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# Newton's approximants and continued fraction expansion of $\frac{1+\sqrt{d}}{2}$

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**Abstract.** Let $d$ be a positive integer such that $d \equiv 1 \pmod 4$ and $d$ is not a perfect square. It is well known that the continued fraction expansion of $\frac{1+\sqrt{d}}{2}$ is periodic and symmetric, and if it has the period length $\ell \le 2$, then all Newton's approximants $R_n = \frac{p_n^2 + \frac{d-1}{4} q_n^2}{q_n(2p_n - q_n)}$ are convergents of $\frac{1+\sqrt{d}}{2}$ and then it holds $R_n = \frac{p_{2n+1}}{q_{2n+1}}$ for all $n \ge 0$. We say that $R_n$ is a good approximant if $R_n$ is a convergent of $\frac{1+\sqrt{d}}{2}$. When $\ell > 2$, then there is a good approximant in the half and at the end of the period. In this paper we prove that being a good approximant is a palindromic and a periodic property. We show that when $\ell > 2$, there are $R_n$'s, which are not good approximants. Further, we define the numbers $j = j(d, n)$ by $R_n = \frac{p_{2n+1+2j}}{q_{2n+1+2j}}$ if $R_n$ is a good approximant; and $b = b(d) = |\{n : 0 \le n \le \ell - 1 \text{ and } R_n \text{ is a good approximant}\}|$. We construct sequences which show that $|j|$ and $b$ are unbounded.
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## 1. Introduction

Let $d \equiv 1 \pmod 4$ be a positive integer which is not a perfect square. The simple continued fraction expansion of $\frac{1+\sqrt{d}}{2}$ has the form

$$\frac{1+\sqrt{d}}{2} = [\, a_0, \overline{a_1, a_2, \ldots, a_{\ell-1}, 2a_0 - 1}\,].$$

Here $\ell = \ell\big(\frac{1+\sqrt{d}}{2}\big)$ denotes the length of the shortest period in the expansion of $\frac{1+\sqrt{d}}{2}$. It is well known (see e.g. [9, §30]) that the sequence $a_1, \ldots, a_{\ell-1}$ is palindromic, i.e. $a_i = a_{\ell-i}$ for $i = 1, \ldots, \ell - 1$. This expansion can be obtained using the following algorithm: $a_0 = \big\lfloor \frac{1+\sqrt{d}}{2} \big\rfloor$, $s_0 = t_0 = 1$,

$$s_{i+1} = 2a_i t_i - s_i, \quad t_{i+1} = \frac{d - s_{i+1}^2}{4t_i}, \quad a_{i+1} = \left\lfloor \frac{s_{i+1} + \sqrt{d}}{2t_{i+1}} \right\rfloor, \quad \text{for } i \ge 0. \tag{1}$$

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The numbers $s_i$ and $t_i$ are also palindromic ([9, Satz 3.32]):

$$s_{i+1} = s_{\ell-i}, \qquad t_i = t_{\ell-i}, \tag{2}$$

for $i = 0, 1, \ldots, \ell - 1$, and when we get:

(i) $s_i = s_{i+1}$, then $\ell = 2i$,

(ii) $t_i = t_{i+1}$, then $\ell = 2i + 1$.

See [9, Satz 3.33].

Let $\frac{p_n}{q_n}$ be the $n$th convergent of $\frac{1+\sqrt{d}}{2}$. Then

$$\frac{1}{(a_{n+1}+2)q_n^2} < \left|\frac{1+\sqrt{d}}{2} - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}q_n^2}.$$

In particular,

$$\left|\frac{1+\sqrt{d}}{2} - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}. \tag{3}$$

Furthermore, if a rational number $\frac{p}{q}$ with $q \geq 1$ satisfies

$$\left|\frac{1+\sqrt{d}}{2} - \frac{p}{q}\right| < \frac{1}{2q^2}, \tag{4}$$

then $\frac{p}{q}$ equals one of the convergents of $\frac{1+\sqrt{d}}{2}$ (for the proof see e.g. [9, §13]).

Newton's iterative method for solving nonlinear equations

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

is another approximation method. Applying this method to the equation $f(x) = x^2 - x - \frac{d-1}{4} = 0$, which has a root $\frac{1+\sqrt{d}}{2}$, we obtain

$$x_{k+1} = \frac{x_k^2 + \frac{d-1}{4}}{2x_k - 1}.$$

We are interested in connections between these two approximation methods. The main question is: if we assume that $x_0$ is a convergent of $\frac{1+\sqrt{d}}{2}$, is $x_1$ also a convergent of $\frac{1+\sqrt{d}}{2}$? If $x_0 = \frac{p_n}{q_n}$, we are asking whether

$$R_n \stackrel{\text{def}}{=} \frac{p_n^2 + \frac{d-1}{4}q_n^2}{q_n(2p_n - q_n)}$$

is a convergent of $\frac{1+\sqrt{d}}{2}$.

The same question was discussed by several authors for $\sqrt{d}$ and $R'_n = \frac{1}{2}(\frac{p_n}{q_n} + \frac{dq_n}{p_n})$. It is well known (see e.g. [1, p. 468]) that

$$R'_{k\ell-1} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}, \quad \text{for } k \geq 1. \tag{5}$$

It was proved by Mikusiński [7] (see also Elezović [4], Sharma [10]) that if $\ell = 2t$, then

$$R'_{kt-1} = \frac{p_{2kt-1}}{q_{2kt-1}}, \quad \text{for } k \geq 1.$$

These results imply that if $\ell(d) \leq 2$, then all approximants $R'_n$ are convergents of $\sqrt{d}$. In 2001, Dujella [2] proved the converse of this result. Namely, if all $R'_n$ approximants are convergents of $\sqrt{d}$, then $\ell(d) \leq 2$. Thus, if $\ell(d) > 2$, we know that some of approximants $R'_n$ are convergents and some of them are not. Using a result of Komatsu [6] from 1999, Dujella showed that being a good approximant is a periodic and a palindromic property, so he defined the number $b$ as the number of good approximants in the period. Formula (5) suggests that $R'_n$ should be convergent whose index is twice as large when it is a good approximant. However, this is not always true, and Dujella defined the number $j$ as a distance from a two times larger index. Dujella also pointed out that $j$ is unbounded. In 2005, Dujella and the author [3] proved that $b$ is unbounded, too.

Moreover, Sharma [10] observed arbitrary quadratic surd $\alpha = c + \sqrt{d}$, $c, d \in \mathbb{Q}$, $d > 0$, $d$ is not a square of a rational number, whose period begins with $a_1$, and $f(x) = x^2 + Ax + B$, such that $f(\alpha) = 0$ ($A = -2c$, $B = c^2 - d$). He showed that for every such $\alpha$ and the corresponding Newton's approximant $N_n = \frac{p_n^2 - Bq_n^2}{q_n(2p_n + Aq_n)}$ it holds

$$N_{k\ell-1} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}, \quad \text{for } k \geq 1, \tag{6}$$

and when $\ell = 2t$ and the period is symmetric (except for the last term), then it holds

$$N_{kt-1} = \frac{p_{2kt-1}}{q_{2kt-1}}, \quad \text{for } k \geq 1. \tag{7}$$

In this paper, we show that analogous results hold in the case of the approximation of $\frac{1+\sqrt{d}}{2}$. We show that every approximant is good if and only if $\ell \leq 2$. We give a much easier way to prove that being a good approximant is a palindromic and a periodic property; we construct a sequence that shows that $j$ could be arbitrarily large, and we prove that for every $b$ there exists $d$ such that $b(d) = b$ and $b(d) > \ell(d)/2$.

## 2. Which convergents may appear?

Sometimes good approximants can be found in places other than the half and the end of the period.

**Example 1.** *Let $d = 324n^2 + 108n - 27$, $n \in \mathbb{N}$. Then we have $\ell = 6$ and $R_1 = \frac{p_3}{q_3}$ and $R_3 = \frac{p_7}{q_7}$. Using algorithm (1) it is straightforward to check that*

$$\frac{1+\sqrt{d}}{2} = [\,9n+1, \overline{1,\ 2n-1,\ 3,\ 2n-1,\ 1,\ 18n+1}\,].$$

*Now the direct computation shows that*

$$\begin{aligned}
R_0 &= 9n+2-\frac{8}{18n+1},\\
R_1 &= 9n+2-\frac{3}{6n+1} = \frac{p_3}{q_3},\\
R_2 &= 9n+2-\frac{6n+1}{12n^2+4n} = \frac{p_5}{q_5},\\
R_3 &= 9n+2-\frac{108n^2+36n}{216n^3+108n^2+6n-1} = \frac{p_7}{q_7},\\
R_4 &= 9n+2-\frac{1296n^4-432n^3-216n^2+60n+5}{2592n^5-432n^4-648n^3+84n^2+34n-1},\\
R_5 &= 9n+2-\frac{1296n^4+864n^3+108n^2-12n+1}{2592n^5+2160n^4+432n^3-23n^2-8n} = \frac{p_{11}}{q_{11}}.
\end{aligned}$$

**Theorem 1.** *If $R_n = \frac{p_k}{q_k}$, then $k$ is odd.*

**Proof**.

$$4\left(R_n - \frac{1+\sqrt{d}}{2}\right) = \left(\frac{2p_n-q_n}{q_n} - \sqrt{d}\right) + \left(\frac{dq_n}{2p_n-q_n} - \sqrt{d}\right)$$
$$= \left(\frac{2p_n-q_n}{q_n} - \sqrt{d}\right) - \frac{q_n\sqrt{d}}{2p_n-q_n}\left(\frac{2p_n-q_n}{q_n} - \sqrt{d}\right) = \frac{q_n}{2p_n-q_n}\left(\frac{2p_n-q_n}{q_n} - \sqrt{d}\right)^2.$$

Since $d \geq 5$, we have

$$\frac{p_n}{q_n} > \frac{1+\sqrt{d}}{2} - \frac{1}{q_n^2} > 1 - \frac{1}{q_n^2} > \frac{1}{2}$$

except if $q_n = 1$. But if $q_n = 1$, then we also have $2p_n \geq 2 > q_n$. Anyway, $2p_n > q_n$, so $R_n > \frac{1+\sqrt{d}}{2}$. Since $\frac{p_k}{q_k} > \frac{1+\sqrt{d}}{2}$ if and only if $k$ is odd, from $R_n = \frac{p_k}{q_k}$ we conclude that $k$ is odd. □

In the same way as in [2], when $R_n$ is the convergent of $\frac{1+\sqrt{d}}{2}$ we can write

$$R_n = \frac{p_{2n+1+2j}}{q_{2n+1+2j}}$$

for an integer $j = j(d,n)$. From (6) and (7) we have $j = 0$ when $\ell\left(\frac{1+\sqrt{d}}{2}\right) \leq 2$. In Example 1, $j$ always equals 0. This suggests that Newton's method converges exactly twice faster, and it gives the convergent with a double index. However, this is not always true.

**Example 2.** *Let* $d = 4n^4 + 16n^3 + 28n^2 + 28n + 13$, $n \in \mathbb{N}$. *We have* $\ell(d) = 7$, $R_0 = \frac{p_3}{q_3}$ *and* $R_5 = \frac{p_9}{q_9}$. *Using algorithm* (1) *it is straightforward to check that*

$$\frac{1+\sqrt{d}}{2} = [\,n^2+2n+2, \overline{2n+2,\ n,\ 1,\ 1,\ n,\ 2n+2,\ 2n^2+4n+3}\,].$$

*Now the direct computation shows that:*

$$R_0 = n^2+2n+2+\frac{n+1}{2n^2+4n+3} = \frac{p_3}{q_3},$$
$$R_1 = n^2+2n+2+\frac{4n^3+12n^2+12n+5}{8n^4+32n^3+52n^2+44n+16},$$
$$R_2 = n^2+2n+2+\frac{4n^5+12n^4+16n^3+13n^2+5n+1}{8n^6+32n^5+60n^4+68n^3+46n^2+18n+3},$$
$$R_3 = n^2+2n+2+\frac{4n^5+20n^4+44n^3+53n^2+35n+10}{(2n^2+4n+3)(4n^4+16n^3+28n^2+26n+11)},$$
$$R_4 = n^2+2n+2+\frac{16n^5+64n^4+116n^3+120n^2+68n+17}{4(2n^2+3n+2)(4n^4+14n^3+22n^2+19n+7)},$$
$$R_5 = n^2+2n+2+\frac{16n^7+80n^6+192n^5+284n^4+272n^3+168n^2+61n+10}{(4n^3+8n^2+8n+3)(8n^5+32n^4+60n^3+66n^2+40n+11)}$$
$$= \frac{p_9}{q_9},$$
$$R_6 = n^2+2n+2+$$
$$\frac{64n^9+448n^8+1536n^7+3344n^6+5040n^5+5424n^4+4152n^3+2176n^2+708n+109}{8(4n^4+12n^3+18n^2+14n+5)(4n^6+20n^5+48n^4+70n^3+64n^2+35n+9)}$$
$$= \frac{p_{13}}{q_{13}}.$$

In Example 2, we have shown that $j$ could be $\pm 1$ ($R_0 = \frac{p_3}{q_3}$ and $R_5 = \frac{p_9}{q_9}$). We shall prove (Theorem 3) that $|j|$ can be arbitrarily large. Let us first show some other interesting details. Let us show that being a good approximant is a periodic and a palindromic property. I.e. $j(d,n) = -j(d,\ell-n-2)$.

## 3. Good approximants are periodic and symmetric

Formula [10, (8)] says: For $n \in \mathbb{N}$ it holds

$$a_0 q_{n\ell-1} = p_{n\ell-1} - q_{n\ell-2}, \tag{8}$$
$$(a_0-1)p_{n\ell-1} = \tfrac{d-1}{4} q_{n\ell-1} - p_{n\ell-2}. \tag{9}$$

**Lemma 1.** *For* $k \in \mathbb{N}$ *and* $i \geq 0$ *it holds*

$$R_{k\ell+i-1} = \frac{R_{k\ell-1}R_{i-1} + \frac{d-1}{4}}{R_{k\ell-1} + R_{i-1} - 1}.$$

**Proof**. We have

$$\begin{aligned}\frac{p_{k\ell+i-1}}{q_{k\ell+i-1}} &= [\,a_0, a_1, \ldots, a_{k\ell-1}, a_0 - 1 + a_0, a_1, \ldots, a_{i-2}, a_{i-1}\,]\\ &= \left[\,a_0, a_1, \ldots, a_{k\ell-1}, a_0 - 1 + \frac{p_{i-1}}{q_{i-1}}\,\right] = \frac{p_{k\ell-1}\big(a_0 - 1 + \frac{p_{i-1}}{q_{i-1}}\big) + p_{k\ell-2}}{q_{k\ell-1}\big(a_0 - 1 + \frac{p_{i-1}}{q_{i-1}}\big) + q_{k\ell-2}}\\ &\overset{(9)}{\underset{(8)}{=}} \frac{p_{k\ell-1}\frac{p_{i-1}}{q_{i-1}} + \frac{d-1}{4}q_{k\ell-1}}{q_{k\ell-1}\big(\frac{p_{i-1}}{q_{i-1}} - 1\big) + p_{k\ell-1}} = \frac{p_{k\ell-1}p_{i-1} + \frac{d-1}{4}q_{k\ell-1}q_{i-1}}{q_{k\ell-1}(p_{i-1} - q_{i-1}/2) + q_{i-1}(p_{k\ell-1} - q_{k\ell-1}/2)}.\end{aligned} \tag{10}$$

Now we have

$$\begin{aligned}&\big(R_{k\ell-1}R_{i-1} + \tfrac{d-1}{4}\big) \cdot q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1})q_{i-1}(2p_{i-1} - q_{i-1})\\ &\quad= (p_{k\ell-1}^2 + \tfrac{d-1}{4}q_{k\ell-1}^2)(p_{i-1}^2 + \tfrac{d-1}{4}q_{i-1}^2) + \tfrac{d-1}{4}q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1})q_{i-1}(2p_{i-1} - q_{i-1})\\ &\quad= (p_{k\ell-1}p_{i-1} + \tfrac{d-1}{4}q_{k\ell-1}q_{i-1})^2 + \tfrac{d-1}{4}\big(q_{k\ell-1}(p_{i-1} - q_{i-1}/2) + q_{i-1}(p_{k\ell-1} - q_{k\ell-1}/2)\big)^2,\end{aligned} \tag{11}$$

$$\begin{aligned}&(R_{k\ell-1} + R_{i-1} - 1) \cdot q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1})q_{i-1}(2p_{i-1} - q_{i-1})\\ &\quad= (p_{k\ell-1}^2 + \tfrac{d-1}{4}q_{k\ell-1}^2)q_{i-1}(2p_{i-1} - q_{i-1}) + (p_{i-1}^2 + \tfrac{d-1}{4}q_{i-1}^2)q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1})\\ &\qquad - q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1})q_{i-1}(2p_{i-1} - q_{i-1})\\ &\quad= \big(q_{k\ell-1}(p_{i-1} - q_{i-1}/2) + q_{i-1}(p_{k\ell-1} - q_{k\ell-1}/2)\big)\cdot\\ &\qquad\cdot \big(2p_{k\ell-1}p_{i-1} + \tfrac{d-1}{2}q_{k\ell-1}q_{i-1} - q_{k\ell-1}(p_{i-1} - q_{i-1}/2) - q_{i-1}(p_{k\ell-1} - q_{k\ell-1}/2)\big),\end{aligned} \tag{12}$$

and the quotient of (11) and (12) is, by (10), equal to $R_{k\ell+i-1}$. □

**Lemma 2.** *For $k \in \mathbb{N}$ and $i \geq 0$ it holds*

$$R_{k\ell-i-1} = \frac{R_{k\ell-1}(1 - R_{i-1}) + \frac{d-1}{4}}{R_{k\ell-1} - R_{i-1}}.$$

**Proof**. Using $\frac{x \cdot p_n + p_{n-1}}{x \cdot q_n + q_{n-1}} = [\,a_0, a_1, \ldots, a_{n-1}, a_n, x\,]$ for $x = 0$, $n = k\ell - i$, we have

$$\begin{aligned}\frac{p_{k\ell-i-1}}{q_{k\ell-i-1}} &= \frac{0 \cdot p_{k\ell-i} + p_{k\ell-i-1}}{0 \cdot q_{k\ell-i} + q_{k\ell-i-1}} = [\,a_0, a_1, \ldots, a_{k\ell-i-1}, a_{k\ell-i}, 0\,]\\ &= [\,a_0, a_1, \ldots, a_{k\ell-i}, a_{k\ell-i-1}, 0, -a_{k\ell-i-1}\,]\\ &\;\;\vdots\\ &= [\,a_0, a_1, \ldots, a_{k\ell-i}, a_{k\ell-i-1}, \ldots, a_{k\ell-1}, a_0, 0, -a_0, -a_{k\ell-1}, \ldots, -a_{k\ell-i-1}\,]\\ &= [\,a_0, a_1, \ldots, a_{k\ell-i}, a_{k\ell-i-1}, \ldots, a_{k\ell-1}, a_0, 0, -a_0, -a_1, \ldots, -a_{i-1}\,]\\ &= \left[\,a_0, a_1, \ldots, a_{k\ell-i}, a_{k\ell-i-1}, \ldots, a_{k\ell-1}, a_0 - \frac{p_{i-1}}{q_{i-1}}\,\right]\end{aligned}$$

$$= \frac{p_{k\ell-1}\big(a_0 - \frac{p_{i-1}}{q_{i-1}}\big) + p_{k\ell-2}}{q_{k\ell-1}\big(a_0 - \frac{p_{i-1}}{q_{i-1}}\big) + q_{k\ell-2}} \underset{(8)}{\overset{(9)}{=}} \frac{p_{k\ell-1}\big(1 - \frac{p_{i-1}}{q_{i-1}}\big) + \frac{d-1}{4} q_{k\ell-1}}{p_{k\ell-1} - q_{k\ell-1}\frac{p_{i-1}}{q_{i-1}}}$$
$$= \frac{p_{k\ell-1}(q_{i-1} - p_{i-1}) + \frac{d-1}{4} q_{k\ell-1} q_{i-1}}{p_{k\ell-1} q_{i-1} - q_{k\ell-1} p_{i-1}}. \tag{13}$$

Now we have

$$\begin{aligned}
&\big(R_{k\ell-1}(1 - R_{i-1}) + \tfrac{d-1}{4}\big) \cdot q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1}) q_{i-1}(2p_{i-1} - q_{i-1}) = \\
&= \big(p_{k\ell-1}^2 + \tfrac{d-1}{4} q_{k\ell-1}^2\big)\big(q_{i-1}(2p_{i-1} - q_{i-1}) - p_{i-1}^2 - \tfrac{d-1}{4} q_{i-1}^2\big) + \\
&\quad + \tfrac{d-1}{4} q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1}) q_{i-1}(2p_{i-1} - q_{i-1}) \\
&= -\Big[\big(p_{k\ell-1}(q_{i-1} - p_{i-1}) + \tfrac{d-1}{4} q_{k\ell-1} q_{i-1}\big)^2 + \tfrac{d-1}{4}\big(p_{k\ell-1} q_{i-1} - q_{k\ell-1} p_{i-1}\big)^2\Big], \quad (14)
\end{aligned}$$

$$\begin{aligned}
&(R_{k\ell-1} - R_{i-1}) \cdot q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1}) q_{i-1}(2p_{i-1} - q_{i-1}) = \\
&= (p_{k\ell-1}^2 + \tfrac{d-1}{4} q_{k\ell-1}^2) q_{i-1}(2p_{i-1} - q_{i-1}) - (p_{i-1}^2 + \tfrac{d-1}{4} q_{i-1}^2) q_{k\ell-1}(2p_{k\ell-1} - q_{k\ell-1}) \\
&= -(p_{k\ell-1} q_{i-1} - q_{k\ell-1} p_{i-1}) \cdot \\
&\qquad \cdot \big(2p_{k\ell-1}(q_{i-1} - p_{i-1}) + \tfrac{d-1}{2} q_{k\ell-1} q_{i-1} - p_{k\ell-1} q_{i-1} + q_{k\ell-1} p_{i-1}\big), \quad (15)
\end{aligned}$$

and the quotient of (14) and (15) is, by (13), equal to $R_{k\ell-i-1}$. □

**Lemma 3.** *For arbitrary $a_0$, $a_1, \ldots, a_k$ and $\alpha$ we have*

$$[\,a_k, a_{k-1}, \ldots, a_1, a_0 + \alpha\,] = \frac{p_k + \alpha q_k}{p_{k-1} + \alpha q_{k-1}}.$$

**Proof**. Using $[\,a_k, a_{k-1}, \ldots, a_1, a_0 + \alpha\,] = \big[\,a_k, a_{k-1}, \ldots, a_1, a_0, \frac{1}{\alpha}\,\big]$ we get

$$\begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{k-1} & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\alpha} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}^{\tau}\begin{pmatrix} \frac{1}{\alpha} & 1 \\ 1 & 0 \end{pmatrix},$$

so we have

$$[\,a_k, a_{k-1}, \ldots, a_1, a_0 + \alpha\,] = \frac{\frac{1}{\alpha} p_k + q_k}{\frac{1}{\alpha} p_{k-1} + q_{k-1}} = \frac{p_k + \alpha q_k}{p_{k-1} + \alpha q_{k-1}}.$$

□

**Theorem 2.** *For $i = 0, \ldots, \lfloor \ell/2 \rfloor$ and*

$$\alpha_i = -\frac{q_{2i-1}\big(\frac{d-1}{4} q_{i-1}^2 + p_{i-1}^2\big) - p_{2i-1}(2p_{i-1} - q_{i-1}) q_{i-1}}{q_{2i}\big(\frac{d-1}{4} q_{i-1}^2 + p_{i-1}^2\big) - p_{2i}(2p_{i-1} - q_{i-1}) q_{i-1}} = -\frac{p_{2i-1} - q_{2i-1} R_{i-1}}{p_{2i} - q_{2i} R_{i-1}} \tag{16}$$

*it holds*

$$R_{k\ell+i-1} = \frac{\alpha_i p_{2k\ell+2i} + p_{2k\ell+2i-1}}{\alpha_i q_{2k\ell+2i} + q_{2k\ell+2i-1}}, \text{ for all } k \geq 0, \text{ and}$$
$$R_{k\ell-i-1} = \frac{p_{2k\ell-2i-1} - \alpha_i p_{2k\ell-2i-2}}{q_{2k\ell-2i-1} - \alpha_i q_{2k\ell-2i-2}}, \text{ for all } k \geq 1.$$

**Proof**. Let us first consider the continued fraction expansion of $\alpha_i$.

$$\alpha_i = -\left[0, \frac{p_{2i} - R_{i-1}q_{2i}}{p_{2i-1} - R_{i-1}q_{2i-1}}\right] \overset{\text{Lm. 3}}{=} -[0, a_{2i}, a_{2i-1}, \ldots, a_1, a_0 - R_{i-1}]$$
$$= [0, -a_{2i}, -a_{2i-1}, \ldots, -a_1, -a_0 + R_{i-1}].$$

If $k = 0$, we have

$$\frac{\alpha_i p_{2i} + p_{2i-1}}{\alpha_i q_{2i} + q_{2i-1}} = [a_0, a_1, \ldots, a_{2i-1}, a_{2i}, \alpha_i]$$
$$= [a_0, a_1, \ldots, a_{2i-1}, a_{2i}, 0, -a_{2i}, -a_{2i-1}, \ldots, -a_1, -a_0 + R_{i-1}] = R_{i-1},$$

and if $k > 0$, we have

$$\frac{\alpha_i p_{2k\ell+2i} + p_{2k\ell+2i-1}}{\alpha_i q_{2k\ell+2i} + q_{2k\ell+2i-1}} = [a_0, a_1, \ldots, a_{2k\ell-1}, a_0 - 1 + a_0, a_1, \ldots, a_{2i-1}, a_{2i}, \alpha_i]$$
$$= [a_0, a_1, \ldots, a_{2k\ell-1}, a_0 - 1 + R_{i-1}] = \frac{p_{2k\ell-1}(a_0 - 1 + R_{i-1}) + p_{2k\ell-2}}{q_{2k\ell-1}(a_0 - 1 + R_{i-1}) + q_{2k\ell-2}}$$
$$\underset{(8)}{\overset{(9)}{=}} \frac{p_{2k\ell-1}R_{i-1} + \frac{d-1}{4}q_{2k\ell-1}}{q_{2k\ell-1}(R_{i-1} - 1) + p_{2k\ell-1}} \overset{(6)}{=} \frac{R_{k\ell-1}R_{i-1} + \frac{d-1}{4}}{R_{i-1} - 1 + R_{k\ell-1}} \overset{\text{Lm. 1}}{=} R_{k\ell+i-1},$$

$$\frac{p_{2k\ell-2i-1} - \alpha_i p_{2k\ell-2i-2}}{q_{2k\ell-2i-1} - \alpha_i q_{2k\ell-2i-2}} = \left[a_0, a_1, \ldots, a_{2(k\ell-i)-1}, -\frac{1}{\alpha_i}\right]$$
$$= [a_0, a_1, \ldots, a_{2(k\ell-i)-1}, 0, 0, a_{2i}, a_{2i-1}, \ldots, a_1, a_0 - R_{i-1}]$$
$$= [a_0, a_1, \ldots, a_{2(k\ell-i)-1}, a_{2(k\ell-i)}, a_{2(k\ell-i)+1}, \ldots, a_{2k\ell-1}, a_0 - R_{i-1}]$$
$$= \frac{p_{2k\ell-1}(a_0 - R_{i-1}) + p_{2k\ell-2}}{q_{2k\ell-1}(a_0 - R_{i-1}) + q_{2k\ell-2}} \underset{(8)}{\overset{(9)}{=}} \frac{p_{2k\ell-1}(1 - R_{i-1}) + \frac{d-1}{4}q_{2k\ell-1}}{p_{2k\ell-1} - R_{i-1}q_{2k\ell-1}}$$
$$\overset{(6)}{=} \frac{R_{k\ell-1}(1 - R_{i-1}) + \frac{d-1}{4}}{R_{k\ell-1} - R_{i-1}} \overset{\text{Lm. 2}}{=} R_{k\ell-i-1}.$$

□

**Remark 1.** *Theorem 2 could be proved using the same ideas as in [6], but the ideas in the proof of Theorem 2 can also be used to prove [6, Tm. 1] in an easier way.*

The following Corollary reduces our problem to half-periods.

**Corollary 1.** *For $n = 0, \ldots, \lfloor \ell/2 \rfloor$ and $k \geq 0$*

$$R_{k\ell+n} = \frac{p_{2(k\ell+n)+1+2j}}{q_{2(k\ell+n)+1+2j}} \iff R_{(k+1)\ell-n-2} = \frac{p_{2((k+1)\ell-n-2)+1-2j}}{q_{2((k+1)\ell-n-2)+1-2j}},$$

*or in other words: $j(d, k\ell + n) = j(d, n) = -j(d, \ell - n - 2)$.*

**Proof**. Same as [2, Lemma 3]. □

## 4. How large can $j$ be?

**Lemma 4.**

$$R_{n+1} < R_n. \tag{17}$$

**Proof**. Using $R_n = \frac{1}{4}\big(\frac{2p_n-q_n}{q_n}+\frac{dq_n}{2p_n-q_n}+2\big)$, the statement of the lemma is equivalent to

$$(-1)^n\big(dq_nq_{n+1} - (2p_n - q_n)(2p_{n+1} - q_{n+1})\big) > 0. \tag{18}$$

If $n$ is even, then $\frac{p_n}{q_n} < \frac{\sqrt{d}+1}{2}$ and $\frac{p_{n+1}}{q_{n+1}} > \frac{\sqrt{d}+1}{2}$. Furthermore, since $\frac{p_{n+1}}{q_{n+1}} - \frac{\sqrt{d}+1}{2} < \frac{\sqrt{d}+1}{2} - \frac{p_n}{q_n}$, we have $\frac{2p_n-q_n}{q_n} + \frac{2p_{n+1}-q_{n+1}}{q_{n+1}} < 2\sqrt{d}$. Therefore

$$\frac{2p_n - q_n}{q_n} \cdot \frac{2p_{n+1} - q_{n+1}}{q_{n+1}} < \Big[\Big(\frac{2p_n - q_n}{q_n} + \frac{2p_{n+1} - q_{n+1}}{q_{n+1}}\Big)\Big/2\Big]^2 < d,$$

and (18) holds. If $n$ is odd, the proof is completely analogous. □

**Proposition 1.** *When $\ell\big(\frac{1+\sqrt{d}}{2}\big) > 2$, then for all $n \geq 0$ we have*

$$\big|j(d, n)\big| \leq \frac{\ell - 3}{2}.$$

**Proof**. Let $R_n = \frac{p_{2n+1+2j}}{q_{2n+1+2j}}$. According to Corollary 1, it suffices to consider the case $j > 0$ and $n < \ell$.

Assume first that $\ell$ is even, say $\ell = 2m$. Then $R_{m-1} = \frac{p_{\ell-1}}{q_{\ell-1}}$ and $R_{\ell-1} = \frac{p_{2\ell-1}}{q_{2\ell-1}}$. If $n < m - 1$, using (17) we have $2n + 1 + 2j \leq \ell - 2$, and $2j \leq \ell - 3$. Since $\ell$ is even, we have $j \leq \frac{\ell-4}{2}$. For $n = m - 1$ and $n = \ell - 1$ we have $j = 0$, and for $m-1 < n < \ell-1$ we have $2n+1+2j \leq 2\ell-2$ and $2j \leq 2\ell-3-2n \leq \ell-3$. Thus we have $j \leq \frac{\ell-4}{2}$ again.

Assume now that $\ell$ is odd, say $\ell = 2m + 1$. If for some $n, 0 \leq n < m$ we got $j > \frac{\ell-3}{2}$, i.e. $j \geq m$, we would have $2n + 1 + 2j \geq \ell$. By Corollary 1, we have $R_{\ell-n-2} = \frac{p_{2(\ell-n-2)+1-2j}}{q_{2(\ell-n-2)+1-2j}}$, and $2(\ell - n - 2) + 1 - 2j \leq \ell - 2$. Now from $\frac{p_1}{q_1} > \frac{p_3}{q_3} > \frac{p_5}{q_5} > \dots$ it follows that $R_n > R_{\ell-n-2}$. However, Lemma 4 implies that this is not possible, since $\ell - n - 2 \geq m$. For $m - 1 < n < \ell - 1$, the proof is completely analogous to the even case. □

Let us show now that the Proposition 1 estimate is sharp. If we want $j = j(d, n)$ to be large, the continued fraction expansion should have many small $a_i$'s following $a_n$. Let us first see for fixed $a_i$'s what property $a_0$ should satisfy, in order to get the continued fraction expansion of a number of the form $\frac{1+\sqrt{d}}{2}, d \in \mathbb{N}, d \equiv 1 \pmod 4$.

**Proposition 2.** *Let $\ell \in \mathbb{N}$ and $a_1, a_2, \dots, a_{\ell-1}$ such that $a_1 = a_{\ell-1}, a_2 = a_{\ell-2}, \dots$. Then the number $[\,a_0, \overline{a_1, a_2, \dots, a_{\ell-1}, 2a_0 - 1}\,]$ is of the form $\frac{1+\sqrt{d}}{2}, d \in \mathbb{N}, d \equiv 1$ (mod 4) if and only if*

$$2a_0 \equiv 1 - (-1)^\ell p'_{\ell-2}q'_{\ell-2} \pmod{p'_{\ell-1}}, \tag{19}$$

*where* $\frac{p'_i}{q'_i}$ *are convergents of the number* $[a_1, a_2, \ldots, a_i]$. *Then it holds:*

$$d = 1 + 4\left(a_0^2 - a_0 + \frac{(2a_0 - 1)p'_{\ell-2} + q'_{\ell-2}}{p'_{\ell-1}}\right). \tag{20}$$

**Proof**. Let $\alpha = \frac{1+\sqrt{d}}{2} = [a_0, \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0 - 1}]$. Since $a_0, a_1 \in \mathbb{N}$, we have $\alpha > 1$. Let us observe:

$$\beta = a_0 - 1 + \alpha = [\overline{2a_0 - 1, a_1, a_2, \ldots, a_2, a_1}] = [\overline{b_0, b_1, b_2, \ldots, b_{\ell-2}, b_{\ell-1}}].$$

Since $\beta$ is purely periodic, $\beta$ is reduced, so we have ($\frac{p_i}{q_i}$ are convergents of $\beta$):

$$\beta, \overline{\beta} = \frac{p_{\ell-1} - q_{\ell-2} \pm \sqrt{(p_{\ell-1} - q_{\ell-2})^2 + 4q_{\ell-1}p_{\ell-2}}}{2q_{\ell-1}}.$$

We then have

$$\beta = [\overline{b_0, b_1, b_2, \ldots, b_{\ell-2}, b_{\ell-1}}] \quad \text{and} \quad -1/\overline{\beta} = [\overline{b_{\ell-1}, b_{\ell-2}, \ldots, b_2, b_1, b_0}],$$

i.e. because the expansion is palindromic, we have

$$\beta = [b_0, \overline{b_1, b_2, \ldots, b_2, b_1, b_0}] = [b_0, \beta_1] \quad \text{and} \quad -\overline{\beta} = [0, \overline{b_1, b_2, \ldots, b_2, b_1, b_0}] = [0, \beta_1].$$

We see that $2a_0 - 1 = b_0 = \beta + \overline{\beta}$. Now we have:

$$d = (2\alpha - 1)^2 = (2\beta - 2a_0 + 1)^2 = (\beta - \overline{\beta})^2 = \frac{(p_{\ell-1} - q_{\ell-2})^2 + 4q_{\ell-1}p_{\ell-2}}{q_{\ell-1}^2}.$$

From

$$p_i = (2a_0 - 1)p'_i + q'_i, \qquad q_i = p'_i,$$

we get

$$d = \left(\frac{(2a_0 - 1)p'_{\ell-1} + q'_{\ell-1} - p'_{\ell-2}}{p'_{\ell-1}}\right)^2 + 4\frac{(2a_0 - 1)p'_{\ell-2} + q'_{\ell-2}}{p'_{\ell-1}}.$$

Because the expansion is palindromic, $p'_{\ell-2} = q'_{\ell-1}$, we have

$$d = (2a_0 - 1)^2 + 4\frac{(2a_0 - 1)p'_{\ell-2} + q'_{\ell-2}}{p'_{\ell-1}}. \tag{=20}$$

It is clear that $(2a_0 - 1)^2 \equiv 1 \pmod 4$, so $d$ will be an integer congruent to 1 mod 4, if and only if $p'_{\ell-1} \mid (2a_0 - 1)p'_{\ell-2} + q'_{\ell-2}$, i.e.

$$(2a_0 - 1)p'_{\ell-2} \equiv -q'_{\ell-2} \pmod{p'_{\ell-1}}.$$

From $p_{n-1}q_n - p_nq_{n-1} = (-1)^n$ follows $p'_{\ell-2}q'_{\ell-1} \equiv (-1)^\ell \pmod{p'_{\ell-1}}$, so we have

$$(2a_0 - 1) \equiv -(-1)^\ell q'_{\ell-2}q'_{\ell-1} \pmod{p'_{\ell-1}}.$$

and we get (19) because the expansion is palindromic, i.e. $p'_{\ell-2} = q'_{\ell-1}$. □

**Lemma 5.** *Let $F_k$ denote the $k$-th Fibonacci number, and $F_{-2}=-1, F_{-1}=1, F_0=0$. For $m\in\mathbb{N}$ or $2m\in\mathbb{N}$ when $k\mid 3$, and $d_k(m)=4\big((m\cdot F_k+1)^2+m\cdot F_{k-3}\big)+1$ it holds*

$$\frac{1+\sqrt{d_k(m)}}{2}=[\,m\cdot F_k+1,\overline{\underbrace{1,1,\dots,1,1}_{k-1\ times},\ 2m\cdot F_k+1}\,],$$

*and $\ell\Big(\frac{1+\sqrt{d_k(m)}}{2}\Big)=k$.*

**Proof**. From (19), it follows:

$$2a_0\equiv 1-(-1)^kF_{k-1}F_{k-2}\equiv 1-(-1)^kF_{k-1}(F_k-F_{k-1})\equiv 1+(-1)^kF_{k-1}^2 \pmod{F_k}.$$

Now from Cassini's identity $F_kF_{k-2}-F_{k-1}^2=(-1)^{k-1}$ we have:

$$2a_0\equiv 2 \pmod{F_k},$$

or

$$a_0=\begin{cases}1+m\cdot F_k,\ m\in\mathbb{N}, & \text{when } 3\nmid k,\\ 1+\frac{m}{2}\cdot F_k,\ m\in\mathbb{N}, & \text{when } 3\mid k.\end{cases}$$

From (20) it follows:

$$\begin{aligned} d&=4\bigg((m\cdot F_k+1)^2-m\cdot F_k-1+\frac{(2m\cdot F_k+2-1)F_{k-1}+F_{k-2}}{F_k}\bigg)+1\\ &=4\big((m\cdot F_k+1)^2+m\cdot F_k-2m\cdot F_{k-2}\big)+1=4\big((m\cdot F_k+1)^2+m\cdot F_{k-3}\big)+1,\end{aligned}$$

and when $k\equiv 0 \pmod 3$,

$$d=4\bigg(\Big(\frac{m}{2}\cdot F_k+1\Big)^2+\frac{m}{2}\cdot F_{k-3}\bigg)+1.$$

□

For arbitrary $k$, we find $m$, such that for $d_k(m)$ it holds $R_0=\frac{p_{k-2}}{q_{k-2}}$. We have $\frac{p_{k-2}}{q_{k-2}}=a_0+\frac{F_{k-2}}{F_{k-1}}$. On the other hand, using $a_0=1+m_kF_k$ we have: $R_0=\frac{a_0^2+(1+m_kF_k)^2+m_kF_{k-3}}{2a_0-1}=a_0+\frac{a_0+m_kF_{k-3}}{2a_0-1}$. So we get $R_0=\frac{p_{k-2}}{q_{k-2}}$ if and only if $\frac{F_{k-2}}{F_{k-1}}=\frac{a_0+m_kF_{k-3}}{2a_0-1}$, or

$$m_k=\frac{F_{k-1}-F_{k-2}}{2F_kF_{k-2}-F_{k-1}F_k-F_{k-1}F_{k-3}}=\frac{F_{k-3}}{F_kF_{k-4}-F_{k-1}F_{k-3}}.$$

It remains to see when $m_k$ is an integer, or half of on integer if $3\mid k$. Using Binet's formula $F_n=\frac{(1+\sqrt5)^n-(1-\sqrt5)^n}{2^n\sqrt5}$, we have

$$m_k=\frac{F_{k-3}}{2\cdot(-1)^{k-1}}.$$

So if $k$ is odd, $m_k$ is greater than 0, and when $3\nmid k$, then $F_{k-3}$ has to be even, but this is not possible. So, $k$ has to be of the form $6n+3, n\in\mathbb{N}$. Thus, we just proved:

**Theorem 3.** *Let $k = 6n+3, n \in \mathbb{N}$. For $d_k = (F_k F_{k-3}+2)^2 + 2F_{k-3}^2 + 1$ it holds $R_0 = \frac{p_{k-2}}{q_{k-2}}$.*

**Remark 2.** *From Corollary 1 and (6) we get $R_{k-2} = \frac{p_k}{q_k}$ and $R_{k-1} = \frac{p_{2k-1}}{q_{2k-1}}$, respectively. So these are the only three good approximants.*

**Corollary 2.**

$$\sup\left\{|j(d,n)|\right\} = +\infty,$$

$$\limsup\left\{\frac{|j(d,n)|}{\ell\left(\frac{1+\sqrt{d}}{2}\right)}\right\} = \frac{1}{2}.$$

There remains the question how large $j$ can be compared with $d$. Let

$$d(j) = \min\{d \mid \text{there exists } n \text{ such that } j(d,n) \geq j\}.$$

In Table 1, we list $d(j)$ values for $1 \leq j \leq 104$ such that $d(j) > d(j')$ for $j' < j$. We also give corresponding $n$ and $k$ values such that $R_n = \frac{p_k}{q_k} = \frac{p_{2n+1+2j}}{q_{2n+1+2j}}$.

## 5. Number of good approximants

**Theorem 4.** *$R_n$ is a convergent of $\frac{1+\sqrt{d}}{2}$ for all $n \geq 0$ if and only if $\ell\left(\frac{1+\sqrt{d}}{2}\right) \leq 2$.*

**Proof**. Formulas (6) and (7) imply that if $\ell \leq 2$, then all $R_n$ are convergents of $\frac{\sqrt{d}+1}{2}$.

Assume now that $R_{n-1}$ is a convergent of $\frac{\sqrt{d}+1}{2}$ for all $n \in \mathbb{N}$. Then by Theorems 1, 4 and (6), we must have $R_{n-1} = \frac{p_{2n-1}}{q_{2n-1}}$ for all $n \in \mathbb{N}$, so every $\alpha_n$ in Theorem 2 has to be 0. Thus, for $n = 1$, we should have $R_0 = \frac{p_1}{q_1}$, i.e. $\alpha_1 = 0$. So let $d = 1 + 4(a_0^2 - a_0 + t)$. Then $\frac{1+\sqrt{d}}{2} = [\,a_0, \overline{a_1, a_2, \ldots, a_{\ell-1}, 2a_0 - 1}\,]$. Hence, from (16) it follows:

$$\begin{aligned}
0 &= q_1\left(\frac{d-1}{4}q_0^2 + p_0^2\right) - p_1(2p_0 - q_0)q_0 = \\
&= a_1(a_0^2 - a_0 + t + a_0^2) - (1 + a_0 a_1)(2a_0 - 1) \\
&= a_1 t - 2a_0 + 1,
\end{aligned}$$

that is

$$t = \frac{2a_0 - 1}{a_1}.$$

It is well known [9, p. 107] that $\ell\left(\frac{1+\sqrt{d}}{2}\right) \leq 2$, when $d = 1 + 4(a_0^2 - a_0 + t)$ and $t \mid 2a_0 - 1$. □

Let

$$b(d) = \left|\left\{n \mid 0 \leq n \leq \ell - 1, R_n \text{ is a convergent of } \frac{1+\sqrt{d}}{2}\right\}\right|.$$

| $d(j)$ | $\ell\left(\frac{1+\sqrt{d}}{2}\right)$ | $n$ | $k$ | $j(d,n)$ | $\frac{\ln d(j)}{\ln j(d,n)}$ | $\frac{\sqrt{d(j)}}{j(d,n)}$ | $\frac{j(d,n)}{\ell\left(\frac{1+\sqrt{d}}{2}\right)}$ |
|---|---|---|---|---|---|---|---|
| 57 | 6 | 0 | 3 | 1 | | 7.549834 | 0.166667 |
| 193 | 15 | 2 | 9 | 2 | 7.592457 | 6.946222 | 0.133333 |
| 721 | 36 | 6 | 19 | 3 | 5.989956 | 8.950481 | 0.0833333 |
| 1121 | 28 | 6 | 21 | 4 | 5.0652853 | 8.370335 | 0.142857 |
| 2521 | 85 | 22 | 55 | 5 | 4.866551 | 10.0419122 | 0.0588235 |
| 2641 | 82 | 23 | 59 | 6 | 4.397305 | 8.56511 | 0.0731707 |
| 4201 | 105 | 32 | 79 | 7 | 4.287494 | 9.259303 | 0.0666667 |
| 5401 | 120 | 16 | 49 | 8 | 4.133004 | 9.186437 | 0.0666667 |
| 10 369 | 161 | 63 | 109 | 9 | 4.208298 | 11.314254 | 0.0559006 |
| 12 241 | 167 | 37 | 97 | 11 | 3.925337 | 10.0580957 | 0.0658683 |
| 24 841 | 231 | 62 | 151 | 13 | 3.945595 | 12.123868 | 0.0562771 |
| 33 121 | 340 | 124 | 277 | 14 | 3.943803 | 12.999411 | 0.0411765 |
| 38 689 | 310 | 79 | 189 | 15 | 3.900707 | 13.113013 | 0.0483871 |
| 46 729 | 406 | 163 | 293 | 17 | 3.795027 | 12.715819 | 0.0418719 |
| 52 201 | 345 | 88 | 217 | 20 | 3.626111 | 11.423769 | 0.057971 |
| 66 721 | 413 | 123 | 295 | 24 | 3.495307 | 10.76267 | 0.0581114 |
| 121 369 | 513 | 109 | 271 | 26 | 3.593077 | 13.399252 | 0.0506823 |
| 139 921 | 559 | 158 | 373 | 28 | 3.555854 | 13.359291 | 0.0500894 |
| 203 449 | 879 | 280 | 631 | 35 | 3.437967 | 12.887235 | 0.039818 |
| 212 881 | 907 | 309 | 691 | 36 | 3.423587 | 12.816397 | 0.0396913 |
| 311 761 | 962 | 300 | 685 | 42 | 3.38446 | 13.294181 | 0.043659 |
| 430 081 | 1389 | 436 | 961 | 44 | 3.427875 | 14.904673 | 0.0316775 |
| 503 881 | 1438 | 500 | 907 | 47 | 3.410284 | 15.103101 | 0.0326843 |
| 606 481 | 1266 | 407 | 915 | 50 | 3.403719 | 15.575378 | 0.0394945 |
| 706 729 | 1815 | 539 | 1181 | 51 | 3.425483 | 16.48376 | 0.0280992 |
| 760 369 | 1802 | 559 | 1231 | 56 | 3.364069 | 15.571275 | 0.0310766 |
| 795 409 | 1180 | 346 | 807 | 57 | 3.360485 | 15.646615 | 0.0483051 |
| 990 721 | 1840 | 569 | 1267 | 64 | 3.319687 | 15.552339 | 0.0347826 |
| 1 132 609 | 2256 | 681 | 1507 | 72 | 3.259556 | 14.781126 | 0.0319149 |
| 1 157 641 | 2441 | 727 | 1603 | 74 | 3.243886 | 14.539693 | 0.0303154 |
| 1 318 249 | 2607 | 808 | 1773 | 78 | 3.234509 | 14.719875 | 0.0299194 |
| 1 700 689 | 2856 | 892 | 1951 | 83 | 3.246676 | 15.712105 | 0.0290616 |
| 1 912 681 | 2921 | 838 | 1845 | 84 | 3.264413 | 16.464251 | 0.0287573 |
| 2 058 001 | 3190 | 983 | 2155 | 94 | 3.199714 | 15.26142 | 0.0294671 |
| 2 357 569 | 3224 | 1044 | 2297 | 104 | 3.159325 | 14.763824 | 0.0322581 |

Table 1: $d(j)$ *for* $1 \leq j \leq 104$.

Theorem 4 shows that $\frac{\ell(d)}{b(d)} > 1$ when $\ell > 2$ and $\frac{\ell(d)}{b(d)} = 1$, for $\ell \leq 2$. In Example 1 we showed that for $d = 324n^2 + 108n - 27$ we have $b(d) = 4$ and $\ell(d) = 6$, and in Example 2 we showed that for $d = 4n^4 + 16n^3 + 28n^2 + 28n + 13$ we have $b(d) = 3$ and $\ell(d) = 7$.

Let

$$\ell_b = \min\Big\{\ell \mid \text{there exists } d \text{ such that } \ell\big(\tfrac{1+\sqrt{d}}{2}\big) = \ell \text{ and } b = b(d)\Big\}.$$

According to Theorem 4, we have $\ell_1 = 1$, $\ell_2 = 2$ and $\ell_b > b$ for $b > 2$. From Example 2 it follows that $\ell_3 \leq 7$. From Corollary 1, (6) and (7) it follows that $\ell_b$ and $b$ have the same parity, when $\ell_b < +\infty$. From Example 1, it follows that $\ell_4 = 6$. Let us show that $\ell_3 = 5$.

**Example 3.** *Let $d = 16n^4 + 16n^3 + 12n^2 - 4n + 1$, $n \in \mathbb{N}$. Then $\ell(d) = 5$ and $b(d) = 3$. Using algorithm* (1) *it is straightforward to check that*

$$\frac{1+\sqrt{d}}{2} = [\,2n^2 + n, \overline{1,\ 2n,\ 2n,\ 1,\ 4n^2 + 2n - 1}\,].$$

*Now the direct computation shows:*

$$\begin{aligned}
R_0 &= 2n^2 + n + 1 - \frac{2n-1}{4n^2+2n-1},\\
R_1 &= 2n^2 + n + 1 - \frac{2n}{4n^2+2n+1} = \frac{p_3}{q_3},\\
R_2 &= 2n^2 + n + 1 - \frac{8n^3+8n^2+2n-1}{16n^4+24n^3+16n^2+2n-1} = \frac{p_5}{q_5},\\
R_3 &= 2n^2 + n + 1 - \frac{2n(16n^4+16n^3+12n^2+2n+1)}{(4n^2+2n+1)(16n^4+16n^3+12n^2+1)},\\
R_4 &= 2n^2 + n + 1 - \frac{32n^5+64n^4+64n^3+28n^2+4n-1}{8n(8n^5+20n^4+26n^3+18n^2+7n+1)} = \frac{p_9}{q_9}.
\end{aligned}$$

In Table 2, we list the upper bounds for $\ell_b$, $3 \leq b \leq 100$, obtained by experiments. It is not hard to check that sequences of numbers such that $b = 5$ and $\ell = 9$ or $b = 6$ and $\ell = 10$ exist, but number 945 is the only one found which shows that $\ell_{10} \leq 14$ (and we tested all numbers $\leq 2^{21.5}$). Also, contrary to $\sqrt{d}$, where it holds $\ell_6 = 8$ [2, Exam. 1], for $\frac{1+\sqrt{d}}{2}$ we were not able to find such $d$.

In the next section, we find some sequences which will significantly improve some of the entries in Table 2.

## 6. Sequences with many good approximants

Let us first prove some lemmas.

**Proposition 3.** *Let $d$, $s_n$, $t_n$, $p_n$, $q_n$ be as in Algorithm* (1)*. Then for $n \geq -1$ it holds*

$$\begin{aligned}
(2p_n - q_n)^2 - dq_n^2 &= (-1)^{n+1} 4t_{n+1},\\
(2p_n - q_n)(2p_{n-1} - q_{n-1}) - dq_nq_{n-1} &= (-1)^n 2s_{n+1}.
\end{aligned}$$

| $b$ | $\ell_b \leq$ | $d$ | $\ell_b/b \leq$ | $b$ | $\ell_b \leq$ | $d$ | $\ell_b/b \leq$ |
|---|---|---|---|---|---|---|---|
| 3 | 5 | 41 | 1.66667 | 52 | 180 | 2 414 425 | 3.46154 |
| 4 | 6 | 57 | 1.5 | 53 | 429 | 2 328 625 | 8.09434 |
| 5 | 9 | 353 | 1.8 | 54 | 176 | 554 625 | 3.25926 |
| 6 | 10 | 129 | 1.66667 | 55 | 397 | 1 004 809 | 7.21819 |
| 7 | 13 | 4481 | 1.85714 | 56 | 180 | 1 839 825 | 3.21429 |
| 8 | 14 | 873 | 1.75 | 57 | 471 | 1 977 625 | 8.26316 |
| 9 | 17 | 67 073 | 1.88889 | 58 | 232 | 365 625 | 4.26316 |
| 10 | 14 | 945 | **1.4** | 59 | 499 | 2 601 625 | 8.45763 |
| 11 | 21 | 1 054 721 | 1.9091 | 60 | 210 | 1 388 625 | 3.5 |
| 12 | 20 | 2625 | 1.66667 | 61 | 607 | 2 739 601 | 9.9509 |
| 13 | 33 | 204 425 | 2.53847 | 62 | 246 | 2 660 065 | 3.96775 |
| 14 | 22 | 215 985 | 1.57143 | 63 | 527 | 2 229 625 | 8.36508 |
| 15 | 45 | 127 465 | 3.57143 | 64 | 226 | 2 544 993 | 3.5313 |
| 16 | 28 | 28 665 | 1.75 | 65 | 387 | 1 665 625 | 5.95385 |
| 17 | 31 | 244 205 | 1.82353 | 66 | 260 | 2 165 625 | 3.9394 |
| 18 | 34 | 87 057 | 1.88889 | 67 | 625 | 2 944 201 | 9.32836 |
| 19 | 53 | 2 483 125 | 2.78948 | 68 | 266 | 2 237 625 | 3.91177 |
| 20 | 38 | 1 588 457 | 1.9 | 69 | 679 | 2 586 625 | 9.8406 |
| 21 | 69 | 1 007 165 | 3.28572 | 70 | 340 | 1 517 697 | 4.85715 |
| 22 | 44 | 1 343 433 | 2.28572 | 71 | 763 | 2 193 241 | 10.74648 |
| 23 | 91 | 2 720 801 | 3.95653 | 72 | 298 | 2 721 705 | 4.13889 |
| 24 | 50 | 770 133 | 2.083334 | 73 | 961 | 2 792 425 | 13.16439 |
| 25 | 87 | 2 193 425 | 3.48 | 74 | 310 | 408 969 | 4.18919 |
| 26 | 64 | 190 125 | 2.46154 | 75 | 985 | 1 783 825 | 13.13334 |
| 27 | 95 | 2 632 825 | 3.51852 | 76 | 390 | 1 083 537 | 5.13158 |
| 28 | 60 | 182 457 | 2.14286 | 77 | 993 | 2 751 625 | 12.89611 |
| 29 | 113 | 1 286 305 | 3.89656 | 78 | 400 | 2 768 985 | 5.12821 |
| 30 | 76 | 2 837 097 | 2.53334 | 79 | 1083 | 1 859 425 | 13.70887 |
| 31 | 99 | 1 503 125 | 3.19355 | 80 | 356 | 639 009 | 4.45 |
| 32 | 86 | 235 305 | 2.6875 | 81 | 1075 | 2 188 825 | 13.27161 |
| 33 | 129 | 186 745 | 3.9091 | 82 | 356 | 1 105 425 | 4.34147 |
| 34 | 94 | 133 353 | 2.76471 | 83 | 1131 | 2 394 625 | 13.62651 |
| 35 | 153 | 1 512 745 | 4.37143 | 84 | 398 | 610 929 | 4.7381 |
| 36 | 94 | 174 097 | 2.61112 | 85 | 1187 | 2 602 825 | 13.96471 |
| 37 | 147 | 2 263 105 | 3.973 | 86 | 462 | 2 967 289 | 5.3721 |
| 38 | 112 | 57 321 | 2.94737 | 87 | 1105 | 2 889 625 | 12.70115 |
| 39 | 173 | 614 125 | 4.4359 | 88 | 462 | 1 112 697 | 5.25 |
| 40 | 96 | 2 033 361 | 2.4 | 89 | 1259 | 2 558 425 | 14.14607 |
| 41 | 227 | 2 526 625 | 5.53659 | 90 | 386 | 1 157 625 | 4.28889 |
| 42 | 122 | 677 457 | 2.90477 | 91 | 1409 | 2 766 625 | 15.48352 |
| 43 | 309 | 680 425 | 7.18605 | 92 | 672 | 2 100 249 | 7.30435 |
| 44 | 142 | 2 512 705 | 3.22728 | 93 | 1395 | 2 402 425 | 15.30435 |
| 45 | 243 | 1 743 625 | 5.4 | 94 | 592 | 1 796 977 | 6.29788 |
| 46 | 128 | 2 754 297 | 2.78261 | 95 | 1717 | 2 056 609 | 18.073685 |
| 47 | 273 | 2 815 625 | 5.80852 | 96 | 518 | 2 739 625 | 5.39584 |
| 48 | 166 | 1 962 873 | 3.45834 | 97 | 2013 | 2 903 209 | 20.75258 |
| 49 | 353 | 2 796 625 | 7.20409 | 98 | 530 | 2 268 945 | 5.40817 |
| 50 | 142 | 2 411 937 | 2.84 | 99 | 3495 | 2 869 441 | 35.3031 |
| 51 | 245 | 1 540 625 | 4.80393 | 100 | 746 | 2 718 441 | 7.46 |

Table 2: *Upper bounds for* $\ell_b$, *for* $3 \leq b \leq 100$.

**Proof**. Similarly to [9, §20, III], since $\sqrt{d}$ is irrational, from

$$\frac{1+\sqrt{d}}{2} = \left[a_0, a_1, \dots, a_n, \tfrac{s_{n+1}+\sqrt{d}}{2t_{n+1}}\right] = \frac{\frac{s_{n+1}+\sqrt{d}}{2t_{n+1}}p_n + p_{n-1}}{\frac{s_{n+1}+\sqrt{d}}{2t_{n+1}}q_n + q_{n-1}}$$
$$= \frac{p_n(\sqrt{d}+s_{n+1}) + 2t_{n+1}p_{n-1}}{q_n(\sqrt{d}+s_{n+1}) + 2t_{n+1}q_{n-1}},$$

we get

$$2p_n - q_n = q_n s_{n+1} + 2t_{n+1}q_{n-1} \tag{21}$$
$$dq_n = 2(p_n s_{n+1} + 2t_{n+1}p_{n-1}) - (q_n s_{n+1} + 2t_{n+1}q_{n-1}). \tag{22}$$

Multiplying (21) by $2p_n - q_n$ and (22) by $q_n$, by subtraction we get:

$$(2p_n - q_n)^2 - dq_n^2 = 4t_{n+1}(q_{n-1}p_n - p_{n-1}q_n) = 4t_{n+1}(-1)^{n+1},$$

and multiplying (21) by $2p_{n-1} - q_{n-1}$ and (22) by $q_{n-1}$, by subtraction we get:

$$(2p_n - q_n)(2p_{n-1} - q_{n-1}) - dq_n q_{n-1} = 2s_{n+1}(q_n p_{n-1} - p_n q_{n-1}) = 2s_{n+1}(-1)^n.$$

□

Let $g_n \overset{\text{def}}{=} \gcd\left(p_n^2 + \frac{d-1}{4}q_n^2, q_n(2p_n - q_n)\right)$.

**Lemma 6.** *$g_n$ divides* $\gcd(d, t_{n+1}, s_{n+1}, s_{n+2})$.

**Proof**. Assume first that $q_n$ is odd. Then $g_n$ is odd as well, and we have:

$$\begin{aligned} g_n &= \gcd\left(p_n^2 + \tfrac{d-1}{4}q_n^2, q_n(2p_n - q_n)\right) \\ &= \gcd\left(4(p_n^2 + \tfrac{d-1}{4}q_n^2) - 2q_n(2p_n - q_n), q_n(2p_n - q_n)\right) \\ &= \gcd\left((2p_n - q_n)^2 + dq_n^2, q_n(2p_n - q_n)\right). \end{aligned}$$

Since $q_n$ is odd and $\gcd(p_n, q_n) = 1$, it follows that $\gcd(2p_n - q_n, q_n) = 1$. Thus $g_n$ divides $2p_n - q_n$ and $d$.

If $q_n$ is even, then $p_n$ is odd, and since $d \equiv 1 \pmod 4$, $g_n$ is also odd. So we have:

$$\begin{aligned} g_n &= \gcd\left(p_n^2 + \tfrac{d-1}{4}q_n^2, 2q_n(p_n - \tfrac{q_n}{2})\right) = \gcd\left(p_n^2 + \tfrac{d-1}{4}q_n^2 - q_n(p_n - \tfrac{q_n}{2}), q_n(p_n - \tfrac{q_n}{2})\right) \\ &= \gcd\left((p_n - \tfrac{q_n}{2})^2 + \tfrac{dq_n^2}{4}, q_n(p_n - \tfrac{q_n}{2})\right). \end{aligned}$$

Since $q_n$ is odd and $\gcd(p_n, q_n) = 1$ it follows that $\gcd(2p_n - q_n, q_n) = 2$. So $g_n$ divides $2p_n - q_n$ and $d$.

Proposition 3 implies that $g_n | \gcd(t_{n+1}, s_{n+1}, s_{n+2})$. □

**Proposition 4.** *i) If $a_{n+1} > \frac{\sqrt{2}}{g_n}\sqrt{\sqrt{d}+2}$, then $R_n$ is a convergent of $\frac{1+\sqrt{d}}{2}$.*

*ii) If $a_{n+1} < \frac{1}{g_n}\sqrt{\sqrt{d}-2} - 2$, then $R_n$ is not a convergent of $\frac{1+\sqrt{d}}{2}$.*

**Proof**. Let $R_n = \frac{u}{v}$, $\gcd(u,v)=1$. Then $v = \frac{q_n(2p_n-q_n)}{g_n}$.
i) Let $a_{n+1} > \frac{\sqrt{2}}{g_n}\sqrt{\sqrt{d}+2}$. We have

$$\begin{aligned} R_n - \frac{1+\sqrt{d}}{2} &= \frac{q_n}{2p_n-q_n}\left(\frac{p_n}{q_n}-\frac{1+\sqrt{d}}{2}\right)^2 \\ &< \frac{q_n}{2p_n-q_n}\frac{1}{a_{n+1}^2q_n^4} = \frac{1}{2v^2}\frac{2}{g_n^2a_{n+1}^2}\left(2\frac{p_n}{q_n}-1\right) \\ &< \frac{1}{2v^2}\frac{2}{g_n^2a_{n+1}^2}\left(2\left(\frac{1+\sqrt{d}}{2}+1\right)-1\right) = \frac{1}{2v^2}\frac{2}{g_n^2a_{n+1}^2}(\sqrt{d}+2) < \frac{1}{2v^2}. \end{aligned}$$

From (4), we see that $R_n$ is a convergent.
ii) Let $a_{n+1} < \frac{1}{g_n}\sqrt{\sqrt{d}-2}-2$. Then

$$\begin{aligned} R_n - \frac{1+\sqrt{d}}{2} &= \frac{q_n}{2p_n-q_n}\left(\frac{p_n}{q_n}-\frac{1+\sqrt{d}}{2}\right)^2 > \frac{q_n}{2p_n-q_n}\frac{1}{(a_{n+1}+2)^2q_n^4} \\ &= \frac{1}{v^2}\frac{1}{g_n^2(a_{n+1}+2)^2}\left(2\frac{p_n}{q_n}-1\right) > \frac{1}{v^2}\frac{1}{g_n^2(a_{n+1}+2)^2}\left(2\left(\frac{1+\sqrt{d}}{2}-1\right)-1\right) \\ &= \frac{1}{v^2}\frac{1}{g_n^2(a_{n+1}+2)^2}(\sqrt{d}-2) > \frac{1}{v^2}. \end{aligned}$$

Now (3) proves (ii) of the proposition. □

There are many quadruples $(e,f,g,h)$ such that experimental results show that $d_n = (e\cdot f^n+g)^2+h\cdot f^n$ should have many good approximants (numbers of this form sometimes have a very interesting continued fraction expansion; see e.g. [5, 8, 11]). But it is not easy to show that either (i) or (ii) from Proposition 4 holds for every $n$. However, we found some for which that holds.

**Proposition 5.** *If*

$$d_n = (24\cdot 9^n+1)^2 + 12\cdot 9^n, \tag{23}$$

*then for* $n\in\mathbb{N}$ *we have* $\ell\left(\frac{1+\sqrt{d_n}}{2}\right) = 4n+6$ *and*

$$\frac{1+\sqrt{d_n}}{2} = \Big[12\cdot 9^n+1,\ \overline{8,\ 24\cdot 9^{n-1},\ 8\cdot 9^1,\ 24\cdot 9^{n-2},\ 8\cdot 9^2\ \ldots\ \ldots\ 24\cdot 9,\ 8\cdot 9^{n-1},\ 24,\ 8\cdot 9^n,\ 2,\ 1,\ 2,\ 8\cdot 9^n,\ 24,\ 8\cdot 9^{n-1},\ 24\cdot 9\ \ldots\ \ldots\ 8\cdot 9^2,\ 24\cdot 9^{n-2},\ 8\cdot 9^1,\ 24\cdot 9^{n-1},\ 8,\ 24\cdot 9^n+1}\Big].$$

**Proof**. Let $s_0 = t_0 = 1$, and we have $a_0 = 12\cdot 9^n+1$.

$$\begin{aligned} s_1 &= 12\cdot 9^n+1, & t_1 &= 3\cdot 9^n, & a_1 &= 8, \\ s_2 &= 12\cdot 9^n-1, & t_2 &= 9, & a_2 &= 24\cdot 9^{n-1}. \end{aligned}$$

For $1 \le k \le n$, from

$$s_{2k} = 12 \cdot 9^n - 1, \qquad t_{2k} = 9^k, \qquad a_{2k} = 24 \cdot 9^{n-k},$$

using (1) we have:

$$s_{2k+1} = 12 \cdot 9^n + 1, \quad t_{2k+1} = 3 \cdot 9^{n-k}, \quad a_{2k+1} = \left\lfloor \frac{24 \cdot 9^n + 1}{3 \cdot 9^{n-k}} \right\rfloor = 8 \cdot 9^k,$$
$$s_{2k+2} = 12 \cdot 9^n - 1, \quad t_{2k+2} = 9^{k+1}, \quad a_{2k+2} = \left\lfloor \frac{24 \cdot 9^n}{9^{k+1}} \right\rfloor.$$

For $k < n$ we have:

$$a_{2k+2} = 24 \cdot 9^{n-(k+1)},$$

and for $k = n$:

$$a_{2n+2} = 2,$$
$$s_{2n+3} = 12 \cdot 9^n + 1, \quad t_{2n+3} = 12 \cdot 9^n + 1, \quad a_{2n+3} = \left\lfloor \frac{36 \cdot 9^n + 2}{24 \cdot 9^n + 2} \right\rfloor = 1,$$
$$s_{2n+4} = 12 \cdot 9^n + 1,$$

and since $s_{2n+3} = s_{2n+4}$ we have $\ell = 2(2n+3) = 4n+6$. □

**Lemma 7.** *For sequence* (23) *and* $g_k = \gcd\left(p_k^2 + \frac{d_n - 1}{4} q_k^2, q_k(2p_k - q_k)\right)$, *for* $k = 0$, $1, \ldots, 2n+1, 2n+3, \ldots, 4n+4$ *we have* $g_k = 1$.

**Proof**. From (2) it follows $s_{4n+6} = s_1, s_{4n+5} = s_2, \ldots, t_{4n+5} = t_1, \ldots$, and using Lemma 6, for $k = 0, 1, \ldots, 2n+1, 2n+3, \ldots 2n+4$ we have $g_k \mid \gcd(s_{k+1}, s_{k+2}, t_{k+1}) = 1$. □

**Theorem 5.** *For the sequence* $d_n = (24 \cdot 9^n + 1)^2 + 12 \cdot 9^n$ *we have* $b(d_n) = 2n + 4$.

**Proof**. Using Proposition 5, we have $\ell = 4n + 6$. Using (7) and (6), $R_{2n+2}$ and $R_{4n+5}$ are good approximants. By Corollary 1, it suffices to check approximants $R_k$, $k = 0, 1, \ldots, 2n+1$. By Lemma 7, we have $g_k = 1$. By Proposition 4 (i) $R_k$ is a good approximant if

$$a_{k+1}^2 \ge 48 \cdot 9^n + 8 = 2(24 \cdot 9^n + 4) > 2(\sqrt{d} + 2), \tag{24}$$

and by Proposition 4 (ii) $R_k$ is not a good approximant if

$$a_{k+1} < \sqrt{24 \cdot 9^n + 3} - 2 < \sqrt{\sqrt{d} - 2} - 2. \tag{25}$$

For $i = 2k$, $k = 0, 1, \ldots, n$ using Proposition 5 we have $a_{2k+1} = 8 \cdot 9^k$, so let us see when (24) holds.

$$(8 \cdot 9^k)^2 = 64 \cdot 9^{2k} > 48 \cdot 9^n + 8$$

holds when $2k \geq n$, i.e. $k \geq \lfloor \frac{n+1}{2} \rfloor$. Thus $R_{2\lfloor \frac{n+1}{2} \rfloor}$, $R_{2\lfloor \frac{n+1}{2} \rfloor+2}$, $\dots$, $R_{2n}$ are good approximants. For $2k \leq n-1$, i.e. $k \leq \lfloor \frac{n-1}{2} \rfloor$, (25) holds, thus $R_0, R_2, \dots, R_{2\lfloor \frac{n-1}{2} \rfloor}$ are not good approximants.

For $i = 2k-1$, $k = 1, 2, \dots, n$ we have $a_{2k} = 24 \cdot 9^{n-k}$ and $a_{2n+2} = 2$, so (24) holds when $2k \leq n+1$, i.e. $k \leq \lfloor \frac{n+1}{2} \rfloor$. Thus $R_1$, $R_3$, $\dots$, $R_{2\lfloor \frac{n+1}{2} \rfloor-1}$ are good approximants. For $2k \geq n+2$, i.e. $k \geq \lfloor \frac{n+3}{2} \rfloor$, (25) holds, so $R_{2\lfloor \frac{n+1}{2} \rfloor+1}$, $R_{2\lfloor \frac{n+1}{2} \rfloor+3}$, $\dots$, $R_{2n-1}$ are not good approximants.

Therefore there are exactly $2+2\big(n+1-\lfloor \frac{n+1}{2} \rfloor + \lfloor \frac{n+1}{2} \rfloor\big) = 2+2(n+1) = 2n+4$ good approximants. □

**Corollary 3.** *For sequence* (23) *we have* $\ell(d_n) = 4n+6$ *and* $b(d_n) = 2n+4$ *for every* $n \in \mathbb{N}$. *So for every even positive integer* $b$ *there exists* $d \in \mathbb{N}$, $d \equiv 1 \pmod 4$, $d \neq \square$, *such that* $b(d) = b$ *and* $b(d) > \frac{\ell(d)}{2}$.

**Proof**. For $b = 2$ we have $\ell = 2$, and for $b = 4$, in Table 2 we have number 57, which has $\ell = 6$, and for other even $b$'s we use sequence (23). □

**Proposition 6.** *Let*

$$d_n = (3 \cdot 16^n + 1)^2 + 4 \cdot 16^n. \tag{26}$$

*Then for* $n \in \mathbb{N}$ *it holds* $\ell\big(\frac{1+\sqrt{d_n}}{2}\big) = 4n+1$ *and*

$$\frac{1+\sqrt{d_n}}{2} = \Big[\tfrac{3}{2} \cdot 16^n + 1, \overline{3,\ 3 \cdot 4^{2n-1},\ 3 \cdot 4^1,\ 3 \cdot 4^{2n-2},\ 3 \cdot 4^2,\ \dots}$$
$$\overline{\dots,\ 3 \cdot 4^{n+1},\ 3 \cdot 4^{n-1},\ 3 \cdot 4^n,\ 3 \cdot 4^n,\ 3 \cdot 4^{n-1},\ 3 \cdot 4^{n+1},\ \dots}$$
$$\overline{\dots,\ 3 \cdot 4^2,\ 3 \cdot 4^{2n-2},\ 3 \cdot 4^1,\ 3 \cdot 4^{2n-1},\ 3,\ 3 \cdot 16^n + 1}\Big].$$

**Proof**. Let $s_0 = t_0 = 1$, and we have $a_0 = \frac{3}{2} \cdot 16^n + 1$.

$$s_1 = 3 \cdot 16^n + 1, \qquad t_1 = 4^{2n}, \qquad a_1 = 3 \cdot 4^0.$$

For $0 \leq k < 2n-1$ we get

$$s_{2k+1} = 3 \cdot 16^n + 1, \quad t_{2k+1} = 4^{2n-k}, \qquad a_{2k+1} = 3 \cdot 4^k,$$

and we have:

$$s_{2k+2} = 3 \cdot 16^n - 1, \quad t_{2k+2} = 4^{k+1}, \qquad a_{2k+2} = \left\lfloor \frac{3 \cdot 4^{2n}}{4^{k+1}} \right\rfloor = 3 \cdot 4^{2n-(k+1)},$$
$$s_{2k+3} = 3 \cdot 16^n + 1, \quad t_{2k+3} = 4^{2n-(k+1)}, \quad a_{2k+3} = \left\lfloor \frac{3 \cdot 4^{2n} + 1}{4^{2n-(k+1)}} \right\rfloor = 3 \cdot 4^{k+1},$$

so when $k = n-1$, we have $t_{2k+2} = t_{2k+3}$, thus $\ell = 2(2n-2+2)+1 = 4n+1$. □

**Lemma 8.** *For sequence* (26) *and* $g_k = \gcd\big(p_k^2 + \frac{d_n-1}{4} q_k^2, q_k(2p_k - q_k)\big)$, *for* $k \geq 0$ *we have* $g_k = 1$.

**Proof**. From (2) it follows that $s_{4n-1} = s_{4n+1} = s_{4n+3} = 3 \cdot 16^n + 1$ and $s_{4n-2} = s_{4n} = s_{4n+2} = 3 \cdot 16^n - 1$. Using Lemma 6, we have $g_k \mid \gcd(s_{k+1}, s_{k+2}) = 1$, for every $k \geq 0$. □

**Theorem 6.** *For the sequence* $d_n = (3 \cdot 16^n + 1)^2 + 4 \cdot 16^n$ *we have* $b(d_n) = 2n + 1$.

**Proof**. By Proposition 6, we have $\ell = 4n + 1$. Thus, by (6), $R_{4n}$ is a good approximant. By Lemma 8 we have $g_k = 1$. Using Proposition 4 (i) (divided by 3) $R_k$ is a good approximant if

$$\frac{a_{k+1}}{3} \geq 4^n > \sqrt{\tfrac{2}{3} \cdot 16^n + \tfrac{8}{9}} = \frac{\sqrt{2}}{3} \cdot \sqrt{3 \cdot 16^n + 4} > \frac{\sqrt{2}}{3} \cdot \sqrt{\sqrt{d} + 2}, \tag{27}$$

and by Proposition 4 (ii) $R_k$ is not a good approximant if

$$\frac{a_{k+1}}{3} \leq 4^{n-1} < \sqrt{\tfrac{1}{3} \cdot 16^n - \tfrac{1}{9}} - \tfrac{2}{3} = \tfrac{1}{3} \cdot \sqrt{3 \cdot 16^n - 1} - 2 < \tfrac{1}{3}\sqrt{\sqrt{d} - 2} - 2. \tag{28}$$

For $k = 2i + 1$, $i = 0, 1, 2, \ldots, 2n - 1$, by Proposition 6, we have $a_{2i+1} = 3 \cdot 4^i$, thus for $i = 0, 1, 2, \ldots, n - 1$ (28) holds, so $R_0, R_2, \ldots, R_{2n-2}$ are not good approximants, and for $i = n, n+1, \ldots, 2n-1$ (27) holds, thus $R_{2n}, R_{2n+2}, \ldots, R_{4n-2}$ are good approximants. For $k$ of the form $2i$, using Corollary 1, it follows that $R_1, R_3, \ldots, R_{2n-1}$ are good approximants, and the others are not.

Therefore there are exactly $2n + 1$ good approximants. □

**Corollary 4.** *For sequence* (26) *we have* $\ell(d_n) = 4n+1$ *and* $b(d_n) = 2n+1$ *for every* $n \in \mathbb{N}$. *Thus for every odd positive integer* $b$ *there exists* $d \in \mathbb{N}$, $d \equiv 1 \pmod 4$, $d \neq \square$, *such that* $b(d) = b$ *and* $b(d) > \frac{\ell(d)}{2}$. □

From Corollaries 3 and 4, we immediately obtain the following result.

**Corollary 5.**

$$\sup\left\{\frac{\ell_b}{b} : b \geq 1\right\} \leq 2.$$

□

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