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Combinatorial bases of Feigin–Stoyanovsky's type subspaces of higher-level standard $\tilde{\mathfrak{sl}}(\ell+1,\mathbb{C})$-modules ☆

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A B S T R A C T

Let $\tilde{\mathfrak{g}}$ be an affine Lie algebra of the type $A_\ell^{(1)}$. We find a combinatorial basis of Feigin–Stoyanovsky's type subspace $W(\Lambda)$ given in terms of difference and initial conditions. Linear independence of the generating set is proved inductively by using coefficients of intertwining operators. A basis of the standard $\tilde{\mathfrak{g}}$-module $L(\Lambda)$ is obtained as an "inductive limit" of the basis of $W(\Lambda)$.

1. Introduction

Vertex operator construction of affine Lie algebras has been used to prove and to give a representation-theoretic interpretation of Rogers–Ramanujan-type combinatorial identities. This approach was initiated by J. Lepowsky and R. Wilson [19], and was continued in the works of Lepowsky, M. Primc, A. Meurman and others (cf. [18,21]). An important part of this program was to find monomial bases of standard modules for affine Lie algebras, or some of its subspaces. Knowledge of bases was then used to calculate characters of these spaces, which gave the sum side in the Rogers–Ramanujan-type partition identities.

Later, B. Feigin and A. Stoyanovsky considered what they called a principal subspace of a standard $\tilde{\mathfrak{sl}}(2,\mathbb{C})$-module [10]. On one side, they described the dual of this subspace in terms of symmetric polynomial forms, and on the other they realized this subspace geometrically. From these two realizations they recovered Rogers–Ramanujan and Gordon identities. Furthermore, by representing the whole standard module as an inductive limit of Weyl-group translates of the principal subspace, they obtained a combinatorial basis of the whole module in terms of "infinite monomials".

G. Georgiev extended character formulas for principal subspaces obtained by Feigin and Stoyanovsky to a family of standard $\tilde{\mathfrak{sl}}(\ell+1,\mathbb{C})$-modules [14]. He explicitly constructed combinatorial

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bases of these subspaces and in the proof of linear independence, he used intertwining operators from [7]. More recently, S. Capparelli, J. Lepowsky and A. Milas laid out a program to interpret and obtain Rogers–Ramanujan-type recursions in the setting of vertex operator algebras and affine Lie algebras. They used intertwining operators to construct exact sequences between different principal subspaces for $\tilde{\mathfrak{sl}}(2,\mathbb{C})$ and in this way, obtained Rogers–Ramanujan and Rogers–Selberg recursions for characters of these subspaces [5,6]. As a continuation of this program, C. Calinescu obtained Rogers–Ramanujan-type recursions for some classes of standard modules for $\tilde{\mathfrak{sl}}(\ell+1,\mathbb{C})$ [1,2], and Calinescu, Lepowsky and Milas provided new proofs of presentation theorems for principal subspaces for $\tilde{\mathfrak{sl}}(2,\mathbb{C})$ [3,4].

In parallel with these developments, Primc studied similar subspaces of standard modules for different affine Lie algebras [22,23], which he later called Feigin–Stoyanovsky's type subspaces [24]. He used bases of these subspaces to construct from them bases of the whole standard modules. For $\tilde{\mathfrak{sl}}(\ell+1,\mathbb{C})$, these bases were parameterized by $(k,\ell+1)$-admissible configurations, a combinatorial objects that were introduced and further studied in [8] and [9]. In [24], Primc proved linear independence of the spanning set by using Capparelli–Lepowsky–Milas' approach via intertwining operators and a description of the basis from [8]. These operators and a description of basis were used by M. Jerković to obtain exact sequences of Feigin–Stoyanovsky's type subspaces and recurrence relations for the corresponding characters [15].

In our previous paper [26] we have used ideas of Georgiev, Capparelli, Lepowsky and Milas, and of Primc to construct bases and prove linear independence of Feigin–Stoyanovsky's type subspaces for all basic modules for $\tilde{\mathfrak{sl}}(\ell+1,\mathbb{C})$. In this paper we generalize this result to higher-level standard modules for $\tilde{\mathfrak{sl}}(\ell+1,\mathbb{C})$.

Let $\mathfrak{g}=\mathfrak{sl}(\ell+1,\mathbb{C})$, a simple complex Lie algebra of the type A_ℓ, $\mathfrak{h}\subset\mathfrak{g}$ its Cartan subalgebra, R the corresponding root system. Then one has a root decomposition $\mathfrak{g}=\mathfrak{h}+\sum_{\alpha\in R}\mathfrak{g}_\alpha$. Fix root vectors $x_\alpha\in\mathfrak{g}_\alpha$. Let $\langle\cdot,\cdot\rangle$ be a normalized invariant bilinear form on $\mathfrak{g}$, and by the same symbol denote the induced form on $\mathfrak{g}^*$.

Let $\Pi=\{\alpha_1,\dots,\alpha_\ell\}$ be a basis of the root system R, and $\{\omega_1,\dots,\omega_\ell\}$ the corresponding set of fundamental weights. We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ in the usual way and fix a fundamental weight $\omega=\omega_m$. Set

$$\Gamma=\bigl\{\gamma\in R\bigm|\langle\gamma,\omega\rangle=1\bigr\}=\{\gamma_{ij}\mid i=1,\dots,m;\ j=m,\dots,\ell\},$$

where

$$\gamma_{ij}=\alpha_i+\dots+\alpha_m+\dots+\alpha_j.$$

Set

$$\mathfrak{g}_{\pm1}=\sum_{\alpha\in\pm\Gamma}\mathfrak{g}_\alpha,\qquad \mathfrak{g}_0=\mathfrak{h}\oplus\sum_{\langle\alpha,\omega\rangle=0}\mathfrak{g}_\alpha.$$

Then

$$\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1 \tag{1}$$

is a $\mathbb{Z}$-gradation of $\mathfrak{g}$. This decomposition is illustrated in Fig. 1; the subalgebra $\mathfrak{g}_0$ consists of block-diagonal matrices, while $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ consist of matrices with non-zero entries only in the upper right or lower-left block, respectively. We say that the set Γ is *the set of colors*. For $\gamma\in\Gamma$, we say that a fixed basis element $x_\gamma\in\mathfrak{g}_\gamma$ *is of the color* γ. The set of colors Γ can be pictured as a rectangle with row indices $1,\dots,m$ and column indices $m,\dots,\ell$ (see Fig. 2).

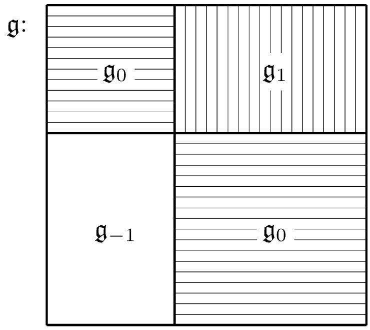

Fig. 1. The $\mathbb{Z}$-gradation of $\mathfrak{g}$.

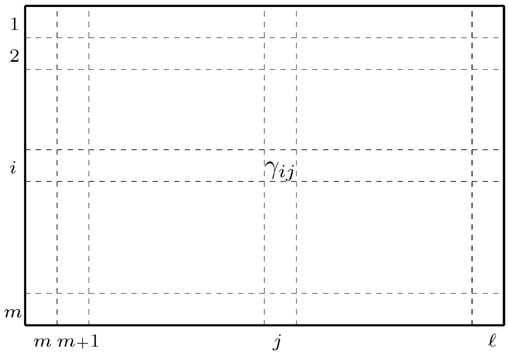

Fig. 2. The set of colors Γ.

Affine Lie algebra associated with $\mathfrak{g}$ is $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where c is the canonical central element, and d is the degree operator. Elements $x_\alpha(n) = x_\alpha \otimes t^n$ are fixed real root vectors. The $\mathbb{Z}$-gradation (1) of $\mathfrak{g}$ induces the $\mathbb{Z}$-gradation of $\tilde{\mathfrak{g}}$:

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1,$$

where $\tilde{\mathfrak{g}}_1 = \mathfrak{g}_1 \otimes \mathbb{C}[t, t^{-1}]$ is a commutative Lie subalgebra with a basis $\{x_\gamma(-r) \mid r \in \mathbb{Z}, \gamma \in \Gamma\}$.

Let $\Lambda_0, \ldots, \Lambda_\ell$ be fundamental weights for $\tilde{\mathfrak{g}}$. For $\Lambda = k_0\Lambda_0 + \cdots + k_\ell\Lambda_\ell$, $k_i \in \mathbb{Z}$, let $L(\Lambda)$ be a standard $\tilde{\mathfrak{g}}$-module of level $k = k_0 + \cdots + k_\ell$, with a fixed highest weight vector v_Λ. A Feigin–Stojanovsky's type subspace is a $\tilde{\mathfrak{g}}_1$-submodule of $L(\Lambda)$ generated with v_Λ,

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1) \cdot v_\Lambda \subset L(\Lambda).$$

We find a basis of the Feigin–Stoyanovsky's type subspace $W(\Lambda)$ consisting of *monomial vectors*

$$\left\{x_{\gamma_1}(-r_1) \cdots x_{\gamma_n}(-r_n) v_\Lambda \;\middle|\; n \in \mathbb{Z}_+;\ \gamma_j \in \Gamma,\ r_j \in \mathbb{N}\right\}$$

whose *monomial parts*

$$x_{\gamma_1}(-r_1) \cdots x_{\gamma_n}(-r_n) \tag{2}$$

satisfy certain combinatorial conditions.

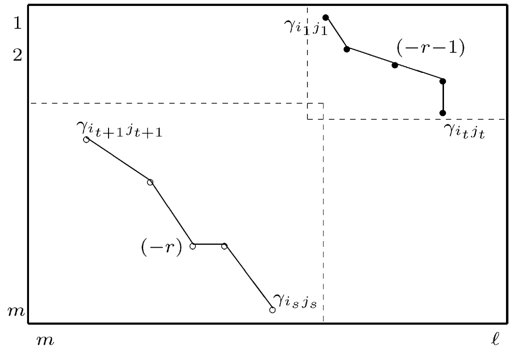

Fig. 3. Difference conditions.

We'll say that an element $x_\gamma(-r) \in \tilde{\mathfrak{g}}_1$ is of a *degree* $-r$ and of a *color* γ. We can look on monomials (2) as colored partitions of $N = -r_1 - \cdots - r_n$. Often we'll use "exponential notation"

$$x_{\gamma_1}(-r_1)^{a_{\gamma_1}^{r_1}} \cdots x_{\gamma_t}(-r_t)^{a_{\gamma_t}^{r_t}} \tag{3}$$

for monomials (2), where we assume that for different indices i, factors $x_{\gamma_i}(-r_i)$ are distinct, and $a_{\gamma_i}^{r_i} \in \mathbb{Z}_+$ are corresponding exponents.

We'll say that a monomial (3) satisfies *difference conditions* for $L(\Lambda)$, if exponents of its factors satisfy the following family of inequalities:

$$a_{i_1 j_1}^{r+1} + \cdots + a_{i_t j_t}^{r+1} + a_{i_{t+1} j_{t+1}}^{r} + \cdots + a_{i_s j_s}^{r} \leqslant k,$$

where $1 \leqslant i_1 \leqslant \cdots \leqslant i_t \leqslant i_{t+1} \leqslant \cdots \leqslant i_s \leqslant m$, $m \leqslant j_{t+1} \leqslant \cdots \leqslant j_s \leqslant j_1 \leqslant \cdots \leqslant j_t \leqslant \ell$ and a_{ij}^r is an exponent of $x_{\gamma_{ij}}(-r)$. We can reformulate this by saying that for any configuration of colors of elements of degree $-r$ and $-r-1$ of the type pictured on Fig. 3, the sum of corresponding exponents must be less than $k+1$.

Similarly, we'll say that a monomial (3) satisfies *initial conditions* for $L(\Lambda)$ if

$$a_{i_1 j_1}^1 + a_{i_2 j_2}^1 + \cdots + a_{i_t j_t}^1 \leqslant k_0 + k_1 + \cdots + k_{i_1 - 1} + k_{j_t + 1} + \cdots + k_\ell,$$

where $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_t \leqslant m$, $m \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_t \leqslant \ell$ and a_{ij}^1 is an exponent of $x_{\gamma_{ij}}(-1)$.

The main result of this paper is

Theorem 14. *Let $L(\Lambda)$ be a standard $\tilde{\mathfrak{g}}$-module. Then the set of monomial vectors whose monomial part satisfies difference and initial conditions for $L(\Lambda)$, is a basis of $W(\Lambda)$.*

Difference conditions on monomials are obtained by observing relations between fields $x_\gamma(z)$, $\gamma \in \Gamma$ on $L(\Lambda)$, while initial conditions are consequences of relations for some modules of lower level.

In the case $k = 1$, these conditions are equivalent to the conditions we have obtained in [26]. Since a standard module of level k can be found inside a k-fold tensor product of modules of level 1, it is natural to ask can a monomial (2) be factorized in such a way so that each factor satisfies difference and initial conditions on the corresponding level 1 module. By combinatorial arguments, we show that the answer to this question is affirmative. This enables us to use tensor products of coefficients of intertwining operators that were constructed in [26] and to inductively prove the linear independence. Following the approach of Primc [22,23] we construct a basis of the whole standard module $L(\Lambda)$ as an "inductive limit" of the basis of $W(\Lambda)$.

In the end, we briefly outline the course of this paper. In Sections 2 to 4 we introduce most of the notation and definitions. In Sections 5 and 6 we briefly sketch the vertex operator construction of basic modules. In Section 7 we recall from [26] the description of a monomial basis of $W(\Lambda)$ in the level 1 case. In Section 8 we describe a monomial basis of modules of level k in terms of difference and initial conditions, and in Section 9 we rewrite these conditions in terms of conditions for level 1 modules. In Section 10 we prove linear independence of the basis. In the last two sections we construct a basis of $L(\Lambda)$ and give a presentation theorem for $W(\Lambda)$.

2. Affine Lie algebras

For $\ell \in \mathbb{N}$, let

$$\mathfrak{g} = \mathfrak{sl}(\ell+1, \mathbb{C}),$$

a simple Lie algebra of the type A_ℓ. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$ and R the corresponding root system. Fix a basis $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ of R. Then we have the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. By R_+ and R_- we denote sets of positive and negative roots, and let θ be the maximal root. Then $\langle x, y\rangle = \operatorname{tr} xy$ is a normalized invariant bilinear form on $\mathfrak{g}$; via $\langle \cdot,\cdot\rangle$ we have an identification $\mathfrak{h} \cong \mathfrak{h}^*$. For each root α fix a root vector $x_\alpha \in \mathfrak{g}_\alpha$.

Let $\{\omega_1, \dots, \omega_\ell\}$ be the set of fundamental weights of $\mathfrak{g}$, $\langle \omega_i, \alpha_j\rangle = \delta_{ij}$, $i, j = 1, \dots, \ell$. Denote by $Q = \sum_{i=1}^{\ell} \mathbb{Z}\alpha_i$ the root lattice, and by $P = \sum_{i=1}^{\ell} \mathbb{Z}\omega_i$ the weight lattice of $\mathfrak{g}$.

Denote by $\tilde{\mathfrak{g}}$ the associated affine Lie algebra

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}\big[t, t^{-1}\big] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Set $x(j) = x \otimes t^j$ for $x \in \mathfrak{g}$, $j \in \mathbb{Z}$. Commutation relations are then given by

$$\begin{aligned}
&\big[x(i), y(j)\big] = [x, y](i+j) + i\langle x, y\rangle \delta_{i+j,0} c,\\
&[c, \tilde{\mathfrak{g}}] = 0,\\
&\big[d, x(j)\big] = jx(j).
\end{aligned}$$

For an element $x \in \mathfrak{g}$ form a formal series $x(z) = \sum_{n\in\mathbb{Z}} x(n) z^{-n-1}$.

Set $\mathfrak{h}^e = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$, $\tilde{\mathfrak{n}}_\pm = \mathfrak{g} \otimes t^{\pm1}\mathbb{C}[t^{\pm1}] \oplus \mathfrak{n}_\pm$. Then we also have the triangular decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h}^e \oplus \tilde{\mathfrak{n}}_+$.

Let $\hat{\Pi} = \{\alpha_0, \alpha_1, \dots, \alpha_\ell\} \subset (\mathfrak{h}^e)^*$ be the set of simple roots of $\tilde{\mathfrak{g}}$. Usual extensions of bilinear forms $\langle \cdot,\cdot\rangle$ onto $\mathfrak{h}^e$ and $(\mathfrak{h}^e)^*$ are denoted by the same symbols (we take $\langle c, d\rangle = 1$). Define fundamental weights $\Lambda_i \in (\mathfrak{h}^e)^*$ by $\langle \Lambda_i, \alpha_j\rangle = \delta_{ij}$ and $\Lambda_i(d) = 0$, $i, j = 0, \dots, \ell$.

Let V be a highest weight module for affine Lie algebra $\tilde{\mathfrak{g}}$. Then V is generated by a highest weight vector v_Λ, for some $\Lambda \in (\mathfrak{h}^e)^*$, such that

$$\begin{aligned}
&h \cdot v_\Lambda = \Lambda(h) v_\Lambda, \quad \text{for } h \in \mathfrak{h}^e,\\
&x \cdot v_\Lambda = 0, \quad \text{for } x \in \tilde{\mathfrak{n}}_+.
\end{aligned}$$

Module V is a direct sum of weight subspaces $V_\mu = \{v \in V \mid h \cdot V = \mu(h) v \text{ for } h \in \mathfrak{h}^e\}$, $\mu \in (\mathfrak{h}^e)^*$.

Standard (i.e. integrable highest weight) $\tilde{\mathfrak{g}}$-module $L(\Lambda)$ is the irreducible highest weight module with dominant integral highest weight Λ, i.e.

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell,$$

where $k_i \in \mathbb{Z}_+$, $i = 0, \dots, \ell$. The central element c acts on $L(\Lambda)$ as multiplication by scalar

$$k = \Lambda(c) = k_0 + k_1 + \cdots + k_\ell,$$

which is called the level of the module $L(\Lambda)$.

3. Feigin–Stoyanovsky's type subspace

Fix a fundamental weight

$$\omega = \omega_m,$$

for some $m \in \{1, \dots, \ell\}$. Set

$$\Gamma = \{\alpha \in R \mid \langle \alpha, \omega \rangle = 1\}.$$

Then we have the induced $\mathbb{Z}$-gradation of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \tag{4}$$

where

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \sum_{\langle \alpha, \omega \rangle = 0} \mathfrak{g}_\alpha,$$

$$\mathfrak{g}_{\pm 1} = \sum_{\alpha \in \pm\Gamma} \mathfrak{g}_\alpha.$$

Subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are commutative, and $\mathfrak{g}_0$ acts on them by adjoint action. The subalgebra $\mathfrak{g}_0$ is reductive with semisimple part $\mathfrak{l}_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$ of the type $A_{m-1} \times A_{\ell-m}$; as a root basis one can take $\{\alpha_1, \dots, \alpha_{m-1}\} \cup \{\alpha_{m+1}, \dots, \alpha_\ell\}$, and the center is equal to $\mathbb{C}\omega$.

Basis of the subalgebra $\mathfrak{g}_1$ can be identified with the set of roots Γ. We call elements $\gamma \in \Gamma$ *colors* and the set Γ *the set of colors*. For $\omega = \omega_m$, the set of colors is

$$\Gamma = \{\gamma_{ij} \mid i = 1, \dots, m;\ j = m, \dots, \ell\}$$

where

$$\gamma_{ij} = \alpha_i + \cdots + \alpha_m + \cdots + \alpha_j. \tag{5}$$

The maximal root θ is equal to $\gamma_{1\ell}$. The set of colors Γ can be pictured as a rectangle with row-indices $1, \dots, m$ and column-indices $m, \dots, \ell$, like in Fig. 2.

Similarly, one also has the induced $\mathbb{Z}$-gradation of affine Lie algebra $\tilde{\mathfrak{g}}$:

$$\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

$$\tilde{\mathfrak{g}}_{\pm 1} = \mathfrak{g}_{\pm 1} \otimes \mathbb{C}[t, t^{-1}],$$

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1.$$

As above, $\tilde{\mathfrak{g}}_{-1}$ and $\tilde{\mathfrak{g}}_1$ are commutative subalgebras, and $\tilde{\mathfrak{g}}_1$ is a $\tilde{\mathfrak{g}}_0$-module.

For a dominant integral weight Λ, we define a *Feigin–Stoyanovsky's type subspace*

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1) \cdot v_\Lambda \subset L(\Lambda).$$

Our objective is to find a combinatorial basis of $W(\Lambda)$. Set

$$\tilde{\mathfrak{g}}_1^+ = \tilde{\mathfrak{g}}_1 \cap \tilde{\mathfrak{n}}_+, \qquad \tilde{\mathfrak{g}}_1^- = \tilde{\mathfrak{g}}_1 \cap \tilde{\mathfrak{n}}_-.$$

Then we have

$$W(\Lambda) = U\left(\tilde{\mathfrak{g}}_1^-\right) \cdot v_\Lambda.$$

By Poincaré–Birkhoff–Witt theorem, we have a spanning set of $W(\Lambda)$ consisting of monomial vectors

$$\left\{x_{\gamma_t}(-r_t)\cdots x_{\gamma_2}(-r_2)x_{\gamma_1}(-r_1)v_\Lambda \;\middle|\; t \in \mathbb{Z}_+;\ \gamma_j \in \Gamma,\ r_j \in \mathbb{N}\right\}. \tag{6}$$

In the end, we say a few words about notation. Set

$$\tilde{\Gamma} = \left\{x_\gamma(-r) \;\middle|\; \gamma \in \Gamma, r \in \mathbb{Z}\right\}, \qquad \tilde{\Gamma}^- = \left\{x_\gamma(-r) \;\middle|\; \gamma \in \Gamma, r \in \mathbb{N}\right\}.$$

Since the subalgebra $\tilde{\mathfrak{g}}_1$ is commutative, we have $U(\tilde{\mathfrak{g}}_1) \cong \mathbb{C}[\tilde{\Gamma}]$ and $U(\tilde{\mathfrak{g}}_1^-) \cong \mathbb{C}[\tilde{\Gamma}^-]$. Hence elements of the spanning set (6) can be identified with monomials from $\mathbb{C}[\tilde{\Gamma}^-]$. Because of this we often refer to elements of $\tilde{\Gamma}^-$ as to *variables*, *elements* or *factors* of a monomial.

Monomials from $\mathbb{C}[\tilde{\Gamma}]$ can be identified with *colored partitions*. Let $\pi : \tilde{\Gamma} \to \mathbb{Z}_+$ be a colored partition (cf. [22, Section 3]). The corresponding monomial $x(\pi) \in \mathbb{C}[\tilde{\Gamma}]$ is

$$x(\pi) = x_{\gamma_t}(-r_t)^{\pi(x_{\gamma_t}(-r_t))} \cdots x_{\gamma_1}(-r_1)^{\pi(x_{\gamma_1}(-r_1))}.$$

From this identification we take notation $x(\pi)$ for the monomials from $\mathbb{C}[\tilde{\Gamma}]$. It will be convenient to define some new monomials by using this identification. Also, our combinatorial conditions for the basis elements will be written in terms of exponents $\pi(x_\gamma(-r))$, which gives a parametrization of the basis by a generalization of the notion of $(k, \ell+1)$-admissible configurations from [8].

4. Order on the set of monomials

We introduce a linear order on the set of monomials.

On the weight and root lattice, we have an order $\prec$ defined in the standard way: for $\mu, \nu \in P$ set $\mu \prec \nu$ if $\mu - \nu$ is an integral linear combination of simple roots α_i, $i = 1, \ldots, \ell$, with non-negative coefficients.

Next, we define a linear order $<$ on the set of colors Γ which is an extension of the order $\prec$. For elements of Γ, $\gamma_{i'j'} \prec \gamma_{ij}$ is equivalent to saying that $i' \geqslant i$ and $j' \leqslant j$. The order $<$ on Γ is defined in the following way:

$$\gamma_{i'j'} < \gamma_{ij} \quad \text{if} \begin{cases} i' > i & \text{or} \\ i' = i, \ j' < j. \end{cases}$$

It is clear that this is a linear order on the set of colors.

On the set of variables $\tilde{\Gamma}$ we define a linear order $<$ so that we compare degrees first, and then colors of variables:

$$x_\alpha(-r) < x_\beta(-r') \quad \text{if} \begin{cases} -r < -r' & \text{or} \\ r = r' & \text{and} \quad \alpha < \beta. \end{cases}$$

We assume that the variables in monomials from $\mathbb{C}[\tilde{\Gamma}]$ are sorted ascendingly from left to right. An order $<$ on the set of monomials is defined as a lexicographic order, where we compare variables from right to left (from the greatest to the lowest one). If $x(\pi)$ and $x(\pi')$ are two monomials,

$$x(\pi) = x_{\gamma_t}(-r_t)x_{\gamma_{t-1}}(-t_{t-1})\cdots x_{\gamma_2}(-r_2)x_{\gamma_1}(-r_1),$$
$$x(\pi') = x_{\gamma'_s}\big(-r'_s\big)x_{\gamma'_{s-1}}\big(-r'_{s-1}\big)\cdots x_{\gamma'_2}\big(-r'_2\big)x_{\gamma'_1}\big(-r'_1\big),$$

then $x(\pi) < x(\pi')$ if there exist $i_0 \in \mathbb{N}$ so that $x_{\gamma_i}(-r_i) = x_{\gamma'_i}(-r'_i)$, for all $i < i_0$, and either $i_0 = t+1 \leqslant s$ or $x_{\gamma_{i_0}}(-r_{i_0}) < x_{\gamma'_{i_0}}(-r'_{i_0})$.

This monomial order is compatible with multiplication:

Proposition 1. *(See [26].) Let*

$$x(\pi_1) \leqslant x(\mu_1) \quad \textit{and} \quad x(\pi_2) \leqslant x(\mu_2).$$

Then

$$x(\pi_1)x(\pi_2) \leqslant x(\mu_1)x(\mu_2),$$

and if one of the first two inequalities is strict, then the last one is also strict.

For a monomial $x(\pi) \in \mathbb{C}[\tilde{\Gamma}]$, we also define a degree and a shape of $x(\pi)$. A *degree* of a monomial is equal to the sum of degrees of its variables. For

$$x(\pi) = x_{\gamma_t}(-r_t)x_{\gamma_{t-1}}(-r_{t-1})\cdots x_{\gamma_2}(-r_2)x_{\gamma_1}(-r_1),$$

its degree is equal to $-r_1 - r_2 - \cdots - r_t$. A *shape* of a monomial is obtained from its colored partition by forgetting colors and considering only degrees of factors. More precisely, for a monomial $x(\pi)$ and its partition $\pi : \{x_\gamma(-r) \mid \gamma \in \Gamma,\ r \in \mathbb{Z}\} \to \mathbb{Z}_+$, the corresponding shape will be

$$s_\pi : \mathbb{Z} \to \mathbb{Z}_+,$$
$$s_\pi(r) = \sum_{\gamma \in \Gamma} \pi\big(x_\gamma(-r)\big).$$

A linear order can also be defined on the set of shapes; we'll say that $s_\pi < s_{\pi'}$ if there exists $r_0 \in \mathbb{Z}$ such that $s_\pi(r) = s_{\pi'}(r)$ for $r < r_0$ and $s_\pi(r_0) < s_{\pi'}(r_0)$.

In the end, for the sake of simplicity, we introduce the following notation:

$$x_{ij}(-r) = x_{\gamma_{ij}}(-r),$$

for $\gamma_{ij} \in \Gamma$, $r \in \mathbb{N}$.

5. Vertex operator construction

We use the vertex operator algebra construction of basic $\tilde{\mathfrak{g}}$-modules (i.e. standard $\tilde{\mathfrak{g}}$-modules of level 1) from [12,25]. We'll sketch this construction here, details can be found in [7,13] or [17].

Consider tensor products

$$V_P = M(1) \otimes \mathbb{C}[P],$$

$$V_Q = M(1) \otimes \mathbb{C}[Q];$$

where $M(1)$ is the Fock space for the Heisenberg subalgebra $\hat{\mathfrak{h}}_{\mathbb{Z}} = \sum_{n\in\mathbb{Z}\setminus\{0\}} \mathfrak{h} \otimes t^n \oplus \mathbb{C}c$, and $\mathbb{C}[P]$ and $\mathbb{C}[Q]$ are group algebras of the weight and root lattice with bases $\{e^\lambda \mid \lambda \in P\}$, and $\{e^\alpha \mid \alpha \in Q\}$, respectively. We identify group elements $e^\lambda = 1 \otimes e^\lambda \in V_P$.

Space V_Q has a structure of vertex operator algebra and V_P is a module for this algebra. Vertex operators corresponding to the group elements $e^\lambda \in \mathbb{C}[P]$ are defined as follows:

$$Y\big(e^\lambda, z\big) = E^-(-\lambda, z)E^+(-\lambda, z) \otimes e^\lambda z^\lambda \epsilon(\lambda, \cdot), \tag{7}$$

where $e^\lambda = 1 \otimes e^\lambda$ is a multiplication operator, $\epsilon_\lambda = 1 \otimes \epsilon(\lambda, \cdot)$ and $\epsilon(\cdot,\cdot)$ is a 2-cocycle (cf. [7]), operator $z^\lambda = 1 \otimes z^\lambda$, $z^\lambda \cdot e^\mu = e^\mu z^{\langle\lambda,\mu\rangle}$ and

$$E^\pm(\lambda, z) = \exp\Bigg(\sum_{m\geqslant 1} \lambda(\pm m) z^{\mp m}/\pm m\Bigg),$$

for $\lambda \in P$.

By using vertex operators, one can define the structure of $\tilde{\mathfrak{g}}$-module on V_P. For $\alpha \in R$ set

$$x_\alpha(z) = Y\big(e^\alpha, z\big),$$

for a properly chosen root vector x_α. Heisenberg subalgebra acts on the Fock space $M(1)$ and c acts as identity. In this way V_Q and $V_Q e^{\omega_j}$, $j = 1, \dots, \ell$ become standard $\tilde{\mathfrak{g}}$-modules of level 1 with highest weight vectors $v_0 = 1$ and $v_j = e^{\omega_j}$, $j = 1, \dots, \ell$,

$$L(\Lambda_0) \cong V_Q \quad \text{and} \quad L(\Lambda_j) \cong V_Q e^{\omega_j}, \ j = 1, \dots, \ell,$$

and

$$V_P \cong L(\Lambda_0) \oplus L(\Lambda_1) \oplus \cdots \oplus L(\Lambda_\ell).$$

We will also be using intertwining operators

$$\mathcal{Y}\big(e^\lambda, z\big) = Y\big(e^\lambda, z\big) e^{i\pi\lambda} c(\cdot, \lambda),$$

for $\lambda \in P$, where $c(\cdot,\cdot)$ is a commutator map (cf. [7]). If $\lambda + \omega_i \equiv \omega_j \mod Q$, then

$$\mathcal{Y}\big(e^\lambda, z\big) : L(\Lambda_i) \to L(\Lambda_j)\{z\}, \tag{8}$$

where $L(\Lambda_j)\{z\}$ is a space of formal series with coefficients in $L(\Lambda_j)$. Here, for convenience, we set $\omega_0 = 0$. Also, for a suitable choice of $\mu \in P$, the operators $\mathcal{Y}(e^\mu, z)$ will commute with $\tilde{\mathfrak{g}}_1$ (cf. [26, Section 8]).

Standard modules of level $k > 1$ can be viewed, by the complete reducibility, as submodules of tensor products of basic modules;

$$L(\Lambda) \subset L(\Lambda_0)^{\otimes k_0} \otimes \cdots \otimes L(\Lambda_\ell)^{\otimes k_\ell},$$

if $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell$, $k = k_0 + k_1 + \cdots + k_\ell$. Highest weight vector of $L(\Lambda)$ is

$$v_\Lambda = v_0^{\otimes k_0} \otimes \cdots \otimes v_\ell^{\otimes k_\ell}.$$

This all can be imbedded into $V_P^{\otimes k}$. One can also define vertex operators corresponding to elements

$$u_1 \otimes \cdots \otimes u_k \in V_P^{\otimes k}$$

as tensor products of vertex operators on the appropriate tensor factors:

$$Y(u_1 \otimes \cdots \otimes u_k, z) = Y(u_1, z) \otimes \cdots \otimes Y(u_k, z).$$

Then $V_Q^{\otimes k} = L(\Lambda_0)^{\otimes k}$ becomes vertex operator algebra with the vacuum vector $1 = 1 \otimes \cdots \otimes 1$, and $V_P^{\otimes k}$ with its subspaces $L(\Lambda_0)^{\otimes k_0} \otimes \cdots \otimes L(\Lambda_\ell)^{\otimes k_\ell}$ become modules for this algebra [7,11].

6. Operator $e(\omega)$

For $\lambda \in P$, e^λ denotes multiplication operator $1 \otimes e^\lambda$ in $V_P = M(1) \otimes \mathbb{C}[P]$. Set

$$e(\lambda) = e^\lambda \epsilon(\cdot, \lambda), \qquad e(\lambda) : V_P \to V_P.$$

Clearly, $e(\lambda)$ is a linear bijection. Its restrictions on basic modules are bijections from one basic module $L(\Lambda_i)$ onto another basic module $L(\Lambda_{i'})$. From the definition of vertex operators $Y(e^\alpha, z)$ for $\alpha \in R$ one gets the following commutation relation

$$Y\big(e^\alpha, z\big)e(\lambda) = e(\lambda) z^{\langle \lambda, \alpha \rangle} Y\big(e^\alpha, z\big),$$

or, in terms of components,

$$x_\alpha(r) e(\lambda) = e(\lambda) x_\alpha\big(r + \langle \lambda, \alpha \rangle\big), \quad r \in \mathbb{Z}. \tag{9}$$

For standard modules of level $k > 1$, one defines operator $e(\lambda)$ on the tensor product of basic modules as a tensor product of the appropriate operators

$$e(\lambda) = e(\lambda) \otimes \cdots \otimes e(\lambda) : \bigotimes_{j=1}^{k} L(\Lambda_{i_j}) \to \bigotimes_{j=1}^{k} L(\Lambda_{i'_j}).$$

Operator $e(\lambda)$ is again a linear bijection, and relation (9) still holds.

For $\lambda = \omega$ and $\gamma \in \Gamma$, the relation (9) becomes

$$x_\gamma(r) e(\omega) = e(\omega) x_\gamma(r + 1).$$

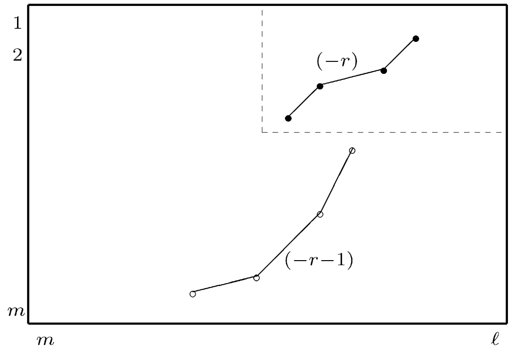

Fig. 4. Difference conditions – the level 1 case.

More generally, for a monomial $x(\pi) \in \mathbb{C}[\tilde{\Gamma}]$,

$$x(\pi)e(\omega) = e(\omega)x\big(\pi^+\big), \tag{10}$$

where $x(\pi^+) \in \mathbb{C}[\tilde{\Gamma}]$ denotes a monomial corresponding to partition

$$\pi^+\big(x_\gamma(r+1)\big) = \pi\big(x_\gamma(r)\big). \tag{11}$$

7. The case $k = 1$

Here we briefly recall the main results from [26] concerning a basis of a Feigin–Stoyanovsky's type subspace of the standard module $L(\Lambda_i)$. It is described in terms of difference and initial conditions.

A monomial $x(\pi)$ *satisfies difference conditions for* $L(\Lambda_i)$ if the following holds:

- if $x(\pi)$ contains elements $x_{pq}(-r)$ and $x_{p'q'}(-r)$, and $\gamma_{p'q'} \leqslant \gamma_{pq}$, then $p' > p$ and $q' < q$,
- if $x(\pi)$ contains elements $x_{pq}(-r)$ and $x_{p'q'}(-r-1)$, then $p' > p$ or $q' < q$.

From this we conclude that colors of elements of the same degree $-r$ inside $x(\pi)$ make a descending sequence; appropriate row-indices strictly increase, while column-indices strictly decrease. Colors of elements of degree $-r-1$ also form a decreasing sequence which is placed below or on the left of the minimal color of elements of degree $-r$ (see Fig. 4).

A monomial $x(\pi)$ *satisfies initial conditions for* $L(\Lambda_i)$ if it doesn't contain any element $x_{pq}(-1) \in \tilde{\mathfrak{g}}_1$ such that $x_{pq}(-1)v_i = 0$. In the case $0 \leqslant i \leqslant m$, $x_{pq}(-1)v_i = 0$ if $p \leqslant i$; in the case $m \leqslant i \leqslant \ell$, $x_{pq}(-1)v_i = 0$ if $q \geqslant i$. Hence, initial conditions imply that the sequence of colors of elements of degree -1 in a monomial $x(\pi)$ lies below the i-th row (if $0 \leqslant i \leqslant m$), or on the left of the i-th column (for $m \leqslant i \leqslant \ell$); see Fig. 5.

8. Difference and initial conditions

8.1. Relations

As in the case $k = 1$ (cf. [26]) we first find relations between fields $x_\gamma(z)$, $\gamma \in \Gamma$ on the standard module $L(\Lambda)$. By equating coefficients of powers of z in these relations, we get relations between monomials. Then we then identify the minimal monomial among these, the so called *leading term* of a relation, and exclude from the spanning set all monomials that contain leading terms. Difference conditions combinatorially describe such monomials in terms of exponentials of factors $x_{pq}(r) \in \tilde{\Gamma}^-$.

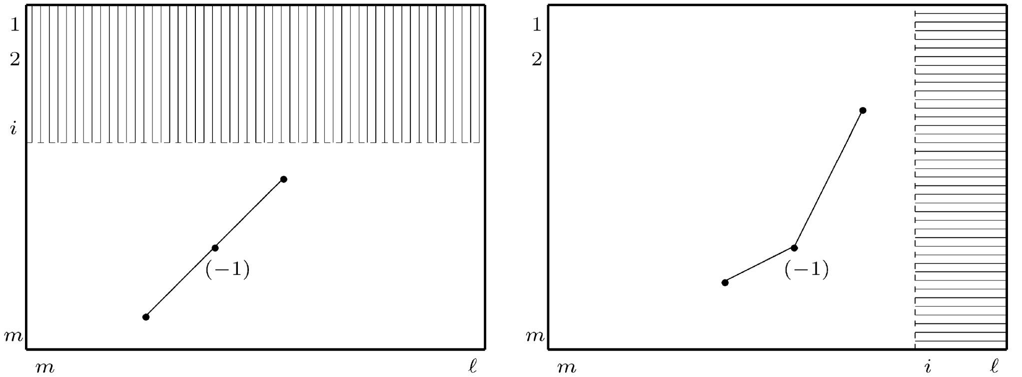

Fig. 5. Initial conditions – the level 1 case.

To obtain relations, we start from a consequence of Frenkel–Kac–Segal vertex operator formula for $L(\Lambda)$

$$x_\theta(z)^{k+1} = 0. \tag{12}$$

Since $\tilde{\mathfrak{g}}_1$ is commutative, the product of fields on the left side is a vertex operator corresponding to the element $x_\theta(-1)^{k+1}1$ in $L(k\Lambda_0) \subset L(\Lambda_0)^{\otimes k}$, and the relation above is equivalent to

$$x_\theta(-1)^{k+1}1 = 0$$

(cf. [16,20,21]; see also [17]).

By acting on this relation with $y \in \mathfrak{l}_0$, one gets

$$\begin{aligned} 0 = y \cdot x_\theta(-1)^{k+1}1 &= \left[y, x_\theta(-1)^{k+1}\right]1 + x_\theta(-1)^{k+1} y \cdot 1 \\ &= \left[y, x_\theta(-1)^{k+1}\right]1. \end{aligned} \tag{13}$$

A commutator $[y, x_\theta(-1)^{k+1}]$ is again an element of $S(\mathfrak{g}_1 \otimes t^{-1}) \subset U(\tilde{\mathfrak{g}}_1)$ and hence from the above equality we obtain another relation between fields $x_\gamma(z)$, $\gamma \in \Gamma$.

Thus we have to study a subrepresentation $V \subset S(\mathfrak{g}_1 \otimes t^{-1})$ of $\mathfrak{l}_0$ generated by a singular vector $x_\theta(-1)^{k+1} = x_{1\ell}(-1)^{k+1}$,

$$V = U(\mathfrak{l}_0) \cdot x_{1\ell}(-1)^{k+1} \subset S\left(\mathfrak{g}_1 \otimes t^{-1}\right) \subset U(\tilde{\mathfrak{g}}_1). \tag{14}$$

The action of $\mathfrak{l}_0$ is defined by the adjoint action of $\mathfrak{l}_0$ on $\tilde{\mathfrak{g}}_1$ (i.e. on $\mathfrak{g}_1$).

Generally, the algebra $\mathfrak{l}_0$ is a direct sum of two simple subalgebras,

$$\mathfrak{l}_0 = \mathfrak{l}_0' \oplus \mathfrak{l}_0'',$$

with the first one being of type A_{m-1}, and the second one of type $A_{\ell-m}$. A basis of $\mathfrak{g}$ can be chosen such that the adjoint action of $\mathfrak{l}_0$ is given by:

$$\begin{aligned} [x_{-\alpha_i}, x_{\gamma_{pq}}] &= \delta_{ip} x_{\gamma_{p+1,q}}, \\ [x_{\alpha_i}, x_{\gamma_{pq}}] &= \delta_{i,p-1} x_{\gamma_{p-1,q}}, \end{aligned} \tag{15}$$

if $i = 1, \ldots, m-1$,

$$[x_{-\alpha_i}, x_{\gamma_{pq}}] = \delta_{iq} x_{\gamma_{p,q-1}},$$
$$[x_{\alpha_i}, x_{\gamma_{pq}}] = \delta_{i,q+1} x_{\gamma_{p,q+1}}, \tag{16}$$

if $i = m+1, \ldots, \ell$. One can say that the first subalgebra $\mathfrak{l}_0'$ acts by changing the row-index, and the second subalgebra $\mathfrak{l}_0''$ by changing the column-index of elements x_γ, $\gamma \in \Gamma$.

From (5) we immediately see that $\theta = \gamma_{1\ell} = \omega_1 + \omega_\ell$ (cf. [27]), and hence the vector $x_{1\ell}(-1)^{k+1}$ is the highest weight vector in V of the weight $(k+1)\theta = (k+1)(\omega_1 + \omega_\ell)$ for $\mathfrak{l}_0$. The highest weight representation of $\mathfrak{l}_0$ can be obtained in another way, by taking tensor product of highest weight representations of its simple subalgebras. Let V_1 be a highest weight representation of $\mathfrak{l}_0'$ with the highest weight $(k+1)\omega_1$. It can be realized as the subspace of homogeneous polynomials of degree $k+1$ in m variables,

$$V_1 = S^{k+1}(x_1, \ldots, x_m) \subset S(x_1, \ldots, x_m).$$

The action of $\mathfrak{l}_0'$ is given for generators $x_{-\alpha_i}, x_{\alpha_i}$; $i = 1, \ldots, m-1$, by

$$x_{-\alpha_i} \mapsto x_{i+1} \frac{\partial}{\partial x_i}, \qquad x_{\alpha_i} \mapsto x_i \frac{\partial}{\partial x_{i+1}}.$$

Similarly, let V_2 a highest weight representation of $\mathfrak{l}_0''$ with the highest weight $(k+1)\omega_\ell$. It can be realized as the space of homogeneous polynomials of degree $k+1$ in $\ell - m + 1$ variables

$$V_2 = S^{k+1}(x_m, \ldots, x_\ell) \subset S(x_m, \ldots, x_\ell),$$

where the action is defined for generators $x_{-\alpha_i}, x_{\alpha_i}$; $i = m+1, \ldots, \ell$, by

$$x_{-\alpha_i} \mapsto x_{i-1} \frac{\partial}{\partial x_i}, \qquad x_{\alpha_i} \mapsto x_i \frac{\partial}{\partial x_{i-1}}.$$

Then

$$V \cong V_1 \otimes V_2.$$

A highest weight vector in V_1 is x_1^{k+1}. From the character formula (cf. [27]) one sees that monomials

$$x_{i_1} x_{i_2} \cdots x_{i_{k+1}}, \quad 1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{k+1} \leqslant m,$$

constitute a basis of V_1 made of weight vectors. Similarly, a monomial x_ℓ^{k+1} is a highest weight vector of V_2, and the basis constitutes of monomials

$$x_{j_1} x_{j_2} \cdots x_{j_{k+1}}, \quad \ell \geqslant j_1 \geqslant j_2 \geqslant \cdots \geqslant j_{k+1} \geqslant m.$$

Hence, the basis of $V_1 \otimes V_2$ is constituted by tensor products

$$x_{i_1} x_{i_2} \cdots x_{i_{k+1}} \otimes x_{j_1} x_{j_2} \cdots x_{j_{k+1}},$$

for $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{k+1} \leqslant m$ and $m \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_{k+1} \leqslant \ell$. The goal is to determine the corresponding basis of V which, in turn, will give us relations on $L(\Lambda)$.

Denote by $\mathbf{p} = (p_1, \ldots, p_{k+1})$, $\mathbf{q} = (q_1, \ldots, q_{k+1})$ sequences of $k+1$ elements from $\{1, \ldots, \ell\}$, and by $\bar{\mathbf{p}} = \{p_1, \ldots, p_{k+1}\}$, $\bar{\mathbf{q}} = \{q_1, \ldots, q_{k+1}\}$ the corresponding multisets.

Lemma 2. *The basis of V is parameterized by $(k+1)$-tuples $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{k+1} \leqslant m$ and $m \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_{k+1} \leqslant \ell$; the corresponding weight vectors are*

$$\sum_{\substack{\bar{\mathbf{p}}=\{i_1,\ldots,i_{k+1}\} \\ \bar{\mathbf{q}}=\{j_1,\ldots,j_{k+1}\}}} C_{\mathbf{pq}} x_{p_1q_1}(-1) x_{p_2q_2}(-1) \cdots x_{p_{k+1}q_{k+1}}(-1), \tag{17}$$

where coefficients $C_{\mathbf{pq}}$ are positive integers.

Proof. One can first act on the highest weight vectors by the $x_{-\alpha_i}$'s, $i \in \{m+1, \ldots, \ell\}$, that would send $x_1^{k+1} \otimes x_\ell^{k+1} \in V_1 \otimes V_2$ to

$$x_1^{k+1} \otimes x_{j_1} x_{j_2} \cdots x_{j_{k+1}}. \tag{18}$$

Acting with the same $x_{-\alpha_i}$'s on the corresponding highest weight vector $x_{1\ell}^{k+1}(-1) \in V$ will give

$$x_{1j_1}(-1) x_{1j_2}(-1) \cdots x_{1j_{k+1}}(-1), \tag{19}$$

or, more precisely, some multiple of that monomial with positive integer coefficient. Next, we act on vectors (18) and (19) with $x_{-\alpha_{i_{k+1}-1}} x_{-\alpha_{i_{k+1}-2}} \cdots x_{-\alpha_1}$; on one side, from (18), we get

$$x_1^k x_{i_{k+1}} \otimes x_{j_1} x_{j_2} \cdots x_{j_{k+1}} \in V_1 \otimes V_2,$$

and on the other, from (19), we obtain

$$\sum_{r=1}^{k+1} x_{1j_1}(-1) \cdots x_{i_{k+1}j_r}(-1) \cdots x_{1j_{k+1}}(-1) \in V.$$

In the second vector for every occurrence of index 1 at the first place in (19), we have a monomial where this index was changed to i_{k+1}. Monomials

$$x_{1j_1}(-1) \cdots x_{1j_{r-1}}(-1) x_{1j_{r+1}}(-1) \cdots x_{1j_{k+1}}(-1)$$

correspond to vectors

$$x_1^k \otimes x_{j_1} \cdots x_{j_{r-1}} x_{j_{r+1}} \cdots x_{j_{k+1}}$$

from the appropriate module $V_1' \otimes V_2'$ of the highest weight $k\theta$. Next, we act on the vector and monomials above by $x_{-\alpha_{i_k-1}} x_{-\alpha_{i_k-2}} \cdots x_{-\alpha_1}$ that would change one occurrence of index 1 into i_k. Since $x_{-\alpha_s} \cdot x_{i_{k+1}j} = 0$ for $s < i_{k+1}$, the proof follows by induction on k. $\square$

Like in (13), from (12) and (17) we obtain

$$\sum_{\substack{\bar{\mathbf{p}}=\{i_1,\ldots,i_{k+1}\} \\ \bar{\mathbf{q}}=\{j_1,\ldots,j_{k+1}\}}} C_{\mathbf{pq}} x_{p_1q_1}(-1) x_{p_2q_2}(-1) \cdots x_{p_{k+1}q_{k+1}}(-1) 1 = 0 \tag{20}$$

in $L(k\Lambda_0) \subset L(\Lambda_0)^{\otimes k}$.

From this we obtain the following family of relations between corresponding vertex operators on $L(\Lambda)$:

Proposition 3. *For* $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{k+1} \leqslant m$ *and* $m \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_{k+1} \leqslant \ell$*, we have*

$$\sum_{\substack{\bar{\mathbf{p}}=\{i_1,\dots,i_{k+1}\}\\ \bar{\mathbf{q}}=\{j_1,\dots,j_{k+1}\}}} C_{\mathbf{pq}} x_{p_1q_1}(z)x_{p_2q_2}(z)\cdots x_{p_{k+1}q_{k+1}}(z) = 0, \tag{21}$$

where $C_{\mathbf{pq}} \in \mathbb{N}$.

8.2. Leading terms

Fix one choice $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{k+1} \leqslant m$ and $m \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_{k+1} \leqslant \ell$ and observe the corresponding relation (21). For every $n \geqslant k+1$, coefficients of powers z^{n-k-1} are infinite sums of monomials:

$$\sum_{\substack{r_1+\cdots+r_{k+1}=n\\ \bar{\mathbf{p}}=\bar{\mathbf{i}},\bar{\mathbf{q}}=\bar{\mathbf{j}}}} C_{pq} x_{p_1q_1}(-r_1)x_{p_2q_2}(-r_2)\cdots x_{p_{k+1}q_{k+1}}(-r_{k+1}) = 0. \tag{22}$$

In each such sum, we identify the minimal monomial in the lexicographical order defined in Section 4. We call this monomial *the leading term* of the relation. Because of the minimality, every monomial that contains a leading term can be excluded from the spanning set (cf. Proposition 4).

All monomials that appear in (22) are of the same length $k+1$ and of the same total degree $-n$. Hence we can consider only those monomials that are of *the minimal shape*, i.e. the ones in which degrees of factors differ for at most 1. The others will be greater than these.

Consider first the case $n = r(k+1)$, for some $r \in \mathbb{N}$. In this case, monomials of minimal shape will have all $k+1$ factors of the same degree $-r$. So we need to find the minimal possible monomial

$$x_{p_1q_1}(-r)x_{p_2q_2}(-r)\cdots x_{p_{k+1}q_{k+1}}(-r)$$

such that $\{p_1,\dots,p_{k+1}\} = \{i_1,\dots,i_{k+1}\}$ and $\{q_1,\dots,q_{k+1}\} = \{j_1,\dots,j_{k+1}\}$. Since all factors are of the same degree, the minimal monomial will be the one that has the minimal configuration of colors of its factors. If we assume that factors of a monomial are sorted ascendingly from left to right, this means that we have to choose the smallest possible color $\gamma_{p_{k+1}q_{k+1}}$ (the greatest color in a monomial), next the smallest possible color $\gamma_{p_kq_k}$, and so on.

Since the row and column-indexes of colors of monomial are fixed ($\{i_1,\dots,i_{k+1}\}$ and $\{j_1,\dots,j_{k+1}\}$, resp.) the greatest color will lie in the i_1-th row, and the smallest possible of them is $\gamma_{i_1j_1}$. The second greatest color lies in the i_2-th row, the smallest possible being $\gamma_{i_2j_2}$.[1] We proceed in the same manner, and obtain a monomial

$$x_{i_1j_1}(-r)\cdots x_{i_kj_k}(-r)x_{i_{k+1}j_{k+1}}(-r). \tag{23}$$

Consider now the configuration of the colors in (23). Each color $\gamma_{i_{t+1}j_{t+1}}$ is placed on the right of, or below, or diagonally on the right and below color $\gamma_{i_tj_t}$ (see Fig. 6). Consequently, we conclude that colors of the leading term lie on a diagonal path as pictured in Fig. 7.

[1] If $i_1 = i_2$, then $\gamma_{i_1j_1} \leqslant \gamma_{i_2j_2}$, but this shouldn't concern us, because in this way we'll certainly obtain the smallest possible choice of colors from the i_1-th row.

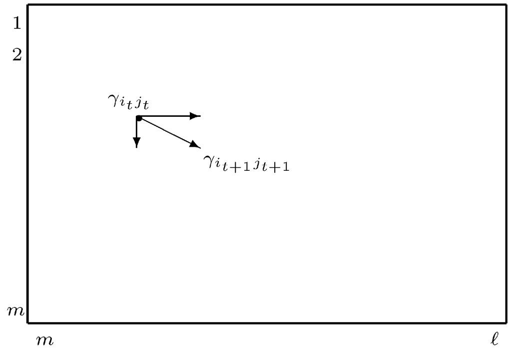

Fig. 6. Configuration of colors of leading terms.

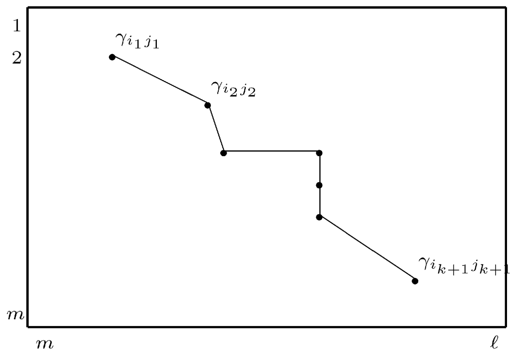

Fig. 7. Configuration of colors of leading terms.

Finally, we can describe the leading terms via exponents of factors. Let $a_{ij}^r = \pi(x_{ij}(-r))$ be an exponent of $x_{ij}(-r)$ in $x(\pi)$. Then the leading terms correspond to solutions of equations

$$a_{i_1 j_1}^r + a_{i_2 j_2}^r + \cdots + a_{i_s j_s}^r = k + 1, \tag{24}$$

where $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_s \leqslant m$ and $m \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_s \leqslant m$. Since exponents a_{ij}^r count the number of occurrences of the factor $x_{ij}(-r)$ in $x(\pi)$, we now don't allow repetitions of colors, $(i_\nu, j_\nu) \neq (i_{\nu+1} j_{\nu+1})$.

Consider next the case $n = r(k+1) + q$. For convenience, we change the setting a bit, so that $1 \leqslant i_1 \leqslant \cdots \leqslant i_q \leqslant i_{q+1} \leqslant \cdots \leqslant i_{k+1} \leqslant m$, $m \leqslant j_{q+1} \leqslant \cdots \leqslant j_{k+1} \leqslant j_1 \leqslant \cdots \leqslant j_q \leqslant \ell$. Monomials of minimal shape in this case are built of q factors of degree $-r-1$ and $k-q+1$ factors of degree $-r$. Since factors of degree $-r$ are greater than factors of degree $-r-1$, the leading term of (23) can be obtained so that first one chooses the smallest possible $(-r)$-part of a monomial, and then the smallest possible $(-r-1)$-part. Hence factors of degree $-r$ are placed in the last $k-q+1$ rows and the first $k-q+1$ columns, and in the rest are placed factors of degree $-r-1$:

$$(-r)\text{-part} \rightsquigarrow (i_{q+1}, i_{q+2}, \ldots, i_{k+1}),\ (j_{q+1}, j_{q+2}, \ldots, j_{k+1}),$$
$$(-r-1)\text{-part} \rightsquigarrow (i_1, i_2, \ldots, i_q),\ (j_1, j_2, \ldots, j_q).$$

We proceed as in the first case; the colors of elements of the same degree will lie on a diagonal path as before, and the configuration of colors is of the type pictured on Fig. 3.

In terms of exponents, this leading terms correspond to solutions of equations

$$a_{i_1 j_1}^{r+1} + \cdots + a_{i_t j_t}^{r+1} + a_{i_{t+1} j_{t+1}}^{r} + \cdots + a_{i_s j_s}^{r} = k + 1, \tag{25}$$

where $1 \leqslant i_1 \leqslant \cdots \leqslant i_t \leqslant i_{t+1} \leqslant \cdots \leqslant i_s \leqslant m$ and $m \leqslant j_{t+1} \leqslant \cdots \leqslant j_s \leqslant j_1 \leqslant \cdots \leqslant j_t \leqslant \ell$, and $a_{ij}^r = \pi(x_{ij}(-r))$, for all $r \in \mathbb{N}$. Again, in this notation, we assume that $(i_\nu, j_\nu) \neq (i_{\nu+1} j_{\nu+1})$ for $\nu \neq t$.

In the end, observe that solutions of (24) are also solutions of some equation of the type (25), simply take $t = 1$ and $i_1 = 1$, $j_1 = \ell$. Hence we can say that all leading terms correspond to solutions of Eqs. (25).

We'll say that a monomial $x(\pi) \in \mathbb{C}[\tilde{\Gamma}^-]$ satisfies *difference conditions* for $L(\Lambda)$, or shortly, that $x(\pi)$ *satisfies DC* for $L(\Lambda)$, if it doesn't contain a leading term of relations (22). More precisely, $x(\pi)$ satisfies difference conditions if

$$a_{i_1 j_1}^{r+1} + \cdots + a_{i_t j_t}^{r+1} + a_{i_{t+1} j_{t+1}}^{r} + \cdots + a_{i_s j_s}^{r} \leqslant k, \tag{26}$$

where $1 \leqslant i_1 \leqslant \cdots \leqslant i_t \leqslant i_{t+1} \leqslant \cdots \leqslant i_s \leqslant m$ and $m \leqslant j_{t+1} \leqslant \cdots \leqslant j_s \leqslant j_1 \leqslant \cdots \leqslant j_t \leqslant \ell$, $a_{ij}^r = \pi(x_{ij}(-r))$, for all $r \in \mathbb{N}$ and $(i_\nu, j_\nu) \neq (i_{\nu+1} j_{\nu+1})$ for $\nu \neq t$.

The following proposition follows by induction from Proposition 1 and the minimality of leading terms.

Proposition 4. *The set*

$$\left\{ x(\pi) v_\Lambda \,\middle|\, x(\pi) \text{ satisfies DC} \right\} \tag{27}$$

spans $W(\Lambda)$.

8.3. Initial conditions

By difference conditions, in a monomial $x(\pi)$, factors of degree $-r$, for $r > 1$, are restricted by factors of degree $-r-1$ and $-r+1$. Exceptions are factors of degree -1, which are restricted only "from below", by factors of degree -2. Initial conditions will play the part of restrictions "from above" on factors of degree -1.

In the case $k = 1$ there were no relations between monomials consisting of factors of degree -1, other than those already used for difference conditions. Thus, the initial conditions demanded only that there are no (-1)-factors that annihilate highest weight vector. However, when the level $k > 1$, generally, there are other relations between such monomials. These relations amount to relations for difference conditions but for modules of level lesser than k.

We'll say that a monomial $x(\pi) \in \mathbb{C}[\tilde{\Gamma}^-]$ satisfies *initial conditions* for $L(\Lambda)$, or shortly, that $x(\pi)$ *satisfies IC* for $L(\Lambda)$, if

$$a_{i_1 j_1}^1 + a_{i_2 j_2}^1 + \cdots + a_{i_t j_t}^1 \leqslant k_0 + k_1 + \cdots + k_{i_t - 1} + k_{j_1 + 1} + \cdots + k_\ell, \tag{28}$$

where $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_t \leqslant m$, $m \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_t \leqslant \ell$, $(i_\nu, j_\nu) \neq (i_{\nu+1}, j_{\nu+1})$ and $a_{ij}^1 = \pi(x_{ij}(-1))$. The sum on the right side of inequality is the sum of k_i's such that at least one $x_{i_\nu j_\nu}(-1)$ doesn't act as 0 on $L(\Lambda_i)$ (cf. Section 7).

Proposition 5. *The set*

$$\left\{ x(\pi) v_\Lambda \,\middle|\, x(\pi) \text{ satisfies IC and DC} \right\}$$

spans $W(\Lambda)$.

Proof. Assume that $x(\pi)$ doesn't satisfy some inequality of the type (28) and set $d=k_0+k_1+\cdots+k_{i_t-1}+k_{j_1+1}+\cdots+k_\ell+1$. Then $x(\pi)$ contains a monomial $x(\pi')$ consisting only of factors of degree -1, such that it also doesn't satisfy that inequality. Furthermore, one can assume that the length of $x(\pi')$ is equal to d. We'll show that we can find monomials $x(\pi'_1),\dots,x(\pi'_s)$ such that $x(\pi')<x(\pi'_i)$, $x(\pi'_i)$'s are of the same degree as $x(\pi')$ and

$$x(\pi')v_\Lambda=C_1x\bigl(\pi'_1\bigr)v_\Lambda+\cdots+C_sx\bigl(\pi'_s\bigr)v_\Lambda,$$

for some $C_i\in\mathbb{C}$. Upon multiplying them with the rest of $x(\pi)$, we obtain monomials $x(\pi_i)$ of the same degree as $x(\pi)$, such that $x(\pi)<x(\pi_i)$ (cf. Proposition 1) and

$$x(\pi)v_\Lambda=C_1x(\pi_1)v_\Lambda+\cdots+C_sx(\pi_s)v_\Lambda,$$

for some $C_i\in\mathbb{C}$. Let

$$x(\pi')=x_{i_1j_1}(-1)\cdots x_{i_dj_d}(-1),$$

$1\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_d\leqslant m$, $m\leqslant j_1\leqslant j_2\leqslant\cdots\leqslant j_d\leqslant\ell$. There are 2 possibilities: $d=k+1$ or $d\leqslant k$. If the first case, $d=k+1$, the initial condition is equivalent to difference condition and the statement follows from relation (22), Proposition 1 and the minimality of leading terms. In the second case, $d\leqslant k$, there exists at least one v_i in the tensor product $v_\Lambda=v_0^{\otimes k_0}\otimes v_1^{\otimes k_1}\otimes\cdots\otimes v_\ell^{\otimes k_\ell}$ that is annihilated by all factors of $x(\pi')$, and $k_i>0$. Set

$$\Lambda'=\sum_{i=0}^{i_d-1}k_i\Lambda_i+\sum_{i=j_1+1}^{\ell}k_i\Lambda_i,$$

$$\Lambda''=\sum_{i=i_d}^{j_1}k_i\Lambda_i=\Lambda-\Lambda'.$$

Denote by $v_{\Lambda'}$ and $v_{\Lambda''}$ highest weight vectors of standard modules $L(\Lambda')$ and $L(\Lambda'')$. Then, by the complete reducibility,

$$L(\Lambda)\subset L(\Lambda')\otimes L(\Lambda''),$$

$$v_\Lambda=v_{\Lambda'}\otimes v_{\Lambda''}.$$

Since all factors of $x(\pi')$ annihilate $v_{\Lambda''}$ (cf. Section 7), we have

$$x(\pi')v_\Lambda=\bigl(x(\pi')v_{\Lambda'}\bigr)\otimes v_{\Lambda''}.$$

Note that $L(\Lambda')$ is a module of level $k'<k$ and $d=k'+1$. From relations (22) for the module $L(\Lambda')$ we obtain monomials $x(\pi'_1),\dots,x(\pi'_s)$ of the same length and degree, such that $x(\pi')v_{\Lambda'}=C_1x(\pi'_1)v_{\Lambda'}+\cdots+C_sx(\pi'_s)v_{\Lambda'}$, $C_i\in\mathbb{C}$, and $x(\pi')<x(\pi'_i)$. Also from these relations, we see that colors of factors of monomials $x(\pi'_i)$ lie in the same rows and columns as colors of $x(\pi')$. Hence, all factors of $x(\pi'_i)$ also act as 0 on $v_{\Lambda''}$. Consequently,

$$x(\pi')v_\Lambda=C_1x\bigl(\pi'_1\bigr)v_\Lambda+\cdots+C_sx\bigl(\pi'_s\bigr)v_\Lambda.\qquad\square$$

9. A combinatorial reduction to the level 1 case

Difference and initial conditions for modules of level $k > 1$ can be restated in terms of difference and initial conditions for modules of level 1. We are going to prove

Theorem 6. *Let $L(\Lambda)$ be a standard module of level k with the highest weight vector $v_\Lambda = v_{i_1} \otimes \cdots \otimes v_{i_k}$, where v_{i_j} are the highest weight vectors of corresponding modules $L(\Lambda_{i_j})$ of level 1. If a monomial $x(\pi) \in \mathbb{C}[\tilde{\Gamma}^-]$ satisfies difference and initial conditions on $L(\Lambda)$, then there exists a factorization*

$$x(\pi) = x\big(\pi^{(1)}\big) \cdots x\big(\pi^{(k)}\big)$$

such that $x(\pi^{(j)})$ satisfies difference and initial conditions on $L(\Lambda_{i_j})$.

Proposition 9 will imply the converse of the theorem. Hence, we'll have an equivalence:

Corollary 7. *With notation as above, a monomial $x(\pi) \in \mathbb{C}[\tilde{\Gamma}^-]$ satisfies difference and initial conditions on $L(\Lambda)$ if and only if there exists a factorization*

$$x(\pi) = x\big(\pi^{(1)}\big) \cdots x\big(\pi^{(k)}\big)$$

such that $x(\pi^{(j)})$ satisfies difference and initial conditions on $L(\Lambda_{i_j})$.

9.1. Difference conditions; $\Lambda = k\Lambda_0$

We first prove Theorem 6 in the special case $\Lambda = k\Lambda_0$ and later for general Λ. In this case initial conditions (28) don't provide any additional relations for $L(k\Lambda_0)$ so we are only considering difference conditions on monomials.

First we recall a simple lemma from [26, Proposition 3] concerning the level 1 case:

Lemma 8. *If $x_\gamma(-j) < x_{\gamma'}(-j') < x_{\gamma''}(-j'')$ are such that monomials $x_\gamma(-j)x_{\gamma'}(-j')$ and $x_{\gamma'}(-j') \times x_{\gamma''}(-j'')$ satisfy difference conditions for level 1 modules, then so does $x_\gamma(-j)x_{\gamma''}(-j'')$, and consequently $x_\gamma(-j)x_{\gamma'}(-j')x_{\gamma''}(-j'')$.*

Define another order on the set of variables:

$$x_{ij}(-r) \sqsubset x_{i'j'}(-r') \quad \text{if} \begin{cases} -r \leqslant -r' - 2, \\ -r = -r' - 1; \quad i > i' \text{ or } j < j', \\ -r = -r'; \quad i > i' \text{ and } j < j'. \end{cases} \tag{29}$$

This is equivalent to saying that $x_{ij}(-r) < x_{i'j'}(-r')$ and a monomial $x_{ij}(-r)x_{i'j'}(-r')$ satisfies difference conditions on modules of level 1. By Lemma 8, relation $\sqsubset$ is transitive, and hence it is a strict partial order on the set of variables.

If we consider monomials $x(\pi) \in \mathbb{C}[\tilde{\Gamma}^-]$ as multisets, then we have the following characterization of monomials satisfying difference conditions on level k modules:

Proposition 9. *A monomial $x(\pi)$ satisfies difference conditions on modules of level k if and only if every subset of $x(\pi)$ in which there are no two elements comparable in the sense of $\sqsubset$, has at most k elements.*

Proof. Let $x_{ij}(-r), x_{i'j'}(-r') \in \tilde{\mathfrak{g}}_1$ be two variables and assume $r \geqslant r'$. By (29), they are incomparable if and only if

$$\begin{cases} -r = -r'; \ i \leqslant i', \ j \leqslant j', \\ -r = -r'; \ i \geqslant i', \ j \geqslant j', \\ -r = -r' - 1; \ i \leqslant i', \ j \geqslant j'. \end{cases} \tag{30}$$

It is now clear that elements whose colors lie on a diagonal path that was considered in (26) are mutually incomparable in the sense of $\sqsubset$. Hence, if $x(\pi)$ doesn't satisfy difference conditions, then it has a subset of at least $k+1$ mutually incomparable elements. Conversely, consider a subset of $x(\pi)$ in which all elements are mutually incomparable. By relation (30), degrees of its elements can differ for at most 1. Assume that they are of degrees $-r$ and $-r-1$. Since the elements of the same degree are incomparable, their colors must all lie on a diagonal path like in (24). Finally, since elements of a different degree aren't comparable these two paths are related like in (26). □

Notice that by Lemma 8 if $\{x_{\gamma_1}(-r_1), x_{\gamma_2}(-r_2), \ldots, x_{\gamma_t}(-r_t)\}$ is a linearly ordered subset,

$$x_{\gamma_1}(-r_1) \sqsubset x_{\gamma_2}(-r_2) \sqsubset \cdots \sqsubset x_{\gamma_t}(-r_t),$$

then the monomial

$$x_{\gamma_1}(-r_1) x_{\gamma_2}(-r_2) \cdots x_{\gamma_t}(-r_t)$$

satisfies difference conditions on modules of level 1. Thus Theorem 6 will be proved when we show that there exists a partition of $x(\pi)$ into k linearly ordered subsets.

9.2. Combinatorial lemma

Let S be a finite set, $|S| = n$. Let $\sqsubset$ be a strict partial order on S. For a subset $X \subset S$, we say that X is *totally disordered* or *discretely ordered* if elements of X are mutually incomparable, i.e. if the restriction $\sqsubset|_{X \times X}$ is an empty set.

Theorem 6 now follows from the following combinatorial lemma:

Lemma 10. *Let $(S, \sqsubset)$ be a finite set with a strict partial order $\sqsubset$. If every totally disordered subset of X consists of at most k elements, then there exists a partition of S into at most k linearly ordered subsets.*

Proof. Let l be the maximal cardinality of a totally disordered subset of S; $l \leqslant k$. We are going to show that there is a partition of S into l linearly ordered subsets.

We prove this by induction on l and on the number of elements of S, $n = |S|$. Distinguish 2 cases:

(i) There exists a subset $\{a_1, \ldots, a_l\} \subset S$ consisting of mutually incomparable elements, such that $a_1, \ldots, a_l$ are neither all maximal elements of S, nor all minimal elements of S.
Because of the maximality of l, every element of S is comparable to some element of $\{a_1, \ldots, a_l\}$. Define subsets

$$G = \{x \in S, \ x \sqsupset a_i \text{ for some } i\},$$

$$D = \{x \in S, \ x \sqsubset a_i \text{ for some } i\}.$$

Since by the hypothesis $a_1, \ldots, a_l$ are neither all maximal elements of S, nor all minimal elements of S, sets G and D are nonempty. Thus we have a partition

$$S = G \cup D \cup \{a_1, \ldots, a_l\}.$$

Set

$$G' = G \cup \{a_1, \dots, a_l\}, \qquad D' = D \cup \{a_1, \dots, a_l\}.$$

Sets G' and D' have less than n elements, so by the induction hypothesis they can be partitioned into linearly ordered subsets. The set $\{a_1, \dots, a_l\}$ is at the same time the set of minimal elements of G', and the set of maximal elements of D'. Hence, linearly ordered subsets of G' end with some of the $a_1, \dots, a_l$, while linearly ordered subsets of D' start with some of the $a_1, \dots, a_l$. By "gluing" appropriate pairs together, we get a partition of S into l linearly ordered subsets.

(ii) The only set with l mutually incomparable elements is either the set of minimal, or the set of maximal elements of S. In this case we cannot construct a partition as we did earlier because either G or D would be empty. Consider 2 cases:

(a) Assume that the only totally disordered subset with l elements is the set of maximal elements of S (analogously for minimal elements). Denote them by $a_1, \dots, a_l$. Choose a linearly ordered subset $\{x_1, x_2, \dots, x_r\} \subset S$ that starts with a_1,

$$a_1 = x_1 \sqsupset x_2 \sqsupset \cdots \sqsupset x_r.$$

The set $S \setminus \{x_1, \dots, x_r\}$ has totally disordered subsets of at most $l-1$ elements, so by the induction hypothesis it can be partitioned into $l-1$ linearly ordered subsets. Together with $\{x_1, x_2, \dots, x_r\}$, this gives a partition of S into l linearly ordered subsets.

(b) Assume that $a_1, \dots, a_l$ are all maximal, and $b_1, \dots, b_l$ all minimal elements of S.
Choose a linearly ordered subset $\{x_1, x_2, \dots, x_r\} \subset S$ that starts with a_1 and ends with some of the b's,

$$a_1 = x_1 \sqsupset x_2 \sqsupset \cdots \sqsupset x_r = b_t.$$

Like in the previous case, the set $S \setminus \{x_1, \dots, x_r\}$ has totally disordered subsets of at most $l-1$ elements. By the induction hypothesis it can be partitioned into $l-1$ linearly ordered subsets, which together with $\{x_1, x_2, \dots, x_r\}$ gives a partition of S into l linearly ordered subsets. $\square$

9.3. Initial conditions

We now prove Theorem 6 in the general case when $\Lambda = k_0\Lambda_0 + \cdots + k_\ell\Lambda_\ell$.

First, let us recall initial conditions for a level 1 module $L(\Lambda_i)$, $i = 1, \dots, \ell$. A monomial $x(\pi)$ satisfies initial conditions for $L(\Lambda_i)$ if colors of elements of degree -1 lie below the i-th row (for $1 \leqslant i \leqslant m$), or on the left of the i-th column (for $m \leqslant i \leqslant \ell$). Note that these conditions can be understood as difference conditions if we add some imaginary elements of degree 0 to $x(\pi)$: for $1 \leqslant i \leqslant m$ add $x_{im}(0)$ to $x(\pi)$, and for $m \leqslant i \leqslant \ell$ add $x_{mi}(0)$ to $x(\pi)$. Then $x(\pi)$ satisfies difference and initial conditions for $L(\Lambda_i)$ if and only if this new monomial satisfies difference conditions for $L(\Lambda_i)$.

This observation generalizes to any level k. Let

$$\Lambda = k_0\Lambda_0 + \cdots + k_\ell\Lambda_\ell, \quad k = k_0 + \cdots + k_\ell.$$

For every $i = 1, \dots, \ell$, we add k_i elements of degree 0 of the appropriate color to $x(\pi)$. Concretely, denote by $x(\pi')$ the monomial

$$x(\pi') = x(\pi) \cdot x_{1m}^{k_1}(0)x_{2m}^{k_2}(0) \cdots x_{mm}^{k_m}(0)x_{m,m+1}^{k_{m+1}}(0) \cdots x_{m\ell}^{k_\ell}(0).$$

Colors of (0)-elements of $x(\pi')$ lie on a diagonal path as pictured on Fig. 8. Consider difference conditions on $x(\pi')$ for elements of degrees -1 and 0. Assume that $x(\pi)$ (and $x(\pi')$) contains elements

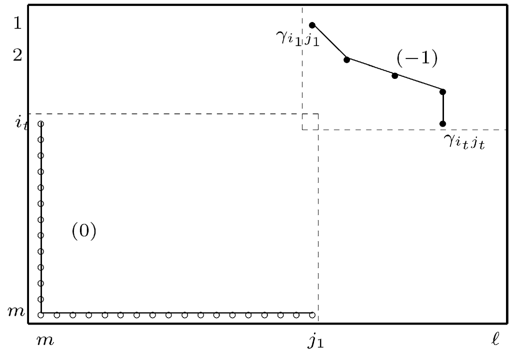

Fig. 8. Initial conditions in terms of difference conditions.

of degree -1 whose colors $\gamma_{i_1 j_1}, \ldots, \gamma_{i_t j_t}$ lie on a diagonal path as on Fig. 8. Let $a^1_{i_1 j_1}, \ldots, a^1_{i_t j_t}$ be the exponents of these elements. For every such choice of (-1)-elements, consider (0)-elements of $x(\pi')$ whose colors lie below the (i_t-1)-st row and on the left of the (j_1+1)-st column – these are the elements $x_{i_t m}(0), x_{i_t+1,m}(0), \ldots, x_{mm}(0), x_{m,m+1}(0), \ldots, x_{mj_1}(0)$, with exponents $k_{i_t}, k_{i_t+1}, \ldots, k_m, \ldots, k_{j_1}$, respectively. By difference conditions (26) for $x(\pi')$, we have

$$a^1_{i_1 j_1} + \cdots + a^1_{i_t j_t} + k_{i_t} + \cdots + k_{j_1} \leqslant k.$$

Then

$$a^1_{i_1 j_1} + \cdots + a^1_{i_t j_t} \leqslant k - k_{i_t} - \cdots - k_{j_1}.$$

Hence

$$a^1_{i_1 j_1} + \cdots + a^1_{i_t j_t} \leqslant k_0 + k_1 + \cdots + k_{i_t-1} + k_{j_1+1} + \cdots + k_\ell.$$

So, we have obtained initial conditions for $L(\Lambda)$ (cf. (28)). We have proved

Proposition 11. *Let $x(\pi)$ and $x(\pi')$ be as above. Then $x(\pi)$ satisfies difference and initial conditions for $W(\Lambda)$ if and only if $x(\pi')$ satisfies difference conditions.*

If $x(\pi')$ satisfies difference conditions then there exists a partition of $x(\pi')$ into k linearly ordered subsets. Elements of degree 0 are mutually incomparable, so they will lie in different subsets of the partition. More precisely, they will be the maximal elements of the corresponding subsets. By removing these (0)-elements from subsets of the partition, we get a partition of $x(\pi)$ into k linearly ordered subsets. Moreover, subsets that've contained (0)-elements corresponding to modules $L(\Lambda_i)$, $i = 0, \ldots, \ell$, satisfy initial conditions on these modules. This finishes the proof of Theorem 6 in the general case.

10. Proof of linear independence

10.1. Intertwining operators

In the level 1 case, the main technical tool in the proof of linear independence of the spanning set of $W(\Lambda)$ was the following proposition (cf. [26])

Proposition 12. *Suppose that a monomial $x(\pi)$ satisfies difference and initial conditions for a level 1 standard module $L(\Lambda_i)$. Write $x(\pi) = x(\pi_1)x(\pi_2)$, where $x(\pi_1)$ is the (-1)-part of a monomial, and $x(\pi_2)$ the rest of the monomial. Then there exists a coefficient $w(\mu)$ of an intertwining operator $\mathcal{Y}(e^{\mu}, z)$,*

$$w(\mu) : L(\Lambda_i) \to L(\Lambda_{i'})$$

for some $i' \in \{0, \dots, \ell\}$, such that:

- *$w(\mu)$ commutes with $\tilde{\mathfrak{g}}_1$,*
- *$w(\mu)x(\pi_1)v_i = Ce(\omega)v_{i'}$, $C \in \mathbb{C}^\times$,*
- *$x(\pi_2^+)$ satisfies IC and DC for $L(\Lambda_{i'})$,*
- *if $x(\pi')$ has a (-1)-part $x(\pi_1')$ greater than $x(\pi_1)$, then $w(\mu)x(\pi')v_i = 0$.*

By using Theorem 6, we are able to generalize this proposition for higher level standard modules. Let $L(\Lambda)$ be a standard module of level k, with the highest weight vector $v_\Lambda = v_{i_1} \otimes \cdots \otimes v_{i_k}$.

Fix a monomial $x(\pi)$ that satisfies difference and initial conditions for $L(\Lambda)$. Let

$$x(\pi) = x(\pi_2)x(\pi_1)$$

be a factorization of $x(\pi)$ such that $x(\pi_1)$ is a (-1)-part, and $x(\pi_2)$ is the rest of the monomial $x(\pi)$. By Theorem 6, there exists a factorization

$$x(\pi) = x\bigl(\pi^{(1)}\bigr) \cdots x\bigl(\pi^{(k)}\bigr),$$

such that $x(\pi^{(j)})$ satisfies difference and initial conditions for $L(\Lambda_{i_j})$. Furthermore, this induces the corresponding factorizations of $x(\pi_1)$ and $x(\pi_2)$:

$$x(\pi_1) = x\bigl(\pi_1^{(1)}\bigr) \cdots x\bigl(\pi_1^{(k)}\bigr), \qquad x(\pi_2) = x\bigl(\pi_2^{(1)}\bigr) \cdots x\bigl(\pi_2^{(k)}\bigr).$$

By Proposition 12, there exist coefficients of intertwining operators $w(\mu_j)$, $j = 1, \dots, k$ such that

$$x\bigl(\pi_1^{(j)}\bigr) v_{i_j} \xrightarrow{w(\mu_j)} C^{(j)} e(\omega) v_{i_j'}, \quad C^{(j)} \in \mathbb{C}^\times.$$

Let $\Lambda' = \Lambda_{i_1'} + \cdots + \Lambda_{i_k'}$, and define an operator $w : L(\Lambda) \to L(\Lambda')$ with

$$w = w(\mu_1) \otimes \cdots \otimes w(\mu_k).$$

Let

$$v_{\Lambda'} = v_{i_1'} \otimes \cdots \otimes v_{i_k'}$$

be the highest weight vector of $L(\Lambda')$. Then

$$x\bigl(\pi_1^{(1)}\bigr) v_{i_1} \otimes \cdots \otimes x\bigl(\pi_1^{(k)}\bigr) v_{i_k} \xrightarrow{w} Ce(\omega) v_{\Lambda'}, \quad C \in \mathbb{C}^\times.$$

Since by Proposition 12, $x(\pi_2^{(j)+})$ satisfy difference and initial conditions for $L(\Lambda_{i_j'})$, then, by Corollary 7, $x(\pi_2^+)$ also satisfies difference and initial conditions for $L(\Lambda')$.

Since $x(\pi_1^{(1)}) v_{i_1} \otimes \cdots \otimes x(\pi_1^{(k)}) v_{i_k}$ is only one of the summands that we get by acting with $x(\pi_1)$ on the tensor product $v_\Lambda = v_{i_1} \otimes \cdots \otimes v_{i_k}$, we need to see what happens with other summands of

$x(\pi_1)v_\Lambda$ when we act on them with the operator w? The other summands of $x(\pi)v_\Lambda$ come from other factorizations of $x(\pi)$. Let

$$x(\pi)=x\big(\nu^{(1)}\big)\cdots x\big(\nu^{(k)}\big),$$

be another factorization of $x(\pi)$, with induced factorizations of $x(\pi_1), x(\pi_2)$. Assume that w doesn't annihilate $x(\nu_1^{(1)})v_{i_1}\otimes\cdots\otimes x(\nu_1^{(k)})v_{i_k}$. Since by Proposition 12, $x(\pi_1^{(j)})$ is maximal for $w(\mu_j)$, we have

$$x\big(\nu_1^{(j)}\big)\leqslant x\big(\pi_1^{(j)}\big),\quad j=1,\dots,k.$$

If some of these inequalities were strict, then by Proposition 1, we would have $x(\pi)<x(\pi)$, which is a contradiction. Hence, all the factors must be equal,

$$x\big(\nu_1^{(j)}\big)=x\big(\pi_1^{(j)}\big),\quad j=1,\dots,k.$$

We conclude that the operator w will not annihilate only those summands of $x(\pi_1)v_\Lambda$ that are equal to $x(\pi_1^{(1)})v_{i_1}\otimes\cdots\otimes x(\pi_1^{(k)})v_{i_k}$, and furthermore,

$$w\cdot x(\pi_1)v_\Lambda=C\cdot\big(w(\mu_1)x\big(\pi_1^{(1)}\big)v_{i_1}\otimes\cdots\otimes w(\mu_k)x\big(\pi_1^{(k)}\big)v_{i_k}\big)=C'e(\omega)v_{\Lambda'},$$

for some $C, C'\in\mathbb{C}^\times$.

In a similar manner, we show that the operator w annihilates all $x(\nu)v_\Lambda$ whose (-1)-part $x(\nu_1)$ is greater than $x(\pi_1)$. If w doesn't annihilate $x(\nu)v_\Lambda$, then there exists a factorization

$$x(\nu)=x\big(\nu^{(1)}\big)\cdots x\big(\nu^{(k)}\big),$$

such that $x(\nu^{(j)})v_{i_j}$ aren't annihilated by $w(\mu_j)$. We also have the induced factorization of the (-1)-part $x(\nu_1)$. By Proposition 12, we have

$$x\big(\nu_1^{(j)}\big)\leqslant x\big(\pi_1^{(j)}\big),\quad j=1,\dots,k,$$

and by Proposition 1, we conclude

$$x(\nu_1)\leqslant x(\pi_1).$$

We have proved a generalization of Proposition 12:

Proposition 13. *Suppose that a monomial $x(\pi)$ satisfies difference and initial conditions for $L(\Lambda)$. Then there exists an operator $w: L(\Lambda)\to L(\Lambda')$, where $L(\Lambda')$ is another standard module of the same level, such that*:

- *w commutes with $\tilde{\mathfrak{g}}_1$,*
- *$w\cdot x(\pi_1)v_\Lambda=Ce(\omega)v_{\Lambda'}$, $C\in\mathbb{C}^\times$,*
- *$x(\pi_2^+)$ satisfies IC and DC for $L(\Lambda')$,*
- *$x(\pi_1)$ is maximal for w, i.e. all monomials $x(\pi')$ such that $wx(\pi')v_\Lambda\neq 0$, have their (-1)-part $x(\pi_1')$ smaller or equal to $x(\pi_1)$.*

10.2. Proof of linear independence

Before we proceed with the proof of linear independence, we introduce some more notation.

For a monomial $x(\pi)$, set $x(\pi_r)$ to be the $(-r)$-part of $x(\pi)$, and $x(\pi_r)=1$ if $x(\pi)$ doesn't contain any element of degree $-r$. Then

$$x(\pi)=x(\pi_n)x(\pi_{n-1})\cdots x(\pi_1),$$

if $x(\pi)$ consists of elements of degree greater than or equal to $-n$. Note that the order on the set of monomials is compatible with the order on the "homogeneous parts": $x(\pi')<x(\pi)$ if and only if there exists $r\in\mathbb{N}$ such that $x(\pi'_j)=x(\pi_j)$, for $j<r$, and $x(\pi'_r)<x(\pi_r)$.

Denote by $x(\pi^{\pm r})$ a monomial corresponding to a partition $\pi^{\pm r}$ defined by

$$\pi^{\pm r}\big(x_\gamma(-j\pm r)\big)=\pi\big(x_\gamma(-j)\big),\quad \gamma\in\Gamma, j\in\mathbb{Z}. \tag{31}$$

Similarly, denote by $x(\overline{\pi^{+r}})$ a monomial obtained from $x(\pi)$ by raising a degree by r for all elements of degree $-j<-r$, and omitting elements of degree $-j\leqslant -r$. Instead of $x(\pi^{+1})$ and $x(\overline{\pi^{+1}})$, we can also write $x(\pi^+)$ and $x(\overline{\pi^+})$, for short (cf. (10) and (11)). Note that $x(\pi^+)$ coincides with $x(\overline{\pi^+})$ if degrees of elements of $x(\pi)$ are less or equal to -2.

We prove linear independence by induction. Let

$$\sum c_\pi x(\pi)v_\Lambda=0. \tag{32}$$

Assume that all monomials in (32) have elements of degree greater or equal to $-n$. Fix a monomial $x(\pi)$ in (32) and assume that

$$c_{\pi'}=0\quad \text{for } x(\pi')<x(\pi).$$

We need to prove that $c_\pi=0$.

By Proposition 13, there exists an operator $w_1: L(\Lambda)\to L(\Lambda')$ such that

- w_1 commutes with $\tilde{\mathfrak{g}}_1$,
- $w_1\cdot x(\pi_1)v_\Lambda=C_1 e(\omega)v_{\Lambda'}$, $C_1\in\mathbb{C}^\times$,
- $x(\overline{\pi^+})$ satisfies DC and IC for $L(\Lambda')$,
- $w_1\cdot x(\pi')v_\Lambda=0$ for $x(\pi'_1)>x(\pi_1)$.

By acting with the operator w_1 on the relation (32), we get

$$\begin{aligned}0&=w_1\sum c_{\pi'}x(\pi')v_\Lambda\\&=w_1\sum_{\pi'_1>\pi_1}c_{\pi'}x(\pi')v_\Lambda+w_1\sum_{\pi'_1<\pi_1}c_{\pi'}x(\pi')v_\Lambda+w_1\sum_{\pi'_1=\pi_1}c_{\pi'}x(\pi')v_\Lambda.\end{aligned}$$

The first sum is annihilated by w_1 because of the maximality of $x(\pi_1)$ for w_1, the second sum is equal to 0 by the induction hypothesis. We obtain

$$\begin{aligned}0&=w_1\sum_{\pi'_1=\pi_1}c_{\pi'}x(\pi')v_\Lambda\\&=\sum_{\pi'_1=\pi_1}c_{\pi'}x\big(\pi'_n\big)\cdots x\big(\pi'_2\big)C_1e(\omega)v_{\Lambda'}\\&=C_1e(\omega)\sum_{\pi'_1=\pi_1}c_{\pi'}x\big(\overline{\pi'^+}\big)v_{\Lambda'}.\end{aligned}$$

Since $e(\omega)$ is an injection, we get

$$\sum_{\pi'_1=\pi_1} c_{\pi'} x\big(\overline{\pi'^{+}}\big) v_{\Lambda'} = 0. \tag{33}$$

Now, for $x(\overline{\pi^{+}})$ there exists an operator $w_2 : L(\Lambda') \to L(\Lambda'')$ such that

- w_2 commutes with $\tilde{\mathfrak{g}}_1$,
- $w_2 \cdot x(\pi_2^{+}) v_{\Lambda'} = C_2 e(\omega) v_{\Lambda''}$, $C_2 \in \mathbb{C}^{\times}$,
- $x(\overline{\pi^{+2}})$ satisfies DC and IC for $L(\Lambda'')$,
- $w_2 \cdot x(\overline{\pi'^{+}}) v_{\Lambda'} = 0$ if $x(\pi'_2) > x(\pi_2)$.

Upon acting with w_2 on the relation (33), we get

$$\begin{aligned} 0 &= w_2 \sum_{\pi'_1=\pi_1} c_{\pi'} x\big(\overline{\pi'^{+}}\big) v_{\Lambda'} \\ &= w_2 \sum_{\substack{\pi'_1=\pi_1 \\ \pi'_2>\pi_2}} c_{\pi'} x\big(\overline{\pi'^{+}}\big) v_{\Lambda'} + w_2 \sum_{\substack{\pi'_1=\pi_1 \\ \pi'_2<\pi_2}} c_{\pi'} x\big(\overline{\pi'^{+}}\big) v_{\Lambda'} + w_2 \sum_{\substack{\pi'_1=\pi_1 \\ \pi'_2=\pi_2}} c_{\pi'} x\big(\overline{\pi'^{+}}\big) v_{\Lambda'}. \end{aligned}$$

As before, the first two sums are equal to 0 because of the action of w_2 and of the induction hypothesis. We obtain

$$\begin{aligned} 0 &= w_2 \sum_{\substack{\pi'_1=\pi_1 \\ \pi'_2=\pi_2}} c_{\pi'} x\big(\overline{\pi'^{+}}\big) v_{\Lambda'} \\ &= \sum_{\substack{\pi'_1=\pi_1 \\ \pi'_2=\pi_2}} c_{\pi'} x\big(\pi_n'^{+}\big) \cdots x\big(\pi_3'^{+}\big) C_2 e(\omega) v_{\Lambda''} \\ &= C_2 e(\omega) \sum_{\substack{\pi'_1=\pi_1 \\ \pi'_2=\pi_2}} c_{\pi'} x\big(\overline{\pi'^{+2}}\big) v_{\Lambda''}. \end{aligned}$$

Since $e(\omega)$ is an injection, we get

$$\sum_{\substack{\pi'_1=\pi_1 \\ \pi'_2=\pi_2}} c_{\pi'} x\big(\overline{\pi'^{+2}}\big) v_{\Lambda''} = 0.$$

We proceed inductively; after n steps, we obtain

$$0 = \sum_{\substack{\pi'_1=\pi_1 \\ \pi'_2=\pi_2 \\ \vdots \\ \pi'_n=\pi_n}} c_{\pi'} x\big(\overline{\pi'^{+n}}\big) v_{\Lambda^{(n)}} = c_{\pi} x\big(\overline{\pi^{+n}}\big) v_{\Lambda^{(n)}} = c_{\pi} v_{\Lambda^{(n)}}$$

and we can conclude that $c_{\pi} = 0$.

Hence we have proved

Theorem 14. *The set*

$$\left\{x(\pi)v_\Lambda \;\middle|\; x(\pi) \text{ satisfies DC and IC for } L(\Lambda)\right\}$$

is a basis of $W(\Lambda)$.

11. Basis of a standard module

Feigin–Stoyanovsky's type subspace $W(\Lambda)$ was implicitly introduced and studied in [22] and [23], where a basis of the whole standard module $L(\Lambda)$ was constructed from a basis of this subspace. We have used this approach in [26] to construct a basis of standard modules of level 1, for any possible choice of $\mathbb{Z}$-gradation (1). By using Corollary 7 we are able to extend this proof to standard modules of higher level.

Set

$$e = \prod_{\gamma\in\Gamma} e^{\gamma} = e^{\sum_{\gamma\in\Gamma}\gamma}.$$

Then

$$e = e^{(\ell+1)\omega} \tag{34}$$

(cf. relation (26) in [26]). From (10), (34) and (31), for a monomial $x(\pi)\in\mathbb{C}[\tilde{\Gamma}]$ we have

$$ex(\pi) = Cx\bigl(\pi^{-\ell-1}\bigr)e, \tag{35}$$

for some $C\in\mathbb{C}^\times$.

The following proposition was proven by Primc (cf. [22, Theorem 8.2] or [23, Proposition 5.2]):

Proposition 15. *Let* $L(\Lambda)_\mu$ *be a weight subspace of* $L(\Lambda)$. *Then there exists an integer* n_0 *such that for any fixed* $n\leqslant n_0$ *the set of vectors*

$$e^n x_{\gamma_1}(r_1)\cdots x_{\gamma_s}(r_s)v_\Lambda \in L(\Lambda)_\mu,$$

where $s\geqslant 0$, $\gamma_1,\dots,\gamma_s\in\Gamma$, $r_1,\dots,r_s\in\mathbb{Z}$, *is a spanning set of* $L(\Lambda)_\mu$. *In particular,*

$$L(\Lambda) = \langle e\rangle U(\tilde{\mathfrak{g}}_1)v_\Lambda.$$

We'll use our results on the basis of $W(\Lambda)$ to prove

Theorem 16. *Let* $L(\Lambda)_\mu$ *be a weight subspace of a standard* $\tilde{\mathfrak{g}}$*-module* $L(\Lambda)$. *Then there exists* $n_0\in\mathbb{Z}$ *such that for any fixed* $n\leqslant n_0$ *the set of vectors*

$$e^n x(\pi)v_\Lambda \in L(\Lambda)_\mu,\quad x(\pi) \text{ satisfies IC and DC for } L(\Lambda), \tag{36}$$

is a basis of $L(\Lambda)_\mu$. *Moreover, for two choices of* $n_1,n_2\leqslant n_0$, *the corresponding two bases are connected by a diagonal matrix.*

We have proved this theorem in [26] for standard modules of level 1. The first part of the theorem directly follows from Proposition 15 and Theorem 14. For the second part of the theorem we have considered in [26] a monomial $x(\mu) \in \mathbb{C}[\tilde{\Gamma}]$ which was defined as the maximal monomial satisfying difference and initial conditions for $L(\Lambda)$ such that its factors are of degree greater or equal to $-r(\frac{1}{m} + \frac{1}{\ell-m+1})$, where r is equal to the smallest common multiple of m and $\ell - m + 1$. For simplicity, set $f = r(\frac{1}{m} + \frac{1}{\ell-m+1}) = r\frac{\ell+1}{m(\ell-m+1)}$ and $N = \frac{\ell+1}{f}$. We showed that the following holds

(i) $e(\omega)^f v_\Lambda = Cx(\mu)v_\Lambda$, for some $C \in \mathbb{C}^\times$,
(ii) f divides $\ell + 1$, i.e. $N \in \mathbb{N}$,
(iii) if a monomial $x(\pi)$ satisfies difference and initial conditions for $L(\Lambda)$, then so does a monomial $x(\pi^{-f})x(\mu)$.

Then we had

$$e(\omega)^f x(\pi)v_\Lambda = x\big(\pi^{-f}\big)e(\omega)^f v_i = Cx\big(\pi^{-f}\big)x(\mu)v_\Lambda.$$

Since $e^\omega x(\pi)v_\Lambda$ and $e(\omega)x(\pi)v_\Lambda$ are proportional, the second part of the theorem followed.

To prove the theorem for higher levels, it is enough to construct a monomial $x(\mu) \in \tilde{\Gamma}^-$ that satisfies properties (i) and (iii). Let $L(\Lambda)$ be a standard module of level k, with the highest weight vector $v_\Lambda = v_{i_1} \otimes \cdots \otimes v_{i_k}$. For each $L(\Lambda_{i_j})$, let $x(\mu^{(j)})$ be as in the previous paragraph. Set

$$x(\mu) = x\big(\mu^{(1)}\big)\cdots x\big(\mu^{(k)}\big).$$

As in the proof of Proposition 13, we have that $x(\mu)$ is the maximal monomial satisfying difference and initial conditions for $L(\Lambda)$ such that its factors are of degree greater or equal to $-f$, and

$$x(\mu)v_\Lambda = C \cdot x\big(\mu^{(1)}\big)v_{i_1} \otimes \cdots \otimes x\big(\mu^{(k)}\big)v_{i_k}.$$

Because of the property (i), we have

$$\begin{aligned} x(\mu)v_\Lambda &= C \cdot e(\omega)^f v_{i_1} \otimes \cdots \otimes e(\omega)^f v_{i_k} \\ &= Ce(\omega)^f v_\Lambda, \end{aligned} \tag{37}$$

for some $C \in \mathbb{N}$. Finally, by the property (iii) for monomials $x(\mu_j)$, and by Corollary 7, it follows that if a monomial $x(\pi)$ satisfies difference and initial conditions for $L(\Lambda)$, then so does a monomial $x(\pi^{-f})x(\mu)$.

Denote $x(\tilde{\mu}) = x(\mu^{(-N+1)f})x(\mu^{(-N+2)f})\cdots x(\mu^{-f})x(\mu)$. By (34) and (37), we have

$$ev_\Lambda = Cx(\tilde{\mu})v_\Lambda, \tag{38}$$

for some $C \in \mathbb{C}^\times$. Now, by (35), we have

$$e^n x(\pi)v_\Lambda = Ce^{n-1}x\big(\pi^{-\ell-1}\big)x(\tilde{\mu})v_\Lambda,$$

and from the definition of $x(\tilde{\mu})$ and from the property (iii) we see that if a monomial $x(\pi)$ satisfies difference and initial conditions, then so does a monomial $x(\pi^{-\ell-1})x(\tilde{\mu})$. Hence the second part of the theorem follows.

In the end, we show how the basis of $L(\Lambda)$ can be described in terms of "semi-infinite monomials" with "periodic tail" like in [10], and a slightly different description is used in [22,23]. By commuting monomials $x(\pi)$ with group elements e^m in (36), we can rewrite the basis of weight subspaces of

$L(\Lambda)$ in terms of vectors of the form $x(\pi)e^n v_\Lambda$ instead of $e^n x(\pi)v_\Lambda$ (with a suitable change of degrees of factors in monomials). Then by (38) and (35) we have

$$x(\pi)e^n v_\Lambda = Cx(\pi)x\big(\tilde{\mu}^{-(n-1)(\ell+1)}\big)e^{n-1}v_\Lambda, \tag{39}$$

for some $C \in \mathbb{C}^\times$. Now take the "limit $n \to -\infty$" in (39) and introduce a formal vector $v_{-\infty} = \lim_{n\to-\infty} e^n v_\Lambda$. In this way, we obtain a description of the basis of $L(\Lambda)$ in terms of "semi-infinite monomials".

12. Presentation of $W(\Lambda)$

By definition, Feigin–Stoyanovsky's type subspace $W(\Lambda)$ is a $\tilde{\mathfrak{g}}_1$-submodule of $L(\Lambda)$ generated by the highest-weight vector v_Λ,

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1)\cdot v_\Lambda.$$

Since $\tilde{\mathfrak{g}}_1$ is commutative, we have $U(\tilde{\mathfrak{g}}_1^-) \cong \mathbb{C}[\tilde{\Gamma}^-]$ and

$$W(\Lambda) = \mathbb{C}\big[\tilde{\Gamma}^-\big]\cdot v_\Lambda.$$

For $1 \leqslant i < m$ and $m < j \leqslant \ell$ let $\mathfrak{g}_{ij} \subset \mathfrak{g}_0$ be the subalgebra generated by the elements $x_{\pm\alpha_t}$, where either $1 \leqslant t \leqslant i-1$ or $j+1 \leqslant t \leqslant \ell$.

Consider a polynomial algebra $\mathbb{C}[\tilde{\Gamma}^-]$, which is also a $\mathfrak{g}_0$-module. Let $J \subset \mathbb{C}[\tilde{\Gamma}^-]$ be the ideal generated by the following sets

$$U(\mathfrak{g}_0)\cdot\Bigg(\sum_{\substack{r_1,\dots,r_{k+1}\geqslant 1\\ r_1+\dots+r_{k+1}=n}} x_\theta(-r_1)\cdots x_\theta(-r_{k+1})\Bigg), \quad \text{for all } n \in \mathbb{N}, \tag{40}$$

and

$$U(\mathfrak{g}_{ij})\cdot\big(x_\theta(-1)^{k_0+\dots+k_{i-1}+k_{j+1}+\dots+k_\ell+1}\big), \quad \text{for all } \begin{array}{l} i=1,\dots,m-1; \\ j=m+1,\dots,\ell. \end{array} \tag{41}$$

Note that elements from (40) and (41) are the ones that appear in the relations between monomial vectors (see Section 8.1 and the proof of Proposition 5).

Theorem 17. *As a vector space, $W(\Lambda)$ is isomorphic to the quotient $\mathbb{C}[\tilde{\Gamma}^-]/J$.*

Proof. Define a mapping

$$\varphi_0 : \mathbb{C}\big[\tilde{\Gamma}^-\big] \to W(\Lambda),$$
$$\varphi_0 : x(\pi) \mapsto x(\pi)\cdot v_\Lambda.$$

Since the ideal J lies in the kernel of φ_0, we can factorize φ_0 to a quotient map

$$\varphi : \mathbb{C}\big[\tilde{\Gamma}^-\big]/J \to W(\Lambda).$$

The map φ is clearly a surjection, since φ_0 is a surjection. We'll show that φ is also an injection. Consider a set

$$\mathcal{B} = \left\{ x(\pi) \;\middle|\; x(\pi) \text{ satisfies DC and IC for } L(\Lambda) \right\} \subset \mathbb{C}\big[\tilde{\Gamma}^-\big]/J.$$

As in the proof of Proposition 4, we see that this set spans $\mathbb{C}[\tilde{\Gamma}^-]/J$. Since φ bijectively maps this set onto

$$\left\{ x(\pi)v_\Lambda \;\middle|\; x(\pi) \text{ satisfies DC and IC for } L(\Lambda) \right\} \subset W(\Lambda),$$

which is a basis of $W(\Lambda)$, we see that $\mathcal{B}$ is also linearly independent. Hence φ maps a basis of $\mathbb{C}[\tilde{\Gamma}^-]/J$ onto a basis of $W(\Lambda)$ and therefore φ is a bijection. $\square$

This kind of presentation of $W(\Lambda)$ was used in [9] in order to obtain fermionic formulas for the character of $W(\Lambda)$. Also, presentation of the Feigin–Stoyanovsky's principal subspace (cf. [3,4,10]) was used in [1,2,5,6], for construction of exact sequences between different principal subspaces from which they obtained recurrence relations for the characters of these spaces.

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