k_2, \qquad c_1 + b_1 \leq k_0 + k_2.$$

By the above lemmas,

$$x_2(-1)v_{\Lambda_1} = 0, \qquad x_2(-1)v_{\Lambda_2} = 0, \qquad x_2(-1)^2 v_{\Lambda_0} = 0,$$

so in the case when $a_1 > k_0$ we have that the vector

$$x_2(-1)^{a_1}\left(v_{\Lambda_1}^{\otimes k_1} \otimes v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2}\right) = x_2(-1)^{a_1 - k_0}\left(v_{\Lambda_1}^{\otimes k_1} \otimes (x_2(-1)v_{\Lambda_0})^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2}\right)$$

equals zero and we may remove it from our spanning set of monomial vectors.

Now assume that the monomial vector $x(\pi)v_\Lambda$ does not satisfy the initial conditions because

$$k'' = b_1 + a_1 > k_0 + k_2.$$

Then we have

$$x_2(z)^{b_1+a_1} = 0 \qquad \text{on} \quad L(k_0\Lambda_0 + k_2\Lambda_2) \subset L(\Lambda_0)^{\otimes k_0} \otimes L(\Lambda_2)^{\otimes k_2},$$

and by the adjoint action of $(x_{-\varepsilon_2})^{b_1}$ we get

$$x_0(z)^{b_1} x_2(z)^{a_1} + \cdots + c_{u,v,t}\, x_{\underline{2}}(z)^u x_0(z)^v x_2(z)^t + \cdots = 0$$

with $a_1 < t$. The coefficient of z^0 gives us

$$R = x_0(-1)^{b_1} x_2(-1)^{a_1} + \cdots + c'_{u,v,t}\, x_{\underline{2}}(-1)^u x_0(-1)^v x_2(-1)^t + \cdots = 0$$

on $L(k_0\Lambda_0 + k_2\Lambda_2)$. The coefficient R is an infinite sum with the leading term

$$x_0(-1)^{b_1} x_2(-1)^{a_1}. \tag{10}$$

In R we have monomials of the form $x_{\gamma_1}(j_1)\cdots x_{\gamma_{k''}}(j_{k''})$ with $j_1 + \cdots + j_{k''} = -k''$, so either $j_1 = \ldots = j_{k''} = -1$ or we have $j_s \geq 0$ for some s. Hence Lemma 4.1 and

$$x_\gamma(j) v_{\Lambda_i} = 0 \qquad \text{for all} \quad \gamma \in \Gamma, \quad j \geq 0, \quad i = 0,1,2,$$

imply

$$Rv_\Lambda = R\big(v_{\Lambda_1}^{\otimes k_1} \otimes \big(v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2}\big)\big) = v_{\Lambda_1}^{\otimes k_1} \otimes R\big(v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2}\big) = 0.$$

Since the monomial (10) is the leading term of the relation $Rv_\Lambda = 0$, we can express

$$x_0(-1)^{b_1} x_2(-1)^{a_1} v_\Lambda$$

as a combination of higher monomial vectors and we may remove it from the spanning set. In a similar way we argue in the case when $c_1 + b_1 > k_0 + k_2$. □

5. Simple current operators

Recall that we have fixed a cominimal coweight $\omega = \omega_1 = \varepsilon_1 \in \mathfrak{h}$. We shall use simple current operators $[\omega]$ on level 1 modules, i.e. linear bijections

$$L(\Lambda_0) \xrightarrow{[\omega]} L(\Lambda_1) \xrightarrow{[\omega]} L(\Lambda_0), \qquad L(\Lambda_2) \xrightarrow{[\omega]} L(\Lambda_2)$$

such that

$$x_\alpha(z)[\omega] = [\omega] z^{\alpha(\omega)} x_\alpha(z) \qquad \text{for all} \quad \alpha \in R,$$

or, written componentwise,

$$x_\alpha(n)[\omega] = [\omega] x_\alpha(n + \alpha(\omega)) \qquad \text{for all} \quad \alpha \in R, \quad n \in \mathbb{Z}. \tag{11}$$

Remark 5.1.
It is easy to see that, up to a scalar multiple, the linear bijection $[\omega]$ between two irreducible modules is uniquely determined by (11). We can prove the existence of such a map in several ways.

In the ADE case, when the lattice construction of level one $\widehat{g}$-modules is available, for minimal weight ω we have $[\omega] \sim e^{\omega}$, see [5, 17] or [28] for notation and details. For level k modules we consider a tensor product of k level one modules and we have

$$[\omega] \sim e^{\omega} \otimes \cdots \otimes e^{\omega},$$

so that (11) holds.

Haisheng Li pointed out that in general the map $[\omega]$ can be interpreted in terms of simple currents. In [8], a module M for a vertex operator algebra V is called a simple current if the tensor functor "$M \boxtimes \cdot$" is a bijection on the set of equivalence classes $\mathrm{Irr}(V)$ of irreducible V-modules. In [8], simple currents M for affine Lie algebras are constructed by deforming vertex operators $Y_V(\cdot, z)$ for simple vertex operator algebras $V = L(k\Lambda_0)$ with formal Laurent series

$$\Delta(\omega, z) = z^{\omega} \exp\left(-\sum_{n>0} \frac{\omega(n)(-z)^{-n}}{n}\right)$$

so that

$$Y_M(\cdot, z) = Y_V(\Delta(\omega, z)\cdot, z).$$

To prove the existence of $[\omega]$ for $B_2^{(1)}$ we may use a related Dong–Li–Mason's result that for a representation $L(\Lambda)$ of $\widehat{\mathfrak{g}}$ on a vector space W, realized by a vertex operator $Y_{L(\Lambda)}(\cdot, z)$, we also have another representation $L(\Lambda')$ on the same vector space W, but realized by a deformed vertex operator

$$Y_{L(\Lambda')}(\cdot, z) = Y_{L(\Lambda)}(\Delta(\omega, z)\cdot, z),$$

cf. [8, 23]. Then

$$[\omega] \colon L(\Lambda) \to L(\Lambda')$$

can be interpreted as the identity map

$$\mathrm{id} \colon W \to W$$

on the vector space W endowed with two different structures of $\widehat{\mathfrak{g}}$-modules, $L(\Lambda)$ and $L(\Lambda')$.

We can also prove the existence of $[\omega]$ for $B_2^{(1)}$ by following the approach of J. Fuchs [16]: a representation $L(\Lambda)$ of $\widehat{\mathfrak{g}}$ on a vector space W, given by

$$\pi \colon \widehat{\mathfrak{g}} \to \mathrm{End}\, W,$$

can be changed to a new representation $L(\Lambda')$ on the same vector space W by considering a composition

$$\pi \circ \sigma \colon \widehat{\mathfrak{g}} \to \mathrm{End}\, W$$

of representation π with an automorphism σ of $\widehat{\mathfrak{g}}$ defined by

$$\sigma(x_\alpha(n)) = x_\alpha(n + \alpha(\omega)) \qquad \text{for all} \quad \alpha \in R, \quad n \in \mathbb{Z}.$$

Then again $[\omega] \colon L(\Lambda) \to L(\Lambda')$ can be interpreted as the identity map on W.

Remark 5.2.
In our later arguments by induction on degree, we use the map $[\omega]$ in essentially the same way as it is used in [5, 17]: we "move" monomial vectors from one space to another and, due to (11), we "lower" their degrees in the process. For this reason we use the same notation $[\omega]$ for all these different maps on different spaces, including the corresponding maps on tensor products of level one modules and on the symmetric algebra of level one modules, cf. equation (12) and Remark 7.1 below. It should be noted that $[\omega]$ "behaves like a group element", cf. Remark 9.2 below.

We fix $v_{\Lambda_0} = \mathbf{1}$ in the vertex operator algebra $L(\Lambda_0)$. Then we have

Lemma 5.3.

With properly normalized v_{Λ_1} and $x_{\underline{2}}$,

(i) $[\omega]v_{\Lambda_0} = v_{\Lambda_1}$,

(ii) $[\omega]v_{\Lambda_1} = x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0}$.

Proof. (i) For a level k standard $\hat{\mathfrak{g}}$-module $L(\Lambda)$ the new module structure $Y_{L(\Lambda')}(\cdot, z) = Y_{L(\Lambda)}(\Delta(\omega, z)\cdot, z)$ gives

$$h(0)[\omega]v_\Lambda = [\omega]\big(h(0) + \langle\omega, h\rangle k\big)v_\Lambda \qquad \text{for} \quad h \in \mathfrak{h}.$$

In particular, $[\omega]v_{\Lambda_0}$ is a weight vector with level 1 weight $\Lambda_1 = \Lambda_0 + \langle\omega, \cdot\rangle$. Relation (11) gives

$$\begin{aligned} x_{-\theta}(1)[\omega]v_{\Lambda_0} &= [\omega]x_{-\theta}(1-\theta(\omega))v_{\Lambda_0} = [\omega]x_{-\theta}(0)v_{\Lambda_0} = 0, \\ x_{\alpha_i}(0)[\omega]v_{\Lambda_0} &= [\omega]x_{\alpha_i}(0+\alpha_i(\omega))v_{\Lambda_0} = [\omega]x_{\alpha_i}(\delta_{i1})v_{\Lambda_0} = 0 \qquad \text{for} \quad i = 1, 2. \end{aligned}$$

Hence $[\omega]v_{\Lambda_0}$ is a highest weight vector and $L(\Lambda_0') = L(\Lambda_1)$.

(ii) Like in (i) we first see that $[\omega]^{-1}x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0}$ is a weight vector with weight Λ_1. By using (11) and Lemma 4.2 we obtain

$$\begin{aligned} x_{-\theta}(1)[\omega]^{-1}x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0} &= [\omega]^{-1}x_{-\theta}(2)x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0} = 0, \\ x_{\alpha_1}(0)[\omega]^{-1}x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0} &= [\omega]^{-1}x_{\alpha_1}(-1)x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0} = 0, \\ x_{\alpha_2}(0)[\omega]^{-1}x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0} &= [\omega]^{-1}x_{\alpha_2}(0)x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0} = 0. \end{aligned}$$

Hence (ii) holds and $L(\Lambda_1') = L(\Lambda_0)$. □

Lemma 5.4.

With properly normalized $[\omega]$, $w_{\underline{2}}$ and x_0, $x_{\underline{2}}$,

(i) $[\omega]v_{\Lambda_2} = x_0(-1)v_{\Lambda_2} = x_2(-1)w_{\underline{2}}$,

(ii) $[\omega]w_{\underline{2}} = x_0(-1)w_{\underline{2}} = x_{\underline{2}}(-1)v_{\Lambda_2}$.

Proof. (i) As in the proof of previous lemma we see that $[\omega]^{-1}x_0(-1)v_{\Lambda_2}$ is a weight vector with weight Λ_2. By using (11) and Lemma 4.3 we obtain

$$\begin{aligned} x_{-\theta}(1)[\omega]^{-1}x_0(-1)v_{\Lambda_2} &= [\omega]^{-1}x_{-\theta}(2)x_0(-1)v_{\Lambda_0} = 0, \\ x_{\alpha_1}(0)[\omega]^{-1}x_0(-1)v_{\Lambda_2} &= [\omega]^{-1}x_{\alpha_1}(-1)x_0(-1)v_{\Lambda_2} = 0, \\ x_{\alpha_2}(0)[\omega]^{-1}x_0(-1)v_{\Lambda_2} &= [\omega]^{-1}x_{\alpha_2}(0)x_0(-1)v_{\Lambda_2} = 0. \end{aligned}$$

Hence, with a proper normalization, $[\omega]^{-1}x_0(-1)v_{\Lambda_2} = v_{\Lambda_2}$. The second equality follows from Lemma 4.3 because

$$0 = x_{-\alpha_2}(0)0 = x_{-\alpha_2}(0)x_2(-1)v_{\Lambda_2} = C'x_0(-1)v_{\Lambda_2} + C''x_2(-1)w_{\underline{2}}$$

for some $C', C'' \neq 0$.

(ii) The first equality follows from (i) by using the fact that $w_{\underline{2}}$ is proportional to $x_{-\alpha_2}(0)v_{\Lambda_2}$ and the fact that $x_{-\alpha_2}(0)$ commutes with $[\omega]$. The second equality follows from Lemma 4.3. □

We define a linear bijection $[\omega]$ on a tensor product of k fundamental modules as

$$[\omega]\otimes\cdots\otimes[\omega]\colon\ \bigotimes_{s=1}^{k} L(\Lambda_{i_s}) \to \bigotimes_{s=1}^{k} L(\Lambda_{i'_s}).$$

It is clear that relation (11) holds for $[\omega]=[\omega]\otimes\cdots\otimes[\omega]$. In particular,

$$x_\gamma(n)[\omega]=[\omega]x_\gamma(n+1) \qquad \text{for} \quad \gamma\in\Gamma. \tag{12}$$

For a colored partition μ we set $\mu^+(x_\gamma(n+1))=\mu(x_\gamma(n))$. Then for monomials, relation (12) reads as

Lemma 5.5.

$x(\mu)[\omega]=[\omega]x(\mu^+)$.

Remark 5.6.

For $x(\mu)=\prod x_\gamma(n)^{m_\gamma(n)}$ we have $x(\mu^+)=\prod x_\gamma(n+1)^{m_\gamma(n)}$, so we may say that $x(\mu^+)$ is obtained from a monomial $x(\mu)$ by "shifting degrees of factors" $x_\gamma(n)\mapsto x_\gamma(n+1)$. Later on we shall also use the notation $\mu^p(x_\gamma(n+p))=\mu(x_\gamma(n))$ for any $p\in\mathbb{Z}$, and we shall write μ^{+p} when we want to emphasize the shift of degrees of factors.

From Lemmas 5.3, 5.4, 4.1, 4.2 and 4.3 we have

$$\begin{aligned}[\omega]\big(v_{\Lambda_0}^{\otimes k_0}\otimes v_{\Lambda_1}^{\otimes k_1}\otimes v_{\Lambda_2}^{\otimes k_2}\big) &= ([\omega]v_{\Lambda_0})^{\otimes k_0}\otimes([\omega]v_{\Lambda_1})^{\otimes k_1}\otimes([\omega]v_{\Lambda_2})^{\otimes k_2} = v_{\Lambda_1}^{\otimes k_0}\otimes\big(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0}\big)^{\otimes k_1}\otimes\big(x_0(-1)v_{\Lambda_2}\big)^{\otimes k_2}\\ &= C\,x_{\underline{2}}(-1)^{k_1}x_0(-1)^{k_2}x_2(-1)^{k_1}\big(v_{\Lambda_1}^{\otimes k_0}\otimes v_{\Lambda_0}^{\otimes k_1}\otimes v_{\Lambda_2}^{\otimes k_2}\big).\end{aligned} \tag{13}$$

For

$$\Lambda=k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2 \qquad \text{set} \qquad \Lambda^*=k_1\Lambda_0+k_0\Lambda_1+k_2\Lambda_2. \tag{14}$$

Then (13) and Lemma 5.5 imply

Proposition 5.7.

$[\omega]\colon L(\Lambda)\to L(\Lambda^*)$ *and* $[\omega]\colon W(\Lambda)\to W(\Lambda^*)$.

This proposition and a construction in [8, 23] show that $[\omega]=[\omega]\otimes\cdots\otimes[\omega]$ is a simple current operator for level k standard modules.

Virasoro algebra operators in a vertex operator algebra are usually denoted by $L(n)$, $n\in\mathbb{Z}$. If we set $L(0)v_\Lambda=C_\Lambda v_\Lambda$, then

$$d=-L(0)+C_\Lambda \qquad \text{on} \quad L(\Lambda).$$

We have the following:

Lemma 5.8.

For elements h of the Cartan subalgebra $\mathfrak{h}$, and the Virasoro algebra element $L(0)$, on level k standard modules we have

(i) $[\omega]^{-n}h(0)[\omega]^n=h(0)+n\langle\omega,h\rangle k$ *for all* $n\in\mathbb{Z}$, *and*

(ii) $[\omega]^{-n}L(0)[\omega]^n=L(0)+n\,\omega(0)+n^2\langle\omega,\omega\rangle k/2$ *for all* $n\in\mathbb{Z}$.

Proof. As it was already said, we can view $[\omega]$ as the identity map on $L(\Lambda)\to L(\Lambda)$, where the target space is given a new module structure $L(\Lambda')$ with a vertex operator

$$Y_{L(\Lambda')}(\cdot,z)=Y_{L(\Lambda)}(\Delta(\omega,z)\cdot,z),\qquad \Delta(\omega,z)=z^{\omega}\exp\left(-\sum_{n>0}\frac{\omega(n)(-z)^{-n}}{n}\right). \tag{15}$$

Then $L(0)[\omega]$ is the coefficient of z^{-2} in the vertex operator

$$Y_{L(\Lambda')}(L(-2)\mathbf{1},z)=Y_{L(\Lambda)}\big(\Delta(\omega,z)L(-2)\mathbf{1},z\big),$$

and only three terms in

$$\Delta(\omega,z)=1-\left(\omega(1)(-z)^{-1}+\frac{\omega(2)(-z)^{-2}}{2}+\cdots\right)+\frac{1}{2!}\left(\omega(1)(-z)^{-1}+\cdots\right)^{2}+\cdots$$

give a contribution to this coefficient

$$L(0)[\omega]=[\omega]\left(L(0)+\omega(0)+\frac{1}{2}\langle\omega,\omega\rangle k\right).$$

Now (ii) follows by induction. Relation (i) is proved in a similar way. □

Remark 5.9.
In the proofs of Lemmas 5.3, 5.4 and 5.8 (i) we suggested the use of deformed vertex operators (15), but all these statements can be proved by using formula (11) as well.

On the other side, the formula in Lemma 5.8 (ii) written for operator d,

$$-d[\omega]^{n}=[\omega]^{n}\left(-d+C_{\Lambda}-C_{\Lambda'}+n\,\omega(0)+\frac{n^{2}}{2}\langle\omega,\omega\rangle k\right)\qquad\text{if}\quad [\omega]^{n}v_{\Lambda}\in L(\Lambda'),$$

contains a term $C_{\Lambda}-C_{\Lambda'}$ which in general depends on $L(0)$. When $\Lambda'=\Lambda$ the power $[\omega]^n$ is a Weyl group translation operator on $L(\Lambda)$ (cf. Lemma 9.1 and Remark 9.2) and a formula for d follows from (29).

6. Coefficients of level 1 intertwining operators

Let V be a vertex operator algebra and let W_1, W_2 and W_3 be three V-modules. Then an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1\,W_2}$ is a formal series

$$\mathcal{Y}(w,z)=\sum_{n\in\mathbb{Q}}w_{n}z^{-n-1},\qquad w\in W_1,$$

with coefficients

$$w_n\in \operatorname{Hom}(W_2,W_3)\qquad\text{for}\quad n\in\mathbb{Q},$$

such that "all the defining properties of a module action that make sense hold", see [14]. In particular, for $v\in V$ we have a commutator formula

$$v_jw_n-w_nv_j=\sum_{i\geq 0}\binom{j}{i}(v_iw)_{n+j-i},$$

where v_j in v_jw_n is a coefficient of the vertex operator $Y_{W_3}(v,z)=\sum v_jz^{-j-1}$ for V-module W_3, v_j in w_nv_j is a coefficient of the vertex operator $Y_{W_2}(v,z)=\sum v_jz^{-j-1}$ for V-module W_2 and v_i in v_iw is a coefficient of the vertex operator

$Y_{W_1}(v,z) = \sum v_i z^{-i-1}$ for V-module W_1. The vector space of all intertwining operators of type $\binom{W_3}{W_1\, W_2}$ is denoted by $I\binom{W_3}{W_1\, W_2}$ and its dimension is called a fusion rule. We have

$$I\begin{pmatrix} W_3 \\ W_1\, W_2 \end{pmatrix} \cong I\begin{pmatrix} W_3 \\ W_2\, W_1 \end{pmatrix} \cong I\begin{pmatrix} W_2' \\ W_1\, W_3' \end{pmatrix},$$

where for a V-module M we denote by M' the contragredient module, see [14]. If W_1 is an irreducible V-module, W_2 a simple current module and

$$W_3 = W_1 \boxtimes W_2,$$

then by [22, Lemma 2.3] the fusion

$$\dim I\begin{pmatrix} W_3 \\ W_1\, W_2 \end{pmatrix} = 1.$$

Lemma 6.1.
Let $\widehat{\mathfrak{g}}$ be an affine Lie algebra and $L(k\Lambda_0)$ a vacuum level k standard $\widehat{\mathfrak{g}}$-module. Let V_1, V_2, V_3 be irreducible modules for vertex operator algebra $V = L(k\Lambda_0)$. Let $\mathcal{Y} \neq 0$ be an intertwining operator of type $\binom{V_3}{V_1\, V_2}$, let W be the top of V_1 and $v \neq 0$ a vector on the top of V_2. Then there is $m \in \mathbb{Q}$ such that the top of V_3 is a $\mathfrak{g}$-module

$$U(\mathfrak{g})\{w_m v : w \in W\}.$$

Proof. By [7, Proposition 11.9] we have $\mathcal{Y}(w,z)v \neq 0$ for $w \neq 0$ and, from the definition of intertwining operators, $w_n v = 0$ for all n large enough. Let

$$m = \max\{n \in \mathbb{Q} : w_n v \neq 0 \text{ for some } w \in W\}.$$

Then we have a nonzero subspace

$$\{w_m v : w \in W\} \subset V_3.$$

For $x_j = x(j)$ in $\widehat{\mathfrak{g}}$ we have a commutator formula

$$x_j w_m - w_m x_j = \sum_{i \geq 0} \binom{j}{i} (x_i w)_{m+j-i}$$

which for $j > 0$ implies

$$x_j(w_m v) = w_m x_j v + \sum_{i \geq 0} \binom{j}{i} (x_i w)_{m+j-i} v = (x_0 w)_{m+j} v = 0$$

because v and w are vectors on the top of modules and m is maximal such that $w_n v$ can be nonzero. Since

$$U(\widehat{\mathfrak{g}}_{\leq 0})\{w_m v : w \in W\} \subset V_3$$

is a $\widehat{\mathfrak{g}}$-invariant subspace of irreducible $\widehat{\mathfrak{g}}$-module V_3, the space $\{w_m v : w \in W\}$ must be a subspace of the top of V_3 and the lemma follows. ☐

Recall that we have fixed vectors $w_2 = v_{\Lambda_2}$ and $w_{\underline{2}}$ with weights ω_2 and $\omega_{\underline{2}}$ in the 4-dimensional spinor $\mathfrak{g}$-module on the top of $L(\Lambda_2)$.

Proposition 6.2.

(i) *With proper scalars λ and μ and an intertwining operator $\mathcal{Y}$ of type*

$$\begin{pmatrix} L(\Lambda_2) \\ L(\Lambda_2)\ L(\Lambda_0) \end{pmatrix} \tag{16}$$

there are coefficients

$$[\omega_2] \quad of \quad \mathcal{Y}(\lambda w_2, z) = \sum_{n\in\mathbb{Q}} (\lambda w_2)_n z^{-n-1}, \qquad [\omega_2]\colon L(\Lambda_0) \to L(\Lambda_2),$$
$$[\omega_{\underline{2}}] \quad of \quad \mathcal{Y}(\mu w_{\underline{2}}, z) = \sum_{n\in\mathbb{Q}} (\mu w_{\underline{2}})_n z^{-n-1}, \qquad [\omega_{\underline{2}}]\colon L(\Lambda_0) \to L(\Lambda_2),$$

which commute with the action of $\widehat{\mathfrak{g}}_1$ and such that

$$[\omega_2] v_{\Lambda_0} = v_{\Lambda_2}, \qquad [\omega_{\underline{2}}] v_{\Lambda_0} = w_{\underline{2}}.$$

(ii) *With proper scalars λ and μ and an intertwining operator $\mathcal{Y}$ of type*

$$\begin{pmatrix} L(\Lambda_1) \\ L(\Lambda_2)\ L(\Lambda_2) \end{pmatrix} \tag{17}$$

there are coefficients

$$[\omega_2] \quad of \quad \mathcal{Y}(\lambda w_2, z) = \sum_{n\in\mathbb{Q}} (\lambda w_2)_n z^{-n-1}, \qquad [\omega_2]\colon L(\Lambda_2) \to L(\Lambda_1),$$
$$[\omega_{\underline{2}}] \quad of \quad \mathcal{Y}(\mu w_{\underline{2}}, z) = \sum_{n\in\mathbb{Q}} (\mu w_{\underline{2}})_n z^{-n-1}, \qquad [\omega_{\underline{2}}]\colon L(\Lambda_2) \to L(\Lambda_1),$$

which commute with the action of $\widehat{\mathfrak{g}}_1$ and such that

$$[\omega_2] v_{\Lambda_2} = 0, \qquad [\omega_2] w_{\underline{2}} = v_{\Lambda_1}, \qquad [\omega_{\underline{2}}] v_{\Lambda_2} = v_{\Lambda_1}, \qquad [\omega_{\underline{2}}] w_{\underline{2}} = 0.$$

Proof. Since $L(\Lambda_2)$ is an $L(\Lambda_0)$-module, we have

$$I\begin{pmatrix} L(\Lambda_2) \\ L(\Lambda_2)\ L(\Lambda_0) \end{pmatrix} \cong I\begin{pmatrix} L(\Lambda_2) \\ L(\Lambda_0)\ L(\Lambda_2) \end{pmatrix}$$

and the space of intertwining operators of this type is 1-dimensional. Since $L(\Lambda_1)$ is a simple current module such that

$$L(\Lambda_1) \boxtimes L(\Lambda_2) = L(\Lambda_2),$$

see [22, 23] or [8], and since both $L(\Lambda_1)$ and $L(\Lambda_2)$ are self-dual, we have

$$I\begin{pmatrix} L(\Lambda_1) \\ L(\Lambda_2)\ L(\Lambda_2) \end{pmatrix} \cong I\begin{pmatrix} L(\Lambda_2) \\ L(\Lambda_2)\ L(\Lambda_1) \end{pmatrix}$$

and the space of intertwining operators of this type is 1-dimensional.

Let $\mathcal{Y} \neq 0$ be an intertwining operator of type (16) and $v = v_{\Lambda_0}$ on the top of $L(\Lambda_0)$. By Lemma 6.1 there is a vector w on the top of $L(\Lambda_2)$ and an integer m such that $w_m v$ is proportional to v_{Λ_2}. It is clear that w is proportional to v_{Λ_2} and we denote by $[\omega_2]=w_m$ the corresponding coefficient of the formal series $\mathcal{Y}(v_{\Lambda_2},z)$. Obviously, for proper normalization of w we have

$$[\omega_2]v_{\Lambda_0}=v_{\Lambda_2}.$$

On the other hand, if we take $w=w_{\underline{2}}$ and the corresponding coefficient $[\omega_{\underline{2}}]=w_m$ of the formal series $\mathcal{Y}(w_{\underline{2}},z)$, with proper normalization we have

$$[\omega_{\underline{2}}]v_{\Lambda_0}=w_{\underline{2}}.$$

Now let $\mathcal{Y}\neq 0$ be an intertwining operator of type (17) and $v=v_{\Lambda_2}$ on the top of $L(\Lambda_2)$. By Lemma 6.1 there is a vector w on the top of $L(\Lambda_2)$ and an integer m such that vector $w_m v$ generates the irreducible 5-dimensional $\mathfrak{g}$-module on the top of $L(\Lambda_1)$. Since the top of $L(\Lambda_2)$ is a 4-dimensional spinor $\mathfrak{g}$-module, $\mathfrak{h}$-weight vectors of the form $w_m v$ can have weights

$$\frac{\varepsilon_1-\varepsilon_2}{2}+\frac{\varepsilon_1+\varepsilon_2}{2},\qquad -\frac{\varepsilon_1-\varepsilon_2}{2}+\frac{\varepsilon_1+\varepsilon_2}{2},\qquad -\frac{\varepsilon_1+\varepsilon_2}{2}+\frac{\varepsilon_1+\varepsilon_2}{2}.$$

In the first case w is proportional to $w_{\underline{2}}$ and $w_m v=Cv_{\Lambda_1}$ for some scalar $C\neq 0$. Vectors in the second and third case can be transformed to the vector $w'_m v=Cv_{\Lambda_1}$ of the first case by acting with Lie algebra $\mathfrak{g}$ elements $x_{\varepsilon_1-\varepsilon_2}$ and x_{ε_1} respectively. So if we take $w=w_{\underline{2}}$ and the corresponding coefficient $[\omega_{\underline{2}}]=w_m$ of the formal series $\mathcal{Y}(w_{\underline{2}},z)$ with proper normalization, we have

$$[\omega_{\underline{2}}]v_{\Lambda_2}=v_1.$$

Inspection of $\mathfrak{h}$-weights in 5-dimensional $\mathfrak{g}$-module on the top of $L(\Lambda_1)$ shows that $[\omega_{\underline{2}}]w_{\underline{2}}=0$. In a similar way we see that for $w=v_{\Lambda_2}$ and the properly normalized corresponding coefficient $[\omega_2]=w_m$ of the formal series $\mathcal{Y}(v_{\Lambda_2},z)$ we have

$$[\omega_2]w_{\underline{2}}=v_{\Lambda_1}\qquad\text{and}\qquad[\omega_2]v_{\Lambda_2}=0.$$

In each of the above cases $[\omega_2]$ and $[\omega_{\underline{2}}]$ are coefficients of $\mathcal{Y}(w,z)$ with w such that

$$x(i)w=0\qquad\text{for all}\quad x\in\mathfrak{g}_1,\quad i\geq 0.$$

Hence the commutation relations for intertwining operators imply

$$x(j)w_m-w_m x(j)=\sum_{i\geq 0}\binom{j}{i}(x(i)w)_{m+j-i}=0\qquad\text{for all}\quad x(j)\in\widehat{\mathfrak{g}}_1.\qquad\square$$

Remark 6.3.

In Introduction we gave a very rough idea how coefficients of intertwining operators can be used in the proof of linear independence of the monomial basis given by Theorem 3.1: with these operators we "move" monomial vectors $x(\pi)v_\Lambda\mapsto x(\pi)v_{\Lambda'}$ from one space to another until we get vectors of the form $x(\pi')[\omega]v_{\Lambda''}$. Since these operators commute with all $x(\pi)$, the only thing that matters is how these operators "move" the highest weight vectors $v_\Lambda\mapsto v_{\Lambda'}$. In our case it is

$$\begin{aligned}
&v_{\Lambda_0}\xrightarrow{[\omega_2]}v_{\Lambda_2}\xrightarrow{[\omega_{\underline{2}}]}v_{\Lambda_1},\qquad [\omega_{\underline{2}}]w_{\underline{2}}=0,\\
&v_{\Lambda_0}\xrightarrow{[\omega_{\underline{2}}]}w_{\underline{2}}\xrightarrow{[\omega_2]}v_{\Lambda_1},\qquad [\omega_2]v_{\Lambda_2}=0.
\end{aligned}$$

For this reason it is convenient to use for different operators the same symbol $[\omega_2]$ which reminds us only that they are obtained as some coefficients of different series $\mathcal{Y}(w_2,z)$, associated with the "same" vector w_2, or, to be precise, associated with the same weight subspace of weight ω_2 of the top of $L(\Lambda_2)$.

7. Proof of linear independence

By Lemma 4.4 the set of monomial vectors

$$x(\pi)v_\Lambda = \cdots x_{\underline{2}}(-j)^{c_j}x_0(-j)^{b_j}x_2(-j)^{a_j}\cdots x_{\underline{2}}(-1)^{c_1}x_0(-1)^{b_1}x_2(-1)^{a_1}v_\Lambda$$

satisfying difference conditions (6) and initial conditions (7) spans $W(\Lambda)$. We prove linear independence of this set by induction on degree

$$-n = \sum_{\gamma\in\Gamma, j\geq 1} -j\cdot\pi(x_\gamma(-j)) = -\big(1a_1+1b_1+1c_1+\cdots+ja_j+jb_j+jc_j+\cdots\big)$$

of monomials $x(\pi)$, considering in a proof all level k modules simultaneously. In the proof we shall briefly write DC for difference conditions (6) and IC for initial conditions (7).

Step 1. The idea of proof is illustrated most clearly in a proof of linear independence of vectors $x(\pi)v_{k\Lambda_1}$ of degree $-n$. As induction hypothesis we assume that vectors $x(\mu)v_{k\Lambda_0}$ of degree greater than $-n$ are linearly independent. Assume that

$$\sum c_\pi x(\pi)v_{k\Lambda_1} = 0. \tag{18}$$

By Lemma 5.3 we have $v_{\Lambda_1} = [\omega]v_{\Lambda_0}$ and hence $v_{k\Lambda_1} = [\omega]v_{k\Lambda_0}$. By Lemma 5.5,

$$\sum c_\pi x(\pi)v_{k\Lambda_1} = \sum c_\pi x(\pi)[\omega]v_{k\Lambda_0} = [\omega]\sum c_\pi x(\pi^+)v_{k\Lambda_0}$$

and injectivity of $[\omega]$ implies

$$\sum c_\pi x(\pi^+)v_{k\Lambda_0} = 0. \tag{19}$$

Monomials $x(\pi)$ in (18) satisfy difference conditions, so, obviously, "shifted by degree" monomials $x(\pi^+)$ in (19) satisfy difference conditions as well. Monomials $x(\pi)$ in (18) satisfy initial conditions for $k\Lambda_1$, i.e., contain no part of the form $x_\alpha(-1)$. But then monomials $x(\pi^+)$ in (19) contain parts of the form $x_\alpha(-j)$, $j\geq 1$, and hence satisfy initial conditions for $k\Lambda_0$. Since monomials $x(\pi^+)$ in (19) have degrees greater than $-n$, the induction hypothesis implies that all $c_\pi = 0$. Hence we proved linear independence of monomial basis vectors for $W(k\Lambda_1)$ of degree $-n$.

Step 2. For $\mathcal{A} = (c_1, b_1, a_1)$ write

$$x(-1)^{\mathcal{A}} = x_{\underline{2}}(-1)^{c_1}x_0(-1)^{b_1}x_2(-1)^{a_1}.$$

Later on it will be convenient to write a monomial $x(\mu)$ as a product

$$\cdots x_{\underline{2}}(-j)^{c_j}x_0(-j)^{b_j}x_2(-j)^{a_j}\cdots x_{\underline{2}}(-1)^{c_1}x_0(-1)^{b_1}x_2(-1)^{a_1} = x(\mu_2)x(-1)^{\mathcal{A}_\mu}.$$

We define a partial order on the set of level k integral dominant weights:

$$\Lambda' = k_0'\Lambda_0 + k_1'\Lambda_1 + k_2'\Lambda_2 \leq \Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$$

if and only if

$$k_0' \leq k_0, \qquad k_0' + k_2' \leq k_0 + k_2.$$

Clearly, $k\Lambda_1$ is the smallest element and $k\Lambda_0$ is the largest element in the set of level k integral dominant weights.

Now we proceed with a proof of linear independence. We assume that vectors $x(\mu)v_{\Lambda'}$ of degree greater than or equal to $-n$ satisfying DC and IC are linearly independent for some set of $\Lambda' \geq k\Lambda_1$. Let Λ be a minimal level k integral weight for which we need to prove linear independence of monomial vectors of degree greater than or equal to $-n$ satisfying DC and IC. Let

$$\sum c_\pi x(\pi) v_\Lambda = 0. \tag{20}$$

Assume that $c_\mu \neq 0$ for some $x(\mu) = x(\mu_2)x(-1)^{\mathcal{A}_\mu}$ for

$$\mathcal{A}_\mu = (c_1, b_1, a_1), \qquad a_1 < k_0,$$

and that a_1 is the smallest power of $x_2(-1)$ appearing in such A_μ. Since $[\omega_2]\colon L(\Lambda_0) \to L(\Lambda_2)$, we have the operator

$$1^{\otimes a_1} \otimes [\omega_2]^{\otimes(k_0-a_1)} \otimes 1^{\otimes(k_2+k_1)} \colon L(\Lambda) \to L(\Lambda'), \qquad v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2} \otimes v_{\Lambda_1}^{\otimes k_1} \mapsto v_{\Lambda_0}^{\otimes a_1} \otimes v_{\Lambda_2}^{\otimes(k_2+k_0-a_1)} \otimes v_{\Lambda_1}^{\otimes k_1},$$

which commutes with the action of $\widehat{\mathfrak{g}}_1$. Note that $\Lambda > \Lambda'$ so that we may use the induction hypothesis for corresponding monomial vectors. If we apply this operator on the sum (20) we get

$$\sum c_\pi x(\pi) v_{\Lambda'} = 0. \tag{21}$$

By Lemmas 4.1, 4.3, 4.2 we have $x_2(-1)v_{\Lambda_2} = 0$, $x_2(-1)v_{\Lambda_1} = 0$, $x_2(-1)^2 v_{\Lambda_0} = 0$, so for any monomial $x(\pi) = x(\pi')x_2(-1)^a$ with $a > a_1$ we have

$$x(\pi)v_{\Lambda'} = x(\pi')x_2(-1)^a \left(v_{\Lambda_0}^{\otimes a_1} \otimes v_{\Lambda_2}^{\otimes(k_2+k_0-a_1)} \otimes v_{\Lambda_1}^{\otimes k_1}\right) = 0.$$

On the other hand, vectors like $x(\mu)v_{\Lambda'}$ besides DC satisfy IC as well, i.e.,

$$a_1 \leq k_0' = a_1, \qquad b_1 + a_1 \leq k_0' + k_2' = k_0 + k_2, \qquad c_1 + b_1 \leq k_0' + k_2' = k_0 + k_2,$$

so by induction hypothesis the coefficient c_μ in linear combination (21) must be zero, a contradiction.

So in (20) we need to consider only monomials with $a_1 = k_0$, i.e., monomials of the form

$$x(\pi) = x(\pi')x_2(-1)^{k_0}.$$

Assume that $c_\mu \neq 0$ for some $x(\mu) = x(\mu_2)x(-1)^{\mathcal{A}_\mu}$ for

$$\mathcal{A}_\mu = (c_1, b_1, a_1), \qquad a_1 = k_0, \quad b_1 + a_1 < k_0 + k_2, \quad c_1 + b_1 < k_0 + k_2.$$

Since $[\omega_2]\colon L(\Lambda_2) \to L(\Lambda_1)$, we have the operator

$$1^{\otimes(k_0+k_2-1)} \otimes [\omega_2] \otimes 1^{\otimes k_1} \colon L(\Lambda) \to L(\Lambda'), \qquad v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2} \otimes v_{\Lambda_1}^{\otimes k_1} \mapsto v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes(k_2-1)} \otimes v_{\Lambda_1}^{\otimes(k_1+1)},$$

which commutes with the action of $\widehat{\mathfrak{g}}_1$. Note that $\Lambda > \Lambda'$ so that we may use the induction hypothesis for corresponding monomial vectors. If we apply this operator on the sum (20) we get

$$\sum c_\pi x(\pi) v_{\Lambda'} = 0. \tag{22}$$

By Lemmas 4.1, 4.3, 4.2 we have $x_2(-1)v_{\Lambda_2} = 0$, $x_2(-1)v_{\Lambda_1} = 0$, $x_2(-1)^2 v_{\Lambda_0} = 0$, so for any monomial $x(\pi) = x(\pi')x_2(-1)^{k_0}$ we have

$$x(\pi)v_{\Lambda'} = x(\pi')x_2(-1)^{k_0}\left(v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes(k_2-1)} \otimes v_{\Lambda_1}^{\otimes(k_1+1)}\right) = C\,x(\pi')\left((x_2(-1)v_{\Lambda_0})^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes(k_2-1)} \otimes v_{\Lambda_1}^{\otimes(k_1+1)}\right)$$

for some $C \neq 0$. If for such $x(\pi) = x(\pi_2)x(-1)^{\mathcal{A}_\pi}$ we have

$$b_1 + a_1 = k_0 + k_2 \qquad \text{or} \qquad c_1 + b_1 = k_0 + k_2,$$

then by Lemma 4.3, $x_{\underline{2}}(-1)^2 v_{\Lambda_2} = x_{\underline{2}}(-1)x_0(-1)v_{\Lambda_2} = x_0(-1)^2 v_{\Lambda_2} = 0$ and by Lemma 4.2, $x_{\underline{2}}(-1)x_0(-1)v_{\Lambda_0} = 0$, so in either case at least one of $x_0(-1)$ or $x_{\underline{2}}(-1)$ must act on one copy of v_{Λ_1}. Hence, by Lemma 4.1, for such $x(\pi)$ there must be

$$x(\pi)v_{\Lambda'} = 0.$$

So in (22) we have only vectors like $x(\mu)v_{\Lambda'}$ which besides DC satisfy IC as well, and by induction hypothesis the coefficient c_μ in linear combination (22) must be zero, a contradiction.

Remark 7.1.
For the rest of the proof it will be convenient to realize $L(\Lambda)$ of level k in a k^{th} component of a symmetric algebra

$$L(\Lambda) \subset S^k(V), \qquad V = L(\Lambda_0) \oplus L(\Lambda_1) \oplus L(\Lambda_2).$$

The operator $[\omega] = S([\omega])$ acts as "a group element" on $S(V)$. On the other hand, operators A and B on V in Lemmas 7.2 and 7.3 below act as derivations on $S(V)$.

Step 3. By the previous step, in the linear combination (20) we need to consider only monomials $x(\pi) = x(\pi_2)x(-1)^{\mathcal{A}_\pi}$ with $a_1 = k_0$ for $\mathcal{A}_\pi = (c_1, b_1, a_1)$ and

$$b_1 + a_1 = k_0 + k_2 \qquad \text{or} \qquad c_1 + b_1 = k_0 + k_2.$$

Assume first we have a monomial vector $x(\pi)v_\Lambda$ such that $a_1 = k_0$, $b_1 + a_1 = k_0 + k_2$ and $c_1 + b_1 \leq k_0 + k_2$. This implies that

$$a_1 = k_0, \qquad b_1 = k_2, \qquad c_1 \leq k_0. \tag{23}$$

As above, Lemmas 4.1, 4.3 and 4.2 imply that

$$\begin{aligned} x(-1)^{\mathcal{A}_\pi} v_\Lambda &= x_{\underline{2}}(-1)^{c_1} x_0(-1)^{k_2} x_2(-1)^{k_0} \left(v_{\Lambda_1}^{k_1} v_{\Lambda_2}^{k_2} v_{\Lambda_0}^{k_0}\right) = C x_{\underline{2}}(-1)^{c_1} \left(v_{\Lambda_1}^{k_1} (x_0(-1)v_{\Lambda_2})^{k_2} (x_2(-1)v_{\Lambda_0})^{k_0}\right) \\ &= C' v_{\Lambda_1}^{k_1} \left(x_0(-1)v_{\Lambda_2}\right)^{k_2} \left(x_2(-1)v_{\Lambda_0}\right)^{k_0 - c_1} \left(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0}\right)^{c_1}. \end{aligned}$$

Let $A\colon V \to V$ be a linear operator

$$A\restriction_{L(\Lambda_0)} = [\omega_{\underline{2}}]\colon\ L(\Lambda_0) \to L(\Lambda_2), \qquad A\restriction_{L(\Lambda_1)\oplus L(\Lambda_2)} = 0,$$

and let A act as a derivation on $S(V)$. By Proposition 6.2, derivation A commutes with the action of $\widehat{\mathfrak{g}}_1$ on the symmetric algebra $S(V)$. Note that $Av_{\Lambda_0} = [\omega_{\underline{2}}]v_{\Lambda_0} = w_{\underline{2}}$ by Proposition 6.2 and $x_{\underline{2}}(-1)w_{\underline{2}} = 0$ by Lemma 4.3, so $A(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0}) = 0$. Hence, by Lemmas 5.3 and 5.4, we have

$$\begin{aligned} A^{k_0-c_1} x(-1)^{\mathcal{A}_\pi} v_\Lambda &= C'' v_{\Lambda_1}^{k_1} (x_0(-1)v_{\Lambda_2})^{k_2} (x_2(-1)w_{\underline{2}})^{k_0-c_1} (x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0})^{c_1} \\ &= C''([\omega]v_{\Lambda_0})^{k_1}([\omega]v_{\Lambda_2})^{k_2}([\omega]v_{\Lambda_2})^{k_0-c_1}([\omega]v_{\Lambda_1})^{c_1} = C''[\omega]\left(v_{\Lambda_0}^{k_1} v_{\Lambda_2}^{k_2} v_{\Lambda_2}^{k_0-c_1} v_{\Lambda_1}^{c_1}\right) = C''[\omega]\left(v_{\Lambda_0}^{k_1} v_{\Lambda_2}^{k_2+k_0-c_1} v_{\Lambda_1}^{c_1}\right). \end{aligned}$$

Since A commutes with the action of $\widehat{\mathfrak{g}}_1$, we have

$$\begin{aligned} A^{k_0-c_1} x(\pi)v_\Lambda &= x(\pi_2)Ax(-1)^{\mathcal{A}_\pi} v_\Lambda = C'' x(\pi_2)[\omega]\left(v_{\Lambda_0}^{k_1} v_{\Lambda_2}^{k_2+k_0-c_1} v_{\Lambda_1}^{c_1}\right) \\ &= C''[\omega]x(\pi_2^+)\left(v_{\Lambda_0}^{k_1} v_{\Lambda_2}^{k_2+k_0-c_1} v_{\Lambda_1}^{c_1}\right) = C''[\omega]x(\pi_2^+)v_{\Lambda'} \end{aligned}$$

for some $C'' \neq 0$. It is clear that "truncated and shifted by degree" monomial $x(\pi_2^+)$ satisfies DC, and IC for $x(\pi_2^+)v_{\Lambda'}$ read

$$a_2 \leq k_1 = k - b_1 - a_1, \qquad b_2 + a_2 \leq k_1 + k_2 + k_0 - c_1, \qquad c_2 + b_2 \leq k_1 + k_2 + k_0 - c_1 = k - c_1.$$

But these are just three difference condition relations which hold for $x(\pi)v_\Lambda$:

$$a_2 + b_1 + a_1 \leq k, \qquad b_2 + a_2 + c_1 \leq k, \qquad c_2 + b_2 + c_1 \leq k.$$

Hence we have proved the following:

Lemma 7.2.
In the case when (23) holds, the monomial vector

$$x(\pi_2^+)v_{\Lambda'} = (C''[\omega])^{-1}A^{k_0-c_1}x(\pi)v_\Lambda$$

satisfies difference conditions (6) and initial conditions (7).

Assume now that we have a monomial vector $x(\pi)v_\Lambda$ such that $a_1 = k_0$ and $b_1 + a_1 \leq k_0 + k_2$ and $c_1 + b_1 = k_0 + k_2$. This implies that

$$a_1 = k_0, \qquad b_1 \leq k_2, \qquad c_1 + b_1 = k_0 + k_2. \tag{24}$$

Like before, Lemmas 4.1, 4.3 and 4.2 imply that

$$\begin{aligned} x(-1)^{\mathcal{A}_\pi}v_\Lambda &= x_{\underline{2}}(-1)^{c_1}x_0(-1)^{b_1}x_2(-1)^{k_0}\left(v_{\Lambda_1}^{k_1}v_{\Lambda_2}^{k_2}v_{\Lambda_0}^{k_0}\right) = C\,x_{\underline{2}}(-1)^{c_1}\left(v_{\Lambda_1}^{k_1}v_{\Lambda_2}^{k_2-b_1}(x_0(-1)v_{\Lambda_2})^{b_1}(x_2(-1)v_{\Lambda_0})^{k_0}\right) \\ &= C'v_{\Lambda_1}^{k_1}\left(x_{\underline{2}}(-1)v_{\Lambda_2}\right)^{k_2-b_1}\left(x_0(-1)v_{\Lambda_2}\right)^{b_1}\left(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0}\right)^{k_0}. \end{aligned}$$

By Lemmas 5.3 and 5.4 we further have

$$x(-1)^{\mathcal{A}_\pi}v_\Lambda = C'([\omega]v_{\Lambda_0})^{k_1}([\omega]w_{\underline{2}})^{k_2-b_1}([\omega]v_{\Lambda_2})^{b_1}([\omega]v_{\Lambda_1})^{k_0} = C'[\omega]\left(v_{\Lambda_0}^{k_1}w_{\underline{2}}^{k_2-b_1}v_{\Lambda_2}^{b_1}v_{\Lambda_1}^{k_0}\right).$$

Hence we have

$$x(\pi)v_\Lambda = x(\pi_2)x(-1)^{\mathcal{A}_\pi}v_\Lambda = C'x(\pi_2)[\omega]\left(v_{\Lambda_0}^{k_1}w_{\underline{2}}^{k_2-b_1}v_{\Lambda_2}^{b_1}v_{\Lambda_1}^{k_0}\right) = C'[\omega]x(\pi_2^+)\left(v_{\Lambda_0}^{k_1}w_{\underline{2}}^{k_2-b_1}v_{\Lambda_2}^{b_1}v_{\Lambda_1}^{k_0}\right).$$

Let $B\colon V \to V$ be a linear operator

$$B\restriction_{L(\Lambda_2)} = [\omega_2]\colon\ L(\Lambda_2) \to L(\Lambda_1), \qquad B\restriction_{L(\Lambda_0)\oplus L(\Lambda_1)} = 0,$$

and let B act as a derivation on $S(V)$. By Proposition 6.2, the derivation B commutes with the action of $\widehat{\mathfrak{g}}_1$ on the symmetric algebra $S(V)$, $Bw_{\underline{2}} = [\omega_2]w_{\underline{2}} = v_{\Lambda_1}$ and $Bv_{\Lambda_2} = [\omega_2]v_{\Lambda_2} = 0$. Hence

$$B^{k_2-b_1}\colon\ v_{\Lambda_0}^{k_1}w_{\underline{2}}^{k_2-b_1}v_{\Lambda_2}^{b_1}v_{\Lambda_1}^{k_0} \mapsto C''v_{\Lambda'} = C''v_{\Lambda_0}^{k_1}v_{\Lambda_1}^{k_0+k_2-b_1}v_{\Lambda_2}^{b_1}.$$

Lemma 7.3.
In the case when (24) holds, the monomial vector

$$x(\pi_2^+)v_{\Lambda'} = B^{k_2-b_1}(C''C'[\omega])^{-1}x(\pi)v_\Lambda$$

satisfies difference conditions (6) and initial conditions (7).

Proof. It is clear that "truncated and shifted by degree" monomial $x(\pi_2^+)$ satisfies DC, and IC for $x(\pi_2^+)v_{\Lambda'}$ read

$$a_2 \leq k_1 = k - c_1 - b_1, \qquad b_2 + a_2 \leq k_1 + b_1, \qquad c_2 + b_2 \leq k_1 + b_1 = k_1 + k_0 + k_2 - c_1 = k - c_1.$$

But these are just three difference condition relations which hold for $x(\pi)v_\Lambda$:

$$a_2 + c_1 + b_1 \leq k, \qquad b_2 + a_2 + c_1 \leq k, \qquad c_2 + b_2 + c_1 \leq k. \qquad \square$$

Now we proceed with the proof of linear independence. As already noted, in the linear combination (20) we need to consider only

$$0 = \sum_{\substack{a_1=k_0,\, b_1+a_1=k_0+k_2>c_1+b_1\\ \text{or}\\ a_1=k_0,\, b_1+a_1\le k_0+k_2=c_1+b_1}} C_{\dots c_1 b_1 a_1} \cdots x_{\underline{2}}(-1)^{c_1} x_0(-1)^{b_1} x_2(-1)^{k_0}\left(v_{\Lambda_1}^{k_1} v_{\Lambda_2}^{k_2} v_{\Lambda_0}^{k_0}\right)$$
$$= \sum_{c_1<k_0} C_{\dots c_1 k_2 k_0} \cdots x_{\underline{2}}(-1)^{c_1} x_0(-1)^{k_2} x_2(-1)^{k_0}\left(v_{\Lambda_1}^{k_1} v_{\Lambda_2}^{k_2} v_{\Lambda_0}^{k_0}\right) + \sum_{b_1\le k_2,\, c_1+b_1=k_0+k_2} C_{\dots c_1 b_1 k_0} \cdots x_{\underline{2}}(-1)^{c_1} x_0(-1)^{b_1} x_2(-1)^{k_0}\left(v_{\Lambda_1}^{k_1} v_{\Lambda_2}^{k_2} v_{\Lambda_0}^{k_0}\right).$$

Note that in the first sum we have vectors of the form

$$x(\pi_2)\left(v_{\Lambda_1}^{k_1}(x_0(-1)v_{\Lambda_2})^{k_2}(x_2(-1)v_{\Lambda_0})^{k_0-c_1}(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0})^{c_1}\right), \tag{25}$$

$k_0-c_1=1,\dots,k_0$, and that in the second sum we have vectors of the form

$$x(\pi_2)\left(v_{\Lambda_1}^{k_1}(x_{\underline{2}}(-1)v_{\Lambda_2})^{k_2-b_1}(x_0(-1)v_{\Lambda_2})^{b_1}(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0})^{k_0}\right), \tag{26}$$

$k_2-b_1=0,\dots,k_2$. In particular, in (25) we see a factor

$$\left(x_2(-1)v_{\Lambda_0}\right)^{k_0-c_1}\left(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0}\right)^{c_1} \qquad \text{for} \quad c_1=0,\dots,k_0-1,$$

and in (26) we see a factor $(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0})^{k_0}$. Hence the operator A^{k_0} annihilates all these terms except the ones with $c_1=0$ and the action on linear combination (20) gives

$$0 = A^{k_0}\sum c_\pi x(\pi)v_\Lambda = [\omega]\sum_{\mathcal{A}_\pi=(0,k_2,k_0)} c_\pi C'' x(\pi_2^+)v_{\Lambda'}.$$

Now Lemma 7.2 and the induction hypothesis imply that $c_\pi=0$ whenever $A_\pi=(0,k_2,k_0)$. In turn this implies that in the first sum of (20) it is enough to consider vectors (25) for $c_1=1,\dots,k_0-1$. Then we apply operator A^{k_0-1} which annihilates all these terms except the ones with $c_1=1$ and the action on linear combination (20) gives

$$0 = A^{k_0}\sum c_\pi x(\pi)v_\Lambda = [\omega]\sum_{\mathcal{A}_\pi=(1,k_2,k_0)} c_\pi C'' x(\pi_2^+)v_{\Lambda'}.$$

Now Lemma 7.2 and the induction hypothesis imply that $c_\pi=0$ whenever $A_\pi=(1,k_2,k_0)$. By proceeding in this way we see that all the coefficients $c_\pi=C_{\dots c_1 b_1 a_1}$ for $c_1<k_0$ in the first sum are equal to zero.

So we are left with the second sum

$$\sum c_\pi x(\pi)v_\Lambda = [\omega]\sum_{\substack{b_1\le k_2\\ c_1+b_1=k_0+k_2}} c_\pi C' x(\pi_2^+)\left(v_{\Lambda_0}^{k_1} w_{\underline{2}}^{k_2-b_1} v_{\Lambda_2}^{b_1} v_{\Lambda_1}^{k_0}\right)=0. \tag{27}$$

This implies

$$\sum_{\substack{b_1\le k_2\\ c_1+b_1=k_0+k_2}} c_\pi C' x(\pi_2^+)\left(v_{\Lambda_0}^{k_1} w_{\underline{2}}^{k_2-b_1} v_{\Lambda_2}^{b_1} v_{\Lambda_1}^{k_0}\right)=0. \tag{28}$$

In (28) we see factors

$$w_{\underline{2}}^{k_2-b_1} v_{\Lambda_2}^{b_1} \qquad \text{for} \quad b_1=0,\dots,k_2.$$

The operator B^{k_0} will annihilate all these terms except the ones with $b_1=0$ and the action on linear combination (28) gives

$$\sum_{\substack{b_1=0\\ c_1+b_1=k_0+k_2}} c_\pi C'C'' x(\pi_2^+)\left(v_{\Lambda_0}^{k_1} v_{\Lambda_1}^{k_0+k_2-b_1} v_{\Lambda_2}^{b_1}\right)=0.$$

Now Lemma 7.3 and the induction hypothesis imply that $c_\pi=0$ whenever $A_\pi=(k_0+k_2,0,k_0)$. In turn this implies that in (28) it is enough to consider vectors for $b_1=1,\dots,k_0$. So next we apply B^{k_0-1} and conclude that $c_\pi=0$ whenever $A_\pi=(k_0+k_2-1,1,k_0)$. By proceeding in this way we see that all the coefficients c_π in the second sum of (27) are equal to zero and our proof of linear independence is complete.

8. Vertex operator formula

For a root α we denote by $\alpha^\vee \in \mathfrak{h}$ a dual root, $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$. In this section for each $\alpha \in R$ we choose $x_\alpha \in \mathfrak{g}_\alpha$ so that $[x_\alpha, x_{-\alpha}] = -\alpha^\vee$ and define on $L(\Lambda)$ a "Weyl group translation" operator e_α by

$$s_\alpha = \exp x_\alpha(0)\exp x_{-\alpha}(0)\exp x_\alpha(0), \qquad s_{\delta-\alpha} = \exp x_{-\alpha}(1)\exp x_\alpha(-1)\exp x_{-\alpha}(1), \qquad e_\alpha = s_{\delta-\alpha}s_\alpha.$$

Then on a standard $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ we have

$$\begin{aligned} &e_\alpha c e_\alpha^{-1} = c, \qquad e_\alpha d e_\alpha^{-1} = d + \alpha^\vee - \frac{1}{2}\langle\alpha^\vee,\alpha^\vee\rangle c, \qquad e_\alpha h e_\alpha^{-1} = h - \langle\alpha^\vee, h\rangle c,\\ &e_\alpha h(j) e_\alpha^{-1} = h(j) \quad \text{for} \quad j \neq 0, \qquad e_\alpha x_\gamma(j) e_\alpha^{-1} = (-1)^{\gamma(\alpha^\vee)} x_\gamma(j - \gamma(\alpha^\vee)) \end{aligned} \tag{29}$$

for $h \in \mathfrak{h}$, $\gamma \in R$ and $j \in \mathbb{Z}$. These formulas are a consequence of the adjoint action of the group element e_α on $\widehat{\mathfrak{g}}$. The map $\alpha^\vee \mapsto e_\alpha$ extends from the dual root system $R^\vee$ to a projective representation of the root lattice $Q(R^\vee)$, cf. [13, 15, 19].

Let $\alpha \in R$. Then $\widehat{\mathfrak{sl}}_2(\alpha)$ defined by (9) is of the type $A_1^{(1)}$ with the canonical central element

$$c_\alpha = \langle x_\alpha, -x_{-\alpha}\rangle c = \frac{2c}{\langle\alpha,\alpha\rangle}.$$

For a standard $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ of level $\Lambda(c) = k$, the restriction to $\widehat{\mathfrak{sl}}_2(\alpha)$ is of level $k_\alpha = \Lambda(c_\alpha) = k$ if $\langle\alpha,\alpha\rangle = 2$ (i.e. if α is a long root) and of level $k_\alpha = 2k$ if $\langle\alpha,\alpha\rangle = 1$ (i.e. if α is a short root). Recall that $zx_\alpha(z) = \sum x_\alpha(n)z^{-n}$ is a formal Laurent series in an indeterminate z with coefficients in $\operatorname{End} L(\Lambda)$. We also define a formal Laurent series $z^{c_\alpha+\alpha^\vee}$ by

$$z^{c_\alpha+\alpha^\vee} v_\mu = v_\mu z^{k_\alpha+\mu(\alpha^\vee)}$$

whenever $v_\mu \in L(\Lambda)$ is a vector of $\mathfrak{h}$-weight μ. Set

$$E^\pm(\alpha, z) = \exp\left(\sum_{i>0} \alpha^\vee(\pm i)\,\frac{z^{\mp i}}{\pm i}\right).$$

Since $x_\alpha(z)^{k_\alpha+1} = 0$ on $L(\Lambda)$, the exponential $\exp(zx_\alpha(z)) = \exp\left(\sum x_\alpha(n)z^{-n}\right)$ is well defined and we have a generalization of the Frenkel–Kac vertex operator formula, cf. [21, Theorem 5.6], [26, Theorem 6.4] or [27, Section 3], for all standard modules:

$$\exp(zx_\alpha(z)) = E^-(-\alpha, z)\exp(-zx_{-\alpha}(z))\,E^+(-\alpha, z)\,e_\alpha z^{c_\alpha+\alpha^\vee}. \tag{30}$$

By (29) the $\mathfrak{h}$-weight components of the vertex operator formula (30) on level k module $L(\Lambda)$ give relations

$$\frac{1}{p!}(z\,x_\alpha(z))^p = \frac{1}{q!}E^-(-\alpha, z)(-zx_{-\alpha}(z))^q E^+(-\alpha, z)\,e_\alpha z^{k_\alpha+\alpha^\vee} \tag{31}$$

for $p, q \geq 0$, $p + q = k_\alpha$. In the level $k = 1$ case, for a long root α and $p = 1, 0$ we have

$$zx_\alpha(z) = E^-(-\alpha, z)E^+(-\alpha, z)\,e_\alpha z^{1+\alpha}, \qquad 1 = E^-(-\alpha, z)(-z\,x_{-\alpha}(z))\,E^+(-\alpha, z)\,e_\alpha z^{1+\alpha}. \tag{32}$$

Since in this case $e_\alpha e_{-\alpha} = -1$, relations (32) are simply the Frenkel–Kac vertex operator formulas

$$x_\alpha(z) = E^-(-\alpha, z)E^+(-\alpha, z)\,e_\alpha z^\alpha, \qquad x_{-\alpha}(z) = E^-(\alpha, z)E^+(\alpha, z)\,e_{-\alpha}z^{-\alpha},$$

see [13, 15]. In fact, the Frenkel–Kac vertex operator formulas for level 1 standard $\widehat{\mathfrak{sl}}_2(\alpha)$-modules imply $x_\alpha(z)^2 = x_{-\alpha}(z)^2 = 0$ and the relation (30), and for higher level k modules we prove (30) simply by applying the "exponentials of Lie algebra elements" on both sides of (30) to tensor product of k copies of level 1 modules.

Denote by $\langle e_\alpha : \alpha \in \Gamma\rangle$ a group of operators on $L(\Lambda)$ generated by all operators e_α, $\alpha \in \Gamma$. Then we have

Lemma 8.1.
$L(\Lambda) = \langle e_\alpha : \alpha \in \Gamma \rangle U(\widehat{\mathfrak{g}}_1) v_\Lambda$.

Proof. First notice that the Lie algebra $\mathfrak{g}$ is generated by $\mathfrak{g}_1 \cup \mathfrak{g}_{-1}$. In particular, $\operatorname{span} \Gamma^\vee = \mathfrak{h}$. Similarly, $[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]$ is generated by $\widehat{\mathfrak{g}}_1 \cup \widehat{\mathfrak{g}}_{-1}$, so we have

$$L(\Lambda) = \{x_1 \cdots x_s v_\Lambda : s \geqslant 0,\, x_i \in \widehat{\mathfrak{g}}_1 \cup \widehat{\mathfrak{g}}_{-1}\}.$$

By using the vertex operator formula (31) for $\alpha \in (-\Gamma)$ and $p = 1$, we may replace each $x_i \in \widehat{\mathfrak{g}}_{-1}$ with a product of elements from

$$\{e_\alpha : \alpha \in (-\Gamma)\} \cup \mathfrak{s} \cup \widehat{\mathfrak{g}}_1,$$

where $\mathfrak{s}$ denotes the Heisenberg subalgebra

$$\mathfrak{s} = \sum_{j \in \mathbb{Z} \setminus \{0\}} \mathfrak{h} \otimes t^j + \mathbb{C}c, \qquad \mathfrak{s}_- = \sum_{j<0} \mathfrak{h} \otimes t^j.$$

Since both group elements e_α and Lie algebra elements from the Heisenberg subalgebra $\mathfrak{s}$ normalize $\widehat{\mathfrak{g}}_1$, we get

$$L(\Lambda_0) = \langle e_\alpha : \alpha \in \Gamma \rangle U(\widehat{\mathfrak{g}}_1) U(\mathfrak{s}_-) v_\Lambda.$$

Now notice that $U(\mathfrak{s}_-)$ is generated by the coefficients of $E^-(-\alpha, z)$ for $\alpha \in \Gamma$. So, using the vertex operator formula (31) for $\alpha \in \Gamma$ and $q = 0$, we may replace elements in $U(\mathfrak{s}_-) v_\Lambda$ by elements in $\langle e_\alpha : \alpha \in \Gamma \rangle U(\widehat{\mathfrak{g}}_1) v_\Lambda$. □

As in [27, Section 5] we set

$$e = e_{\varepsilon_1 - \varepsilon_2} e_{\varepsilon_1} e_{\varepsilon_1 + \varepsilon_2} = \prod_{\alpha \in \Gamma} e_\alpha. \tag{33}$$

Proposition 8.2.
Let $L(\Lambda)_\mu$ be a weight subspace of $L(\Lambda)$. Then there exists an integer m_0 such that for any fixed $m \leqslant m_0$ the set of vectors

$$e^m x_{\beta_1}(j_1) \cdots x_{\beta_s}(j_s) v_\Lambda \in L(\Lambda)_\mu,$$

where $s \geqslant 0$, $\beta_1, \ldots, \beta_s \in \Gamma$, $j_1, \ldots, j_s \in \mathbb{Z}$, is a spanning set of $L(\Lambda)_\mu$. In particular,

$$L(\Lambda) = \langle e \rangle W(\Lambda).$$

Proof. Since $\dim L(\Lambda)_\mu < \infty$, by Lemma 8.1 we may choose a finite spanning set of vectors of the form

$$\left(\prod_{\alpha \in \Gamma} e_\alpha\right)^m \prod_{\alpha \in \Gamma} e_\alpha^{p_\alpha} x_{\beta_1}(j_1) \cdots x_{\beta_r}(j_r) v_\Lambda,$$

$r \geqslant 0$, $x_{\beta_i}(j_i) \in \widehat{\mathfrak{g}}_1$, m fixed for all vectors. Clearly there exists m_0 such that if we choose $m \leqslant m_0$, then all $p_\alpha \geqslant 0$ for all vectors. Since e_α normalize $\widehat{\mathfrak{g}}_1$, we have a spanning set of vectors of the form

$$e^m x_{\beta_1}(j_1') \cdots x_{\beta_r}(j_r') \prod_{\alpha \in \Gamma} e_\alpha^{p_\alpha} v_\Lambda.$$

Now in a finite number of steps we replace each $e_\alpha v_\Lambda$ by an element from $U(\widehat{\mathfrak{g}}_1) v_\Lambda$ using coefficients of $z^{k_\alpha + \Lambda(\alpha^\vee)}$ in the vertex operator formula (31) for $\alpha \in \Gamma$ and $q = 0$. □

9. Bases consisting of semi-infinite monomials

In (14) we have set $\Lambda^* = k_1\Lambda_0 + k_0\Lambda_1 + k_2\Lambda_2$ for $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$. Note that $\Lambda^{**} = \Lambda$. The relation (13) applied twice together with Lemma 5.5 gives

$$\begin{aligned}[t][\omega]^2 v_\Lambda = [\omega]^2\big(v_{\Lambda_0}^{\otimes k_0}\otimes v_{\Lambda_1}^{\otimes k_1}\otimes v_{\Lambda_2}^{\otimes k_2}\big) &= C'[\omega]x_{\underline{2}}(-1)^{k_1}x_0(-1)^{k_2}x_2(-1)^{k_1}\big(v_{\Lambda_1}^{\otimes k_0}\otimes v_{\Lambda_0}^{\otimes k_1}\otimes v_{\Lambda_2}^{\otimes k_2}\big)\\ &= C'x_{\underline{2}}(-2)^{k_1}x_0(-2)^{k_2}x_2(-2)^{k_1}[\omega]\big(v_{\Lambda_1}^{\otimes k_0}\otimes v_{\Lambda_0}^{\otimes k_1}\otimes v_{\Lambda_2}^{\otimes k_2}\big)\\ &= C\,x_{\underline{2}}(-2)^{k_1}x_0(-2)^{k_2}x_2(-2)^{k_1}x_{\underline{2}}(-1)^{k_0}x_0(-1)^{k_2}x_2(-1)^{k_0}v_\Lambda\end{aligned} \tag{34}$$

for some $C = C_\Lambda \neq 0$. If we set

$$x(\kappa_\Lambda) = x_{\underline{2}}(-2)^{k_1}x_0(-2)^{k_2}x_2(-2)^{k_1}x_{\underline{2}}(-1)^{k_0}x_0(-1)^{k_2}x_2(-1)^{k_0},$$

then (34) reads

$$[\omega]^2 v_\Lambda = C_\Lambda x(\kappa_\Lambda)v_\Lambda. \tag{35}$$

This relation and Lemma 5.5 imply

$$[\omega]^2\colon L(\Lambda)\to L(\Lambda) \qquad \text{and} \qquad [\omega]^2\colon W(\Lambda)\to W(\Lambda).$$

Lemma 9.1.

$e = C[\omega]^4$ for some $C \neq 0$.

Proof. Since $(\varepsilon_1-\varepsilon_2)^\vee + \varepsilon_1^\vee + (\varepsilon_1+\varepsilon_2)^\vee = 4\varepsilon_1$, relations (29) and (11) imply

$$e\,x_{\pm\gamma}(j)\,e^{-1} = x_\gamma(j\mp 4) \qquad \text{and} \qquad [\omega]^4 x_{\pm\gamma}(j)[\omega]^{-4} = x_\gamma(j\mp 4)$$

for all $\gamma\in\Gamma$. So $e[\omega]^{-4}$ commutes with the action of $\hat{\mathfrak{g}}$ and must be proportional to the identity operator on $L(\Lambda)$. □

Remark 9.2.

Roughly speaking, the above lemma states that the simple current operator $[\omega]$ is a "fourth root" of inner automorphism e. By choosing $e = \prod_{\alpha\in\Gamma} e_\alpha$ we follow the notation in [27], but our arguments would work in the same way if we have chosen the inner automorphism e to be

$$e_{\varepsilon_1-\varepsilon_2}e_{\varepsilon_1+\varepsilon_2} = C'e_{\varepsilon_1} = C[\omega]^2$$

for some $C', C \neq 0$.

Theorem 9.3.

Let $L(\Lambda)_\mu$ be a weight subspace of a level k standard $B_2^{(1)}$-module $L(\Lambda)$. Then there exists an integer m_0 such that for any fixed $m \le m_0$ the set of vectors

$$C_\Lambda^{-m}[\omega]^{2m}x(\pi)v_\Lambda \in L(\Lambda)_\mu,$$

such that monomial vectors $x(\pi)v_\Lambda \in W(\Lambda)$ satisfy difference conditions (6) and initial conditions (7), is a basis of $L(\Lambda)_\mu$. Moreover, for two choices of $m_1, m_2 \le m_0$ the corresponding two bases are equal.

Proof. By Proposition 8.2 vectors of the form

$$e^m x(\pi)v_\Lambda \in L(\Lambda)_\mu$$

span $L(\Lambda)_\mu$ for a given small enough m, so, by Theorem 3.1, monomial vectors satisfying DC and IC will form a basis. By Lemma 9.1 we can replace e^m with $[\omega]^{4m}$. Note that by Lemma 5.5, the notation from Remark 5.6 and (35)

$$C_\Lambda^{-m}[\omega]^{2m}x(\pi)v_\Lambda = [\omega]^{2m-2}x(\pi^{-2})[\omega]^2 v_\Lambda = C_\Lambda^{-m+1}[\omega]^{2(m-1)}x(\pi^{-2})x(\kappa_\Lambda)v_\Lambda \tag{36}$$

and the monomial vector $x(\pi)v_\Lambda$ satisfies DC and IC if and only if the monomial vector $x(\pi^{-2})x(\kappa_\Lambda)v_\Lambda$ satisfies DC and IC. We can iterate this process:

$$C_\Lambda^{-m}[\omega]^{2m}x(\pi)v_\Lambda = \ldots = C_\Lambda^{-m+2}[\omega]^{2(m-2)}x(\pi^{-4})x(\kappa_\Lambda^{-2})x(\kappa_\Lambda)v_\Lambda = \ldots$$

Hence for different choices of integers $m, m-1, m-2, \ldots$ we always get the same basis vector (36), only written in a different way. □

Remark 9.4.

One may think of Theorem 9.3 as a vertex operator construction for an arbitrary standard $\hat{\mathfrak{g}}$-module $L(\Lambda)$. And while the basis constructed by using an inner automorphism and three commutative currents is relatively simple, the action of $\hat{\mathfrak{g}}$ is given by a complicated implicit use of the vertex operator formula (30).

In the level $k=1$ case, linear independence in this theorem is proved in [27] for the basic representation $L(\Lambda_0)$ by writing basis elements as semi-infinite monomials and then "counting" them by using crystal base character formula [20]. Such semi-infinite monomials interpretation is possible for all standard $B_2^{(1)}$-modules, like in [29] for $A_1^{(1)}$: for fixed Λ and $m \in \mathbb{Z}$ set

$$v_{-m} = C_\Lambda^m[\omega]^{-2m}v_\Lambda.$$

From Lemma 5.8 we see that the $\mathfrak{h}$-weight of v_{-m} is $\Lambda|\mathfrak{h} - 2mk\varepsilon_1$ and the degree of v_{-m} is $2m\Lambda(\varepsilon_1) - 2m^2k$. By using Lemma 5.5 and (35), as in (36) we get

$$v_{-m} = x\left(\kappa_\Lambda^{+2(m+1)}\right)v_{-m-1} = x\left(\kappa_\Lambda^{+2(m+1)}\right)x\left(\kappa_\Lambda^{+2(m+2)}\right)v_{-m-2} = \ldots \tag{37}$$

So by "taking a limit" we see that the vector v_{-m} can be represented by a semi-infinite quasi-periodic monomial

$$v_{-m} \sim x\left(\kappa_\Lambda^{+2(m+1)}\right)x\left(\kappa_\Lambda^{+2(m+2)}\right)\cdots = \prod_{p=1}^{\infty} x\left(\kappa_\Lambda^{+2(m+p)}\right),$$

or written in more detail,

$$\begin{aligned} v_{-m} \sim\ & x_{\underline{2}}(2m)^{k_1}x_0(2m)^{k_2}x_2(2m)^{k_1}x_{\underline{2}}(2m+1)^{k_0}x_0(2m+1)^{k_2}x_2(2m+1)^{k_0} \\ & x_{\underline{2}}(2m+2)^{k_1}x_0(2m+2)^{k_2}x_2(2m+2)^{k_1}x_{\underline{2}}(2m+3)^{k_0}x_0(2m+3)^{k_2}x_2(2m+3)^{k_0}\cdots \end{aligned}$$

Now we can write basis elements of $L(\Lambda)_\mu$ given by Theorem 9.3 as

$$C_\Lambda^m[\omega]^{-2m}x(\pi)v_\Lambda = x(\pi^{+2m})C_\Lambda^m[\omega]^{-2m}v_\Lambda = x(\pi^{+2m})v_{-m}. \tag{38}$$

Then (37) implies

$$x(\pi^{+2m})v_{-m} = x(\pi^{+2m})x\left(\kappa_\Lambda^{+2(m+1)}\right)v_{-m-1} = x(\pi^{+2m})x\left(\kappa_\Lambda^{+2(m+1)}\right)x\left(\kappa_\Lambda^{+2(m+2)}\right)v_{-m-2} = \ldots$$

and we see that our basis vector (38) can be represented by a semi-infinite monomial with quasi-periodic tail

$$x(\pi^{+2m})v_{-m} \sim x(\pi^{+2m})x\left(\kappa_\Lambda^{+2(m+1)}\right)x\left(\kappa_\Lambda^{+2(m+2)}\right)x\left(\kappa_\Lambda^{+2(m+3)}\right)\cdots$$

Hence we have

Corollary 9.5.
We can parametrize a basis of level k standard $B_2^{(1)}$-module $L(\Lambda)$,

$$\Lambda = \Lambda_0 k_0 + \Lambda_1 k_1 + \Lambda_2 k_2, \qquad k = k_0 + k_1 + k_2,$$

by semi-infinite monomials

$$\prod_{j\in\mathbb{Z}} x_{\underline{2}}(-j)^{c_j} x_0(-j)^{b_j} x_2(-j)^{a_j}, \qquad c_j = b_j = a_j = 0 \quad \text{for} \quad -j \ll 0,$$

with quasi-periodic tail with the period of length 6*,*

$$(\ldots, c_{-2n}, b_{-2n}, a_{-2n}, c_{-2n-1}, b_{-2n-1}, a_{-2n-1}, \ldots) = (\ldots, k_1, k_2, k_1, k_0, k_2, k_0, \ldots),$$

$n \gg 0$, satisfying for all $j \in \mathbb{Z}$ difference conditions

$$c_{j+1} + b_{j+1} + c_j \le k, \qquad b_{j+1} + a_{j+1} + c_j \le k, \qquad a_{j+1} + c_j + b_j \le k, \qquad a_{j+1} + b_j + a_j \le k.$$

Note that for semi-infinite monomials the initial conditions follow from the form of quasi-periodic tail and the difference conditions.

10. Presentation of $W(\Lambda)$

Theorem 10.1.
Let $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$ and $k = k_0 + k_1 + k_2$. Let

$$\mathcal{P} = \mathbb{C}[x_{\underline{2}}(j), x_0(j), x_2(j) : j \le -1]$$

and let $\mathcal{I}_\Lambda$ be the ideal in the polynomial algebra $\mathcal{P}$ generated by the set of polynomials

$$\bigcup_{n\le -k-1} U(\mathfrak{g}_0)\cdot\left(\sum_{\substack{j_1,\ldots,j_{k+1}\le -1\\ j_1+\cdots+j_{k+1}=n}} x_2(j_1)\cdots x_2(j_{k+1})\right) \cup \{x_2(-1)^{k_0+1}\} \cup U(\mathfrak{g}_0)\cdot x_2(-1)^{k_0+k_2+1},$$

where $\cdot$ denotes the adjoint action of $\mathfrak{g}_0$ on $\mathcal{P}$. Then, as vector spaces,

$$W(\Lambda) \cong \mathcal{P}/\mathcal{I}_\Lambda.$$

Proof. Since $\mathcal{P} \subset S(\widehat{\mathfrak{g}}_1) = U(\widehat{\mathfrak{g}}_1)$, we have a linear map

$$f\colon \mathcal{P} \to W(\Lambda), \qquad f\colon x(\pi) \mapsto x(\pi)v_\Lambda.$$

Since $x(j)v_\Lambda = 0$ for $x \in \mathfrak{g}_1$ and $j \ge 0$, relations $U(\mathfrak{g}_0)\cdot x_\theta(z)^{k+1} = 0$ on $L(\Lambda)$ imply

$$\bigcup_{n\le -k-1} U(\mathfrak{g}_0)\cdot\left(\sum_{\substack{j_1,\ldots,j_{k+1}\le -1\\ j_1+\cdots+j_{k+1}=n}} x_2(j_1)\cdots x_2(j_{k+1})\right) \subset \ker f.$$

From the proof of Lemma 4.4 we see that

$$\{x_2(-1)^{k_0+1}\} \cup U(\mathfrak{g}_0)\cdot x_2(-1)^{k_0+k_2+1} \subset \ker f.$$

Hence we have a surjective linear map

$$g\colon \mathcal{P}/\mathcal{I}_\Lambda \to W(\Lambda).$$

On the quotient $\mathcal{P}/\mathcal{I}_\Lambda$ we have relations

$$U(\mathfrak{g}_0)\cdot\left(\sum_{\substack{j_1,\dots,j_{k+1}\leq -1\\ j_1+\cdots+j_{k+1}=n}} x_2(j_1)\cdots x_2(j_{k+1})\right)=0 \qquad \text{for all} \quad n\leq -k-1,$$
$$x_2(-1)^{k_0+1}=0 \qquad \text{and} \qquad U(\mathfrak{g}_0)\cdot x_2(-1)^{k_0+k_2+1}=0.$$

As in the proof of Lemma 4.4 we see that monomials $x(\pi)\in\mathcal{P}$ satisfying DC and IC span the quotient $\mathcal{P}/\mathcal{I}_\Lambda$. Since g maps this spanning set to a basis of $W(\Lambda)$, monomials $x(\pi)\in\mathcal{P}$ satisfying DC and IC are a basis of $\mathcal{P}/\mathcal{I}_\Lambda$ and g is an isomorphism. □

11. A connection with monomial bases of standard $A_1^{(1)}$-modules

Let now $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{C})$ with the standard basis e,h,f. Then we have monomial bases of standard $\hat{\mathfrak{g}}$-modules constructed in [11, 24, 25]:

For integral dominant $\Lambda=k_0\Lambda_0+k_1\Lambda_1$ of level $k=k_0+k_1$ the set of finite monomial vectors

$$x(\pi)v_\Lambda=\cdots f(-j)^{c_j}h(-j)^{b_j}e(-j)^{a_j}\cdots f(-1)^{c_1}h(-1)^{b_1}e(-1)^{a_1}f(0)^{c_0}v_\Lambda$$

satisfying difference conditions

$$c_{j+1}+b_{j+1}+c_j\leq k,\qquad b_{j+1}+a_{j+1}+c_j\leq k,\qquad a_{j+1}+c_j+b_j\leq k,\qquad a_{j+1}+b_j+a_j\leq k$$

for all $j\geq 0$, and initial conditions $a_1\leq k_0$ and $c_0\leq k_1$, is a basis of standard $\hat{\mathfrak{g}}$-module $L(\Lambda)$.

These difference and initial conditions for $A_1^{(1)}$-module $L(k\Lambda_0)$ coincide with difference conditions (6) and initial conditions (7) for $B_2^{(1)}$ subspace $W(k\Lambda_0)$. Moreover, the result of E. Feigin [12, Theorem 3.1] implies that $W(k\Lambda_0)$ for $B_2^{(1)}$ and $L(k\Lambda_0)$ for $A_1^{(1)}$ have the same presentation:

Let k be a positive integer. Let

$$\mathcal{P}=\mathbb{C}[f(j),h(j),e(j):j\leq -1]$$

and let $\mathcal{I}_{k\Lambda_0}$ be the ideal in the polynomial algebra $\mathcal{P}$ generated by polynomials

$$\bigcup_{n\leq -k-1} U(\mathfrak{g})\cdot\left(\sum_{\substack{j_1,\dots,j_{k+1}\leq -1\\ j_1+\cdots+j_{k+1}=n}} e(j_1)\cdots e(j_{k+1})\right)$$

(here $\cdot$ denotes the adjoint action of $\mathfrak{g}$ on $\mathcal{P}$). Then, as $\mathbb{Z}$-graded vector spaces and $\mathfrak{g}$-modules,

$$L(k\Lambda_0)\cong\mathcal{P}/\mathcal{I}_{k\Lambda_0}.$$

Due to this coincidence, E. Feigin's fermionic formula [12, Theorem 3.2] for $A_1^{(1)}$-module $L(k\Lambda_0)$ is also a character formula of Feigin–Stoyanovsky type subspace $W(k\Lambda_0)$ for $B_2^{(1)}$.

Acknowledgements

Research is partially supported by the Ministry of Science and Technology of the Republic of Croatia, grant 037-0372794-2806.

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