PRIMITIVE BLOCK DESIGNS WITH AUTOMORPHISM GROUP PSL(2,q)

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This is a joint research with J. Mandić and S. Braić.

Our aim and results so far

In the current research we are concerned with construction and description of a specific class of **primitive t-designs**.

We aim to determine, up to isomorphism and complementation, all nontrivial primitive t-designs with $PSL(2, q)$ as an automorphism group. The number of such designs for each q we denote by $npd(q)$.

The obtained designs we describe by their base block and the full automorphism group.

Our results so far include completely solving the problem in case when a block stabiliser is not in the fifth Aschbacher's class, in particular, when q is a prime. For prime q we have determined the $npd(q)$.

Basic notions and facts

A $t-(v,k,\lambda)$ **design** is a pair $D=(\Omega,\mathcal{B})$, where Ω is a set of v points, $\mathcal{B}$ a set of k-element subsets of Ω called blocks, such that any t points are contained in exactly λ blocks, for some $t \leq k$ and $\lambda > 0$. If $t=2$ a $t-(v,k,\lambda)$ **design** is called (v,k,λ)-**block design**.

An isomorphism of t-designs $D=(\Omega,\mathcal{B})$ and $D'=(\Omega,\mathcal{B}')$ is a permutation of Ω which sends blocks of D to blocks of D'. An isomorphism from D to itself is called automorphism. By $AutD$ we denote the group of all automorphisms of D.

A $t-$ design D we call **primitive** if $AutD$ acts primitively on both point and block set.

Basic notions and facts

Primitive action is a transitive action with certain additional property.

Proposition

Let G be a permutation group acting transitively on a set Ω with at least two points. G is primitive if and only if each point stabiliser $G_\omega, \omega \in \Omega$, is a maximal subgroup of G.

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Proposition

Overgroup of a primitive group is primitive.

Construction method

Theorem

*Let G be a permutation group on the finite set Ω, let $B \subset \Omega$ be a k-subset with at least two elements and let $G_B \leq G$ be a setwise stabiliser of B. If G is t-**homogeneous** and $k \geq t$, then $D = (\Omega, B^G = \{B^\gamma \mid \gamma \in G\})$ is a t-**design** with $b = \left|B^G\right| = |G| \,/\, |G_B|$ blocks and*

$$\lambda = b\binom{k}{t} \Big/ \binom{v}{t} = |G|\binom{k}{t} \Big/ |G_B|\binom{v}{t}.$$

The set B is called a **base block for** D.

As the set of points we take the **projective line** $\Omega = PG(1,q) = \{\infty\} \cup GF(q)$ of cardinality $q+1; q = p^f$.
We consider the action of **projective groups** G,

$$PSL(2,q) \leq G \leq P\Gamma L(2,q), \qquad (1)$$

on Ω. Our theoretical considerations we restrict to $\boxed{q \geq 13, \quad q \neq 23.}$
Henceforth we denote $PSL(2,q)$ by T; $AutT = P\Gamma L(2,q)$. T is the group of all fractional linear transformations

$$t_{a,b,c,d} : z \mapsto \frac{az+b}{cz+d}$$

of the projective line, $a,b,c,d \in GF(q)$, where $ad - bc$ is a square.
Let ξ be a primitive element of $GF(q)$ and $\delta = t_{\xi,0,0,1}$. Let ϕ be the map on Ω, $\phi : z \mapsto z^p$. Then we have
$PGL(2,q) = \langle PSL(2,q), \delta \rangle = \langle T, \delta \rangle$,
$P\Gamma L(2,q) = \langle PSL(2,q), \delta, \phi \rangle = \langle T, \delta, \phi \rangle$, and we define
$P\Sigma L(2,q) = \langle PSL(2,q), \phi \rangle = \langle T, \phi \rangle$.

Construction method

It is well-known that T-action on projective line is

$$\begin{cases} \text{2-homogeneous if } q \equiv 1 \pmod 4, \\ \text{3-homogeneous if } q \equiv 3 \pmod 4 \text{ or } q \text{ is even;} \end{cases}$$

$PGL(2, q)$ acts 3-homogeneously for all q. Therefore, according to the previous theorem, we obtain at least 2-designs by the following **three-step construction**:

- Select a group G from (1);

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- Select a group G from (1);
- Select a subset $B \subset \Omega$ as a base block;
- Obtain the set of all blocks by G-action on the base block, i. e. $\mathcal{B} = B^G$.

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- We denote it by $D(G, \Omega, B)$.
- Base block B in this case is a union of orbits of some maximal subgroup of G.

Minimal group for performing construction

It is of interest to perform design construction with minimal possible among the projective groups (1), and still obtain all desired primitive designs.
Eventually, the majority of our constructions boils down to construction with automorphism group T, i.e. to taking maximal subgroups of T as setwise stabilisers of a base block B. Namely, the following theorem on maximal subgroups of the projective groups holds [1][1].

[1] M. Giudici, *Maximal subgroups of almost simple groups with socle PSL(2,q)*, arXiv:math/0703685v1 [math.GR], 2007.

Minimal group for performing construction

Theorem (Giudici)

Let $T = PSL(2,q) \leq G \leq P\Gamma L(2,q)$ and let M be a maximal subgroup of G which does not contain T. Then either $M \cap T$ is maximal in T, or G and M are given in the Table (2).

G	M (novelty)
$PGL(2,7)$	$N_G(D_6) = D_{12}$
$PGL(2,7)$	$N_G(D_8) = D_{16}$
$PGL(2,9)$	$N_G(D_{10}) = D_{20}$
$PGL(2,9)$	$N_G(D_8) = D_{16}$
M_{10}	$N_G(D_{10}) = C_5 \rtimes C_4$
M_{10}	$N_G(D_8) = C_8 \rtimes C_2$
$P\Gamma L(2,9)$	$N_G(D_{10}) = C_{10} \rtimes C_4$
$P\Gamma L(2,9)$	$N_G(D_8)$
$PGL(2,11)$	$N_G(D_{10}) = D_{20}$
$PGL(2,q), q = p \equiv \pm 11, 19 \pmod{40}$	$N_G(A_4) = S_4$

(2)

Only the last row of the Table (2) is in our range of theoretical considerations, i.e. to $q \geq 13$.
From the last theorem and table (2) we easily see the following.

Corollary

Let $PGL(2, q) \leq G \leq P\Gamma L(2, q)$ and suppose that M is a maximal subgroup of G. Then $M \cap PGL(2, q)$ is maximal in $PGL(2, q)$.

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This means that for construction of all primitive designs with projective automorphism groups, besides taking maximal subgroups of T as setwise base block stabilisers, we only need to consider maximal subgroups of $PGL(2, q)$ as base block stabilisers.

In table (3), second column, we list all groups which have to be considered as base block stabilisers. They are denoted by H. Nine isomorphism types of maximal subgroups of T occupy the first nine rows. $d = \gcd(2, q-1)$.

Type	H (block stabiliser)	Asch.	G^{MIN}	G^{MAX}
1	$C_p^f \rtimes C_{\frac{q-1}{d}}$ (the stabiliser of a point)	C_1	$PSL(2,q)$	$P\Gamma L(2,q)$
2	$D_{\frac{2(q-1)}{d}}$ (the setwise stabiliser of two points)	C_2	$PSL(2,q)$	$P\Gamma L(2,q)$
3	$D_{\frac{2(q+1)}{d}}$	C_3	$PSL(2,q)$	$P\Gamma L(2,q)$
4	$PGL(2,q_0), q = q_0^2$ (2 conj. cl. if q odd)	C_5	$PSL(2,q)$	$P\Sigma L(2,q)$
5	$PSL(2,q_0), q = q_0^r, q_0 \neq 2, r$ odd prime	C_5	$PSL(2,q)$	$P\Gamma L(2,q)$
6	$A_5, q = p^2$, p odd (2 conjugacy classes)	C_9	$PSL(2,q)$	$P\Sigma L(2,q)$
7	$A_5, q = p$ (2 conjugacy classes)	C_9	$PSL(2,q)$	$PSL(2,q)$
8	$A_4, q = p$	C_6	$PSL(2,q)$	$PGL(2,q)$
9	$S_4, q = p$ (2 conjugacy classes)	C_6	$PSL(2,q)$	$PSL(2,q)$
10	$S_4, q = p$ (novelty)	C_6	$PGL(2,q)$	$PGL(2,q)$

(3)

Minimal group for performing construction

The tenth row relates to H not maximal in T (the case when socle T does not act primitively on blocks); it is the novelty from the last row of table (2). G^{MIN} denotes minimal group having H as maximal subgroup. Group G^{MAX} will be explained in the next section. Note that in the tenth row we have $G^{MIN} = G^{MAX} = PGL(2, q)$.

For a given isomorphism type of maximal subgroup of T there exist at most two conjugacy classes of subgroups (which fuse to one conjugacy class in $PGL(2, q)$).

Minimal group for performing construction

To accomplish the construction of specified designs it suffices to do the following.

- Compose base blocks as all possible unions of H-orbits, H of type 2 through 10. [Namely, taking orbits of H-type 1 would lead to a symmetric design of rank 2, and by Kantor's theorem such a design on projective line with automorphism group $PSL(2, q)$ does not exist.]

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- Generate the block set $\mathcal{B}$ from each base block B by the action of

$$\begin{cases} T \text{ on } B, & \text{for } H \text{ of type 2 through 9 } (q \geq 13); \\ PGL(2,q) \text{ on } B, & \text{for } H \text{ of type 10.} \end{cases}$$

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From here onwards (if not stated otherwise) by H we denote a maximal subgroup of T.

Preliminary analysis of designs

Regarding $AutD$ for any D, and possible isomorphism of any two designs $D(G,\Omega,B)$ and $D'(G,\Omega,B')$, the following holds:

1. $AutD \leq P\Gamma L(2,q)$;
2. if $\pi : D \longrightarrow D'$ is an isomorphism of designs, then $\pi \in P\Gamma L(2,q)$.

These assertions are interrelated. Regarding the first, using [2][2] one can check that it holds for $q \geq 13$, $q \neq 23$. [For these values of q the only subgroups of S_{q+1} containing $PSL(2,q)$ are S_{q+1}, A_{q+1}, and subgroups of $P\Gamma L(2,q)$ containing $PSL(2,q)$.]

For $q = 23$, $PSL(2,23) < M_{24}$ holds. However, computer search shows that a primitive design with socle $PSL(2,23)$ does not exist.

The second assertion can be proved using the first.

[2] M.W. Liebeck, C.E. Praeger and J. Saxl, *A Classification of the Maximal Subgroups of the Finite Alternating and Symmetric Groups*, Journal of Algebra **111** (1987), 365-383.

Preliminary analysis of designs

Isomorphisms from $P\Gamma L(2,q)$ preserve T as the group which generates the block set.

$$\left(B^T\right)^{\pi} = B^{T\pi} = B^{\pi\pi^{-1}T\pi} = \left(B^{\pi}\right)^{T}, \qquad \pi \in P\Gamma L(2,q) \tag{4}$$

For a given design $D(T,\Omega,B)$ let's call D-**stabiliser** the set of block stabilisers for all blocks of D :

$$stb(D) = \{\, T_{B'} \,|\, B' \in B^T = \mathcal{B}\}.$$

It is the set of all subgroups conjugated to some block stabiliser $H = T_B$, i.e.

$$stb(D) = T_B^T = H^T = \{\, H^g \,|\, g \in T\}.$$

For an isomorphism $\pi : D \longrightarrow D'$ it follows that $(stb(D))^{\pi} = stb(D')$. We now focus on isomorphisms $\pi \in P\Gamma L(2,q)$ which preserve $stb(D)$ and define

$$G^{MAX} = \{\, \pi \in P\Gamma L(2,q) |\, \left(H^T\right)^{\pi} = H^T\}.$$

Preliminary analysis of designs

Obviously,

$$G^{MAX} = \begin{cases} P\Gamma L(2,q), \text{ if there exists one conj. cl. of } H \text{ in } T; \\ P\Sigma L(2,q), \text{ if } q \text{ is odd and there exist two conj. cl. of } H \text{ in } T. \end{cases}$$

It proves that

$$G^{MAX} = T \cdot N_{G^{MAX}}(H).$$

$N_{G^{MAX}}(H)$ acts on the set of H-orbits and, accordingly, on the block set of the design.

Designs $D_1 = D(T, \Omega, B_1)$ and $D_2 = D(T, \Omega, B_2)$, with $T_{B_1} = T_{B_2} = H$, are isomorphic if and only if there exists $\pi \in N_{G^{MAX}}(H)$ so that $B_1^{\pi} = B_2$.

Description of designs obtained for different H-types

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Description of designs obtained for different H-types

- For H-types 2, 3 and 4 we explicitly found H-orbits.
- For H-types 6 through 10 we found cyclic structure of groups H and $N_{G^{MAX}}(H)$, which enables us to determine full automorphism groups and the number of designs up to isomorphism.
- For H-type 5 we only managed to partly solve the problem by finding cyclic structure for H and not for $N_{G^{MAX}}(H) = N_{P\Gamma L(2,q)}(H)$.

- ***H*-type 2** (a dihedral group)

Proposition

Let $q \geq 13$ and let there exist a block design D, the socle of $AutD$ being $PSL(2,q)$. If the base block stabiliser is in the second Aschbacher's class, then $q \equiv 1 (\mathrm{mod}\, 4)$, D is a 2-$(q+1, \frac{q-1}{2}, \frac{(q-1)(q-3)}{8})$ design up to complementation, and $AutD = P\Sigma L(2,q)$.

$H = T_{\{0,\infty\}} = \left\{ x \longmapsto ax : a \in F_q^{(2)} \right\} \rtimes \left\langle x \longmapsto \frac{-1}{x} \right\rangle$. $G^{MAX} = P\Gamma L_2(q)$.

Let's denote $K = N_{G^{MAX}}(H) = P\Gamma L_2(q)_{\{0,\infty\}}$.

The orbits of subgroup $\left\{ x \longmapsto ax \middle| a \in F_q^{(2)} \right\}$ are $\{\infty\}$,$\{0\}$, $F_q^{(2)}$ and $F_q^* \diagdown F_q^{(2)}$. Consequently,

$$H\text{-orbits are} \begin{cases} \{0,\infty\} \text{ and } F_q^*, & \text{for } q \text{ even or } q \equiv 3 \pmod 4; \\ \{0,\infty\},\ F_q^{(2)} \text{ and } F_q^* \diagdown F_q^{(2)}, & \text{for } q \equiv 1 \pmod 4. \end{cases}$$

Obviously, nontrivial 2-designs exist for $q \equiv 1 \pmod 4$, $q \geq 13$. Up to complementation it remains to consider base blocks consisting of one orbit each, that being $B_1 = F_q^{(2)}$ and $B_2 = F_q^* \diagdown F_q^{(2)}$.

The mapping $x \longmapsto \xi x, \xi \in F_q^* \diagdown F_q^{(2)}$ maps the orbit $F_q^{(2)}$ into $F_q^* \diagdown F_q^{(2)}$, so up to isomorphism there exists

a unique $2 - \left(q+1, \frac{q-1}{2}, \frac{(q-1)(q-3)}{8} \right)$ design

$D = D\left(G, \Omega, F_q^{(2)} \right)$, $q \equiv 1 \pmod 4$, $q \geq 13$, for all G in formula (1).

$AutD = P\Sigma L(2,q)$. D is flag-transitive.

Simple examples of design description

- ***H*-type 8**
  $H \cong A_4$;
  $q = p \equiv 13, 37, 43, 53, 67, 77, 83, 107\,(120)$ (Giudici);
  $G^{MAX} =$ (one conjugacy class of H in T)
  $= P\Gamma L(2,q) = (q = p) = PGL(2,q)$.

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- all nonidentity elements of A_4 and S_4 are fixed-point-free or have two fixed points;
- $(\forall \omega \in \Omega)$ H_ω and K_ω are cyclic, $|H_\omega| \mid \frac{q-1}{2}$ and $|K_\omega| \mid (q-1)$;
- analysis of transitive action of A_4 and S_4 on m points: possible orbit lengths are

$$\begin{cases} A_4 : m \in \{4, 6, 12\}; \\ S_4 : m \in \{6, 8, 12, 24\}. \end{cases}$$

Now suppose that cyclic structure of A_4 (S_4) is $4^a 6^b 12^c$ ($6^a 8^b 12^c 24^d$). Solving equation

$$\begin{array}{ll} 4a+6b+12c=q+1, & a,b,c\in\mathbb{N}_0 \\ (6a+8b+12c+24d=q+1, & a,b,c,d\in\mathbb{N}_0) \end{array} \quad (5)$$

finally gives the desired cyclic structure as given in the second (third) column of table (6). The results for A_4 influence solving equation for S_4. The last column is obtained from cyclic structures by combinatorial enumeration, taking into account all possibilities for design construction up to isomorphism and complementation.

$q=p\equiv$	$H\cong A_4$	$K\cong S_4$	partial $npd\,(q)$
$53,77\,(120)$	$6^1 12^{\frac{q-5}{12}}$	$6^1 24^{\frac{q-5}{24}}$	$2^{\frac{q-17}{12}}+2^{\frac{q-29}{24}}-1$
$83,107\,(120)$	$12^{\frac{q+1}{12}}$	$12^1 24^{\frac{q-11}{24}}$	$2^{\frac{q-23}{12}}+2^{\frac{q-35}{24}}-1$
$13,37\,(120)$	$4^2 6^1 12^{\frac{q-13}{12}}$	$6^1 8^1 24^{\frac{q-13}{24}}$	$2^{\frac{q-1}{12}}+2^{\frac{q-13}{24}}-1$
$43,67\,(120)$	$4^2 12^{\frac{q-7}{12}}$	$8^1 12^1 24^{\frac{q-19}{24}}$	$2^{\frac{q-7}{12}}+2^{\frac{q-19}{24}}-1$

(6)

Simple examples of design description

Now, because we have found the full automorphism groups for all obtained designs from cyclic structures, the result may be described in the following way.

1. If $q \equiv 3 \pmod 4$, we obtain the series of $3-\left(q+1, k, \frac{k(k-1)(k-2)}{24}\right)$ designs D with $PSL(2,q) \leq AutD \leq PGL(2,q)$.

2. If $q \equiv 1 \pmod 4$, we obtain

$$\begin{cases} \text{the series of } 2-\left(q+1, k, \frac{(q-1)k(k-1)}{24}\right) \text{ designs } D,\ AutD = PSL(2,q) \\ \text{or} \\ \text{the series of } 3-\left(q+1, k, \frac{k(k-1)(k-2)}{24}\right) \text{ designs } D,\ AutD = PGL(2,q). \end{cases}$$

Survey of results

Our results so far are incomplete only if a block stabiliser H is of type 5. In particular, they are complete for prime values of q. For $q = p$ we counted the number of nontrivial primitive t-designs, up to isomorphism and complementation, by summing up the results for all H-types. The proof of the following proposition is pure combinatorics.

Proposition

If q is prime, $q \geq 7$, then the following formulae hold:

1. $q \equiv 1\,(120) \Rightarrow npd\,(q) = 2^{\frac{q+59}{60}} + 2^{\frac{q+23}{24}}$
2. $q \equiv 7, 103\,(120) \Rightarrow npd\,(q) = 2^{\frac{q-7}{24}} - 1$
3. $q \equiv 11\,(120) \Rightarrow npd\,(q) = 2^{\frac{q-11}{60}} + 2^{\frac{q-11}{24}} - 2$
4. $q \equiv 13, 37\,(120) \Rightarrow npd\,(q) = 2^{\frac{q-1}{12}} + 2^{\frac{q-13}{24}} + 1$
5. $q \equiv 17, 113\,(120) \Rightarrow npd\,(q) = 2^{\frac{q+7}{24}} + 1$
6. $q \equiv 19\,(120) \Rightarrow npd\,(q) = 2^{\frac{q-19}{60}} + 2^{\frac{q+5}{24}} - 2$
7. $q \equiv 23, 47\,(120) \Rightarrow npd\,(q) = 2^{\frac{q-23}{24}} - 1$
8. $q \equiv 29\,(120) \Rightarrow npd\,(q) = 2^{\frac{q-29}{60}} + 2^{\frac{q-5}{24}}$
9. $q \equiv 31\,(120) \Rightarrow npd\,(q) = 2^{\frac{q+29}{60}} + 2^{\frac{q-7}{24}} - 2$
10. $q \equiv 41\,(120) \Rightarrow npd\,(q) = 2^{\frac{q+19}{60}} + 2^{\frac{q+7}{24}}$
11. $q \equiv 43, 67\,(120) \Rightarrow npd\,(q) = 2^{\frac{q-7}{12}} + 2^{\frac{q-19}{24}} - 1$
12. $q \equiv 49\,(120) \Rightarrow npd\,(q) = 2^{\frac{q+11}{60}} + 2^{\frac{q+23}{24}}$

Proposition (continuing)

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14. $q \equiv 59\,(120) \Rightarrow npd\,(q) = 2^{\frac{q-59}{60}} + 2^{\frac{q-11}{24}} - 2$
15. $q \equiv 61\,(120) \Rightarrow npd\,(q) = 2^{\frac{q+59}{60}} + 2^{\frac{q+11}{24}}$
16. $q \equiv 71\,(120) \Rightarrow npd\,(q) = 2^{\frac{q-11}{60}} + 2^{\frac{q-23}{24}} - 2$
17. $q \equiv 73, 97\,(120) \Rightarrow npd\,(q) = 2^{\frac{q+23}{24}} + 1$
18. $q \equiv 79\,(120) \Rightarrow npd\,(q) = 2^{\frac{q-19}{60}} + 2^{\frac{q-7}{24}} - 2$
19. $q \equiv 83, 107\,(120) \Rightarrow npd\,(q) = 2^{\frac{q-23}{12}} + 2^{\frac{q-35}{24}} - 1$
20. $q \equiv 89\,(120) \Rightarrow npd\,(q) = 2^{\frac{q-29}{60}} + 2^{\frac{q+7}{24}}$
21. $q \equiv 91\,(120) \Rightarrow npd\,(q) = 2^{\frac{q+29}{60}} + 2^{\frac{q+5}{24}} - 2$
22. $q \equiv 101\,(120) \Rightarrow npd\,(q) = 2^{\frac{q+19}{60}} + 2^{\frac{q-5}{24}}$
23. $q \equiv 109\,(120) \Rightarrow npd\,(q) = 2^{\frac{q+11}{60}} + 2^{\frac{q+11}{24}}$
24. $q \equiv 119\,(120) \Rightarrow npd\,(q) = 2^{\frac{q-59}{60}} + 2^{\frac{q-23}{24}} - 2$

Survey of results

The formulae given in the previous proposition are verified for $q \leq 103$ through exhaustive computer search. In fact, the designs for $q = 4, 5, 7, 8, 9, 11$, and 23 we obtained using computer.

q	4	5	7	8	9	11	13	16	17	19	23	25
$npd(q)$	**0**	**1**	**0**	**0**	**2**	**0**	**4**	**1**	**3**	**1**	**0**	**3**

q	27	29	31	32	37	41	43	47	49	53	59	61
$npd(q)$	**2**	**3**	**2**	**0**	**11**	**6**	**9**	**1**	**4**	**11**	**3**	**12**

q	64	67	71	73	79	81	83	89	97	101	103
$npd(q)$	**2**	**35**	**4**	**17**	**8**	**3**	**35**	**18**	**33**	**20**	**15**

The constructed designs and related documentation are available at the site:

http://www.pmfst.hr/~sbraic/t-designs/

Survey of results

Proposition

Let $q \geq 4$. Then $npd(q) = 0$ if and only if $q = 7, 11, 23$ or $q = 2^r$, where r is a prime.

Proof.

$\boxed{\Leftarrow}$ *Simply counting using obtained results for all H types and previous proposition.*

$\boxed{\Rightarrow}$ *Let $npd(q) = 0$.*

- If $q = p$ we simply solve the equalities $npd\,(q) = 0$ in previous proposition. The case $q = 5$ we solved by computor.

- If $q = p^f$, $f \geq 2$ then there exists a prime $r|\ f$ so that $q = q_0^r$ ($q_0 = p^{f/r}$) and it is known that 3-designs $D(T, \Omega, \{\infty\} \cup GF(q_0))$ called spherical geometries exist[a]. This kind of spherical geometry is not primitive design only in case $p = 2$ and f prime.

[a] C.J. Colbourn, J.H. Dinitz, Eds., *Handbook of combinatorial designs*, Second Edition, CRC Press, New York, 2007.

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