ON THE SIZE OF SETS IN A POLYNOMIAL VARIANT OF A PROBLEM OF DIOPHANTUS

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ABSTRACT. In this paper, we prove that there does not exist a set of 8 polynomials (not all constant) with coefficients in an algebraically closed field of characteristic 0 with the property that the product of any two of its distinct elements plus 1 is a perfect square.

1. INTRODUCTION

Diophantus of Alexandria [1] first studied the problem of finding sets with the property that the product of any two of its distinct elements increased by one is a perfect square. Such a set consisting of m elements is therefore called a Diophantine m-tuple. The first Diophantine quadruple of rational numbers $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$ was found by Diophantus himself, while the first Diophantine quadruple of integers $\{1, 3, 8, 120\}$ was found by Fermat. In the case of rational numbers, the first Diophantine quintuple was found by Euler and a few Diophantine sextuples were recently found by Gibbs [10] (see also [11, 3]), but no upper bound for the size of such sets is known. Recently, the first author [2] showed that there does not exist a Diophantine sextuple and there are only finitely many Diophantine quintuples over the integers.

Many generalizations of this problem were considered since then, for example by adding a fixed integer n instead of 1, looking at kth powers instead of squares or considering the problem over other domains than $\mathbb{Z}$ or $\mathbb{Q}$. So we define:

Definition 1. *Let $m \geq 2$, $k \geq 2$ and R be a commutative ring with 1. A kth power Diophantine m-tuple in R is a set $\{a_1, ..., a_m\}$ consisting of m different nonzero elements from R such that $a_i a_j + 1$ is a kth power of an element of R for $1 \leq i < j \leq m$. Moreover, a set $\{a_1, ..., a_m\}$ of m different nonzero elements from R is called a pure power Diophantine m-tuple if $a_i a_j + 1$ is a kth power of an element of R for some $k \geq 2$ and all $1 \leq i < j \leq m$.*

In case of R a polynomial ring, it is usually assumed that not all polynomials $a_1, \ldots, a_m$ are constant. The first polynomial variant of the above problem was studied by Jones [13, 14] and it was for the case $R = \mathbb{Z}[X]$ and $k = 2$. For this case, Dujella and Fuchs [5] proved that there does not exist a second power Diophantine 5-tuple. Moreover, they proved that all second power Diophantine quadruples $\{a, b, c, d\}$ in $\mathbb{Z}[X]$ are regular i.e. they satisfy $(a + b - c - d)^2 = 4(ab + 1)(cd + 1)$, which is not true in $\mathbb{C}[X]$ as we will show in this paper. For other variants of the case $R = \mathbb{Z}[X]$ and $k = 2$ see [4, 7, 8].

Dujella and Luca considered the case $k \geq 3$ and $R = \mathbb{K}[X]$, where $\mathbb{K}$ is an algebraically closed field of characteristic 0. They proved [9] that $m \leq 5$ for $k = 3$, $m \leq 4$ for $k = 4$, $m \leq 3$ for $k \geq 5$ and $m \leq 2$ for $k \geq 5$ and k is even. Using many results from [9], Dujella, Fuchs and Luca [6] proved that there does not exist a second power Diophantine 11-tuple in $\mathbb{K}[X]$, i.e. $m \leq 10$ for $k = 2$. They also proved [6, Theorem 2] that there does not exist a pure power Diophantine quintuple where all perfect powers which appear are ≥ 7. As a combination of the previous results for fixed exponent and Ramsey theory [12], the same authors proved [6, Theorem 3] that $m \leq 2 \cdot 10^9$ for a pure power Diophantine m-tuple in $\mathbb{K}[X]$. Thus, they established an unconditional analogue of the result of Luca [15] for the positive integers, obtained under the assumption of the ABC-conjecture. Let us note that in the case $R = \mathbb{K}[X]$ the assumption that not all the polynomials in a kth power Diophantine m-tuple $\{a_1, ..., a_m\}$ are constant is very natural, since in an algebraically closed field $\mathbb{K}$, any m-tuple of constant polynomials is a kth power Diophantine m-tuple for any $k \geq 2$. We will also assume this for the rest of this paper. It follows [9, Lemma 1] that at most one of the polynomials a_i for $i = 1, ..., m$ is constant. The same conclusion is true, with little modification of the proof, for pure power Diophantine m-tuple in $\mathbb{K}[X]$.

The first goal of this paper is to improve the upper bound [6, Theorem 1] for the size of a second power Diophantine m-tuple in $\mathbb{K}[X]$. We have the following theorem:

Theorem 1. *There does not exist a second power Diophantine* 8*-tuple in* $\mathbb{K}[X]$*, i.e.*

$$m \leq 7 \ for \ k = 2.$$

We will prove Theorem 1 under the assumption that $\mathbb{K}$ is an algebraically closed field of characteristic 0. However, this will immediately imply that the statement of Theorem 1 is true for any field $\mathbb{K}$ of characteristic 0. For the proof of Theorem 1, we combine a gap principle with an upper bound for the degrees of the elements of a second power Diophantine quadruple in $\mathbb{K}[X]$. The mentioned upper bound [6, Proposition 1] is obtained by reducing the problem to a system of Pellian equations. The solutions to these Pellian equations lie in finitely many binary recurrent sequences so the problem is reduced to finding the intersections of these sequences. Here, we also follow that approach and we use many results from [9]. The gap principle we obtain is an improvement of the one [6, Lemma 2] used in the proof of [6, Theorem 1]. It follows from a careful analysis of the elements of the binary recurrence sequences with small indices. An interesting result of that analysis is that we discovered the existence of irregular second power Diophantine quadruples in $\mathbb{K}[X]$. For any choice of the root $\sqrt{-3}$ of X^2+3 in $\mathbb{K}$, the set

$$\mathcal{D}_p = \left\{ \frac{\sqrt{-3}}{2}, -\frac{2\sqrt{-3}}{3}(p^2-1), \frac{-3+\sqrt{-3}}{3}p^2 + \frac{2\sqrt{-3}}{3}, \frac{3+\sqrt{-3}}{3}p^2 + \frac{2\sqrt{-3}}{3} \right\},$$

with $p \in \mathbb{K}[X]$ a nonconstant polynomial, is an irregular polynomial Diophantine quadruple (see Proposition 1).

As a consequence of Theorem 1, we get as a second result an improvement of an upper bound [6, Theorem 3] for the size of a set with the property that the product of any two elements plus 1 is a pure power. We prove:

Theorem 2. *If $\{a_1, ..., a_m\}$ is a pure power Diophantine m-tuple in $\mathbb{K}[X]$, then $m < 2 \cdot 10^7$.*

The proof of this theorem runs along the same line as the proof of [6, Theorem 3]. As an upper bound, we get the Ramsey number $R(8, 6, 4, 5; 2)$. The parameters in this Ramsey number come from the cases $k = 2, 3, 5$ and from [6, Theorem 2].

In Section 4, we prove Theorem 1 and Theorem 2. This part is an improvement of the corresponding parts of [6], due to new gap principles developed in Section 3. These gap principles follow from the analysis of the elements of binary recurrence sequences. We start this analysis in Section 2 by studying the intersections of the above mentioned sequences. Here, we follow the strategy used in [2] in the integer case.

2. RELATIONS BETWEEN m AND n

Before we start our analysis we recall the method of how the problem of extending a second power Diophantine triple $\{a, b, c\}$ in $\mathbb{K}[X]$ to a second power Diophantine quadruple $\{a, b, c, d\}$ is reduced to the resolution of a system of Pellian equations. For brevity, instead of second power Diophantine m-tuple in $\mathbb{K}[X]$ we shall refer to a polynomial Diophantine m-tuple. In what follows, let a, b, c, d be polynomials in $\mathbb{K}[X]$. Denote by $\alpha, \beta, \gamma, \delta$ the degrees of a, b, c, d, respectively, and assume that $\alpha \leq \beta \leq \gamma \leq \delta$.

Let

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2 \tag{1}$$

and $ad + 1 = x^2$, $bd + 1 = y^2$, $cd + 1 = z^2$. Eliminating d, we get

$$az^2 - cx^2 = a - c, \tag{2}$$

$$bz^2 - cy^2 = b - c. \tag{3}$$

By [9, Lemma 4], there exist a nonnegative integer m and a solution (Z_0, X_0) of (2) such that $\deg(Z_0) \leq \frac{3\gamma-\alpha}{4}$, $\deg(X_0) \leq \frac{\alpha+\gamma}{4}$ and

$$z\sqrt{a} + x\sqrt{c} = (Z_0\sqrt{a} + X_0\sqrt{c})(s + \sqrt{ac})^m.$$

Also, there exist a nonnegative integer n and a solution (Z_1, Y_1) of (3) such that $\deg(Z_1) \leq \frac{3\gamma-\beta}{4}$, $\deg(Y_1) \leq \frac{\beta+\gamma}{4}$ and

$$z\sqrt{b} + y\sqrt{c} = (Z_1\sqrt{b} + Y_1\sqrt{c})(t + \sqrt{bc})^n.$$

Hence, $z = V_m = W_n$, where the sequences $(V_m)_{m\geq 0}$ and $(W_n)_{n\geq 0}$ are defined by

$$V_0 = Z_0, \quad V_1 = sZ_0 + cX_0, \quad V_{m+2} = 2sV_{m+1} - V_m, \tag{4}$$

$$W_0 = Z_1, \quad W_1 = tZ_1 + cY_1, \quad W_{n+2} = 2tW_{n+1} - W_n. \tag{5}$$

By [9, Lemma 5], it follows that

$$\deg(V_m) = (m-1)\frac{\alpha+\gamma}{2} + \deg(V_1) \tag{6}$$

for $m \geq 1$ and

$$\frac{\gamma}{2} \leq \deg(V_1) \leq \frac{\alpha+5\gamma}{4}. \tag{7}$$

Similarly,

$$\deg(W_n) = (n-1)\frac{\beta+\gamma}{2} + \deg(W_1) \tag{8}$$

for $n \geq 1$ and

$$\frac{\gamma}{2} \leq \deg(W_1) \leq \frac{\beta+5\gamma}{4}. \tag{9}$$

In the rest of the paper, we will use several lemmas from [9] and [6] which illustrate the properties of the sequences $(V_m)_{m\geq 0}$ and $(W_n)_{n\geq 0}$. In this section, we prove an unconditional relationship between m and n, when $V_m = W_n$.

Lemma 1. *If $V_m = W_n$, then $n-1 \leq m \leq 2n+1$.*

Proof. From (6)–(9), we obtain

$$(m-1)\frac{\alpha+\gamma}{2} + \frac{\gamma}{2} \leq \deg(V_m) \leq (m-1)\frac{\alpha+\gamma}{2} + \frac{\alpha+5\gamma}{4} \tag{10}$$

for all $m \geq 1$ and

$$(n-1)\frac{\beta+\gamma}{2} + \frac{\gamma}{2} \leq \deg(W_n) \leq (n-1)\frac{\beta+\gamma}{2} + \frac{\beta+5\gamma}{4} \tag{11}$$

for all $n \geq 1$. Since $\deg(V_m) = \deg(W_n)$, it follows from (10) and (11) that

$$(m-1)\frac{\alpha+\gamma}{2} + \frac{\gamma}{2} \leq (n-1)\frac{\beta+\gamma}{2} + \frac{\beta+5\gamma}{4}.$$

Since $\alpha \geq 0$ and $\beta \leq \gamma$, this implies that $(m-1)\frac{\gamma}{2} \leq n\gamma$ and finally

$$m \leq 2n+1.$$

Likewise, from (10) and (11) we obtain

$$(n-1)\frac{\beta+\gamma}{2} + \frac{\gamma}{2} \leq (m-1)\frac{\alpha+\gamma}{2} + \frac{\alpha+5\gamma}{4}.$$

Since $\alpha \leq \beta < 3\beta$, this implies that $(n-1)\frac{\beta+\gamma}{2} < (m-1)\frac{\beta+\gamma}{2} + \frac{3}{4}(\beta+\gamma)$, and we get

$$m \geq n-1.$$

□

3. GAP PRINCIPLES

In this section, we prove a gap principle which is an improvement of the one established in [2, Lemma 2] and which is used in the proof of Theorem 1. First we develop a gap principle which comes from studying the equality $V_m = W_n$ for small values of m and n. For this, we need two lemmas and the definition of a regular polynomial Diophantine quadruple.

Lemma 2. *Let $\{a,b,c\}$ be a polynomial Diophantine triple. Denote by d_+ the polynomial with larger degree and by d_- the polynomial with smaller degree among the polynomials*

$$a+b+c+2abc \pm 2rst,$$

where r, s and t are polynomials for which (1) holds. Then $\deg(d_-) < \gamma$.

Proof. Using (1), we conclude that $\deg(d_+) = \alpha + \beta + \gamma$. From

$$d_+ \cdot d_- = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc - 4,$$

it follows that $\deg(d_-) < \gamma$. □

Definition 2. *A polynomial Diophantine quadruple* $\{a, b, c, d\}$ *is called regular if*

$$(a + b - c - d)^2 = 4(ab + 1)(cd + 1),$$

or, equivalently, if either $d = d_+$ *or* $d = d_-$.

It holds

$$ad_\pm + 1 = u^2, \quad bd_\pm + 1 = v^2, \quad cd_\pm + 1 = w^2, \tag{12}$$

where $u = at \pm rs$, $v = bs \pm rt$, $w = cr \pm st$. Moreover, we have that

$$c = a + b + d_\pm + 2(abd_\pm \pm ruv). \tag{13}$$

For $d_- = 0$, by (13), we have $c = a + b \pm 2r$. If $d_- \neq 0$, then from [6, Lemma 1], it follows that $\deg(d_-) = 0$. The existence of $d_\pm$ implies that $V_m = W_n$ has non-trivial solutions. Since d_- has degree $< \gamma$ and $\deg(V_m), \deg(W_n) \geq \gamma$ for $m, n \geq 2$ it must arise from $V_m = W_n$ with $m, n \in \{0, 1\}$.

The following lemma is a more precise version of [6, Lemma 4], where only a suitable version for their application was proved.

Lemma 3.
1) *If* $V_{2m} = W_{2n}$, *then* $Z_0 = Z_1$.
2) *If* $V_{2m+1} = W_{2n}$, *then either* $(Z_0, Z_1) = (\pm 1, \pm s)$ *or* $(Z_0, Z_1) = (\pm s, \pm 1)$ *or* $Z_1 = sZ_0 + cX_0$ *or* $Z_1 = sZ_0 - cX_0$.
3) *If* $V_{2m} = W_{2n+1}$, *then either* $(Z_0, Z_1) = (\pm t, \pm 1)$ *or* $Z_0 = tZ_1 + cY_1$ *or* $Z_0 = tZ_1 - cY_1$.
4) *If* $V_{2m+1} = W_{2n+1}$, *then either* $(Z_0, Z_1) = (\pm 1, \pm cr \pm st)$ *or* $(Z_0, Z_1) = (\pm cr \pm st, \pm 1)$ *or* $sZ_0 + cX_0 = tZ_1 \pm cY_1$ *or* $sZ_0 - cX_0 = tZ_1 \pm cY_1$.

Proof.
1) From [6, Lemma 3], we have $Z_0 \equiv Z_1 \pmod{c}$. Since $\deg(Z_0) < \gamma$ and $\deg(Z_1) < \gamma$, we conclude that $Z_0 = Z_1$.
2) From [6, Lemma 3], we have $Z_1 \equiv sZ_0 \pmod{c}$. Assume that $Z_0 = \pm 1$. Hence,

$$Z_1 \equiv \pm s \pmod{c}. \tag{14}$$

If $\alpha < \gamma$, then $Z_1 = \pm s$. If $\alpha = \beta = \gamma$, then from [6, Lemma 1] it follows that $c = a + b \pm 2r$ and we have $\pm s \equiv \pm t \pmod{c}$. Multiplying this congruence by s, we obtain $\pm 1 \equiv \pm st \pmod{c}$. Now, multiplying (14) by t we get

$$tZ_1 \pm cY_1 \equiv \pm 1 \pmod{c}. \tag{15}$$

If $Z_1 = \pm 1$ then, by (3), $Y_1 = \pm 1$ and from (5) it follows that $\deg(W_1) \leq \gamma$. Similarly, $\deg(V_1) \leq \gamma$. From (6)–(9), it follows that $2m + 1 = 2n$, which is not possible. We conclude that $Z_1 \neq \pm 1$. Now, from [9, Lemma 5], $\deg(Z_1) \geq \frac{\gamma}{2}$ and $\deg(Y_1) \geq \frac{\beta}{2}$. Hence, Y_1 is not constant. Since

$$(cY_1 + tZ_1)(cY_1 - tZ_1) = c^2 - bc - Z_1^2, \tag{16}$$

we conclude that one of the polynomials $cY_1 \pm tZ_1$ has degree less than γ. For that polynomial, (15) becomes an equation. Notice that $\deg(Z_1) \le \frac{3\gamma-\beta}{4} = \frac{\gamma}{2}$, so $\deg(Z_1) = \frac{\gamma}{2}$. From [9, Lemma 4], it follows that $\deg(W_0) = \deg(Z_1) \le \deg(tZ_1 \pm cY_1)$, so we have a contradiction.

Assume now that $Z_0 \neq \pm 1$. By [9, Lemma 5], we have $\deg(Z_0) \ge \frac{\gamma}{2}$ and $\deg(X_0) \ge \frac{\alpha}{2}$. Let $\alpha = 0$ and X_0 be constant. Denote $e := \frac{(X_0^2-1)}{a}$. Then $\{a, e, c\}$ is a Diophantine triple and [9, Lemma 1] implies that $a = e$. Now, $X_0^2 = a^2 + 1$, and by (2) we obtain $Z_0 = \pm s$. Hence, $Z_1 \equiv \pm 1 \pmod{c}$ and we have $Z_1 = \pm 1$. Assume now that X_0 is not constant. Since

$$(cX_0 + sZ_0)(cX_0 - sZ_0) = c^2 - ac - Z_0^2, \tag{17}$$

we conclude that one of the polynomials $cX_0 \pm sZ_0$ has degree less that γ and they are both congruent to Z_1 modulo c. Hence, one of these polynomials is equal to Z_1.

3) This case is completely analogous to case 2), except that β cannot be equal to 0.

4) From [6, Lemma 3], we have $sZ_0 \equiv tZ_1 \pmod{c}$. If X_0 and Y_1 are not constant, then one of the polynomials $cX_0 \pm sZ_0$ and one of the polynomials $cY_1 \pm tZ_1$ have degrees less than γ. These two polynomials are congruent modulo c, thus, they have to be equal.

If $Z_0 = \pm 1$, then $\pm s \equiv tZ_1 \pmod{c}$. Multiplying this congruence by t we obtain $\pm st \pm cr \equiv Z_1 \pmod{c}$. Since

$$(\pm st - cr)(\pm st + cr) = ac + bc + 1 - c^2,$$

one of the polynomials $\pm st \pm cr$ has degree less then γ and the other has degree equal to $\gamma + \frac{\alpha+\beta}{2}$. Hence, $\pm st \pm cr = Z_1$ and $\deg(Z_1) \le \gamma - \frac{\alpha+\beta}{2}$. Also, notice that we have $cd_- + 1 = Z_1^2$. If $\deg(Z_1) < \gamma - \frac{\alpha+\beta}{2}$, it must be that $\beta = \gamma$ so now $\deg(Z_1) < \frac{\gamma}{2} - \frac{\alpha}{2}$. If $\alpha = \beta = \gamma$ we obtain a contradiction $\deg(Z_1) < 0$. If $\alpha < \beta$, it follows that $\deg(Z_1) < \frac{\gamma}{2}$ and we get $d_- = 0$, $Z_1 = \pm 1$. If $\deg(Z_1) = \gamma - \frac{\alpha+\beta}{2}$, from $\deg(Z_1) \le \frac{3\gamma-\beta}{4}$ it follows that $\gamma \le \beta + 2\alpha$. If $\beta = \gamma$ and $\alpha > 0$, then $\deg(Z_1) < \frac{\gamma}{2}$ so $d_- = 0$ and $Z_1 = \pm 1$. If $\beta = \gamma$ and $\alpha = 0$, then $\deg(Z_1) = \frac{\gamma}{2}$. Now we have $\deg(d_-) = 0$ so $d_- = a$ and $Z_1 = \pm s$.

Assume now that $Z_0 \neq \pm 1$ and X_0 is constant. As above, $Z_0 = \pm s$ and we have $\pm 1 \equiv tZ_1 \pmod{c}$. Multiplying this congruence by t, it follows that $\pm t \equiv Z_1 \pmod{c}$. If $\beta < \gamma$, then $Z_1 = \pm t$ and $Y_1^2 = b^2 + 1$, a contradiction. Let $\beta = \gamma$. Notice that $\pm 1 \equiv tZ_1 \pm cY_1 \pmod{c}$. If $\deg(Z_1) \ge \frac{\gamma}{2}$, then by [9, Lemma 5], $\deg(Y_1) \ge \frac{\beta}{2}$ and Y_1 is not constant. As above, we obtain a contradiction. Consider now the general case $\alpha \le \beta \le \gamma$ and $Z_1 = \pm 1$. Multiplying the congruence $sZ_0 \equiv tZ_1 \pmod{c}$ by s, we obtain $Z_0 \equiv \pm st \pmod{c}$ so, as above, $Z_0 = \pm st \pm cr$ and $\deg(Z_0) \le \gamma - \frac{\alpha+\beta}{2}$. Here, we have $Z_0 = \pm s$. When $\beta = \gamma$ we can also have $Z_0 = \pm 1$. □

In the proof od Lemma 3, we obtained the following result which will be used several times in the proof of Proposition 1.

Lemma 4. *If $\alpha = 0$ and X_0 is constant, then $X_0^2 = a^2 + 1$ and $Z_0 = \pm s$.*

Now we are ready to examine the equation $V_m = W_n$ for small indices m and n.

Proposition 1. *Let* $S = \{a, b, c\}$ *be a polynomial Diophantine triple. Assume that* $V_m = W_n$ *and define* $d = \dfrac{V_m^2 - 1}{c}$*. If* $\{0, 1, 2\} \cap \{m, n\} \neq \emptyset$*, then either* $\deg(d) < \gamma$ *or* $d = d_+$ *or* $d = \dfrac{3+\sqrt{-3}}{3}p^2 + \dfrac{2\sqrt{-3}}{3}$*, in the special case* $S = \left\{\dfrac{\sqrt{-3}}{2}, -\dfrac{2\sqrt{-3}}{3}(p^2-1), \dfrac{-3+\sqrt{-3}}{3}p^2 + \dfrac{2\sqrt{-3}}{3}\right\}$ *with* $p \in \mathbb{K}[X]$ *a nonconstant polynomial.*

Proof. From Lemma 1 and the condition $\{0, 1, 2\} \cap \{m, n\} \neq \emptyset$ it follows that

$$(m, n) \in \{(0, 0), (0, 1), (1, 0), (1, 1), (1, 2), (2, 1), (3, 1), (2, 2), (2, 3), (3, 2), (4, 2), (5, 2)\}.$$

1) If $0 \in \{m, n\}$, then from (4) or (5) we have $z = Z_0$ or $z = Z_1$. Since $\deg(Z_0) < \gamma$ and $\deg(Z_1) < \gamma$, from $cd + 1 = z^2$ we obtain

$$\deg(d) < \gamma.$$

2) If $(m, n) = (1, 1)$, then $z = V_1 = W_1$. From (4) and (5) we have

$$z = sZ_0 + cX_0 = tZ_1 + cY_1. \tag{18}$$

Assume first that X_0 and Y_1 are not constants. By Lemma 3, we have the equation

$$sZ_0 \pm cX_0 = tZ_1 \pm cY_1.$$

We consider the four possibilities. If

$$sZ_0 + cX_0 = tZ_1 + cY_1,$$

then from (18) we obtain $\deg(z) < \gamma$. As in the case $0 \in \{m, n\}$, we conclude that $\deg(d) < \gamma$. If

$$sZ_0 + cX_0 = tZ_1 - cY_1,$$

then combining this with (18) we obtain $2cY_1 = 0$, a contradiction. If

$$sZ_0 - cX_0 = tZ_1 + cY_1,$$

then combining this with (18) we obtain $2cX_0 = 0$, again a contradiction. The last possibility is

$$sZ_0 - cX_0 = tZ_1 - cY_1.$$

This equation together with (18) yields $sZ_0 = tZ_1$ and $X_0 = Y_1$. Inserting this into (2) and (3), we obtain

$$(b-a)s^2 = (bZ_1^2 - aZ_0^2)s^2 = Z_1^2(abc + b - abc - a) = Z_1^2(b-a).$$

Therefore, $Z_1 = \pm s$, $Z_0 = \pm t$ and $X_0 = Y_1 = \pm r$. Hence, $z = V_1 = \pm st \pm cr$, and

$$\begin{aligned} d = \frac{z^2-1}{c} &= \frac{abc^2 + ac + bc + 1 \pm 2stcr + abc^2 + c^2 - 1}{c} \\ &= a + b + c + 2abc \pm 2rst. \end{aligned}$$

From Lemma 2, we conclude that either $d = d_+$ or $\deg(d) < \gamma$.

Assume that $(Z_0, Z_1) = (\pm 1, \pm cr \pm st)$. From (2), we have $X_0 = \pm 1$ and from (18) it follows that

$$z = \pm s \pm c = tZ_1 + cY_1,$$

so $\deg(z) \le \gamma$. If $\deg(z) < \gamma$, then $\deg(d) < \gamma$. Let $\deg(z) = \deg(tZ_1 + cY_1) = \gamma$. Then Y_1 must be a constant. Let us note that $cd_- + 1 = Z_1^2$ and, from (3), we have $bd_- + 1 = Y_1^2$. Now we conclude that $d_- = 0$, $Y_1 = \pm 1$ and $Z_1 = \pm 1$. From [6, Lemma 1], it follows that $c = a + b \pm 2r$, so we have $c = \pm s \pm t$. Also, by (18), $\pm s \pm c = \pm t \pm c$. If $\pm s \pm t = 0$, then from (1) we obtain $a = b$, a contradiction. If $\pm s \pm t = \pm 2c$, then combining this relation with $c = \pm s \pm t$ we obtain $c = 0$ or $c|s$, again a contradiction.

Assume now that $(Z_0, Z_1) = (\pm cr \pm st, \pm 1)$. From (3), we have $Y_1 = \pm 1$ and from (18) it follows that

$$z = sZ_0 + cX_0 = \pm t \pm c,$$

so $\deg(z) \leq \gamma$. If $\deg(z) < \gamma$ then $\deg(d) < \gamma$, so we only deal with $\deg(z) = \deg(sZ_0 + cX_0) = \gamma$. Here, X_0 is a constant. Observe that $cd_- + 1 = Z_0^2$ and by (2) $ad_- + 1 = X_0^2$. If a is not a constant, we have a contradiction as above. If a is a constant, then from Lemma 4 it follows that $X_0^2 = a^2 + 1$ and $Z_0 = \pm s$. Moreover, $\beta = \gamma$ and $\alpha = 0$. From $s(\pm cr \pm st) + cX_0 = \pm t \pm c$, we obtain

$$c(\pm at \pm rs) \pm t + cX_0 = \pm t \pm c,$$

where $ad_- + 1 = (\pm at \pm rs)^2$. Hence, we have $2cX_0 = \pm c$ so $X_0 = \pm\frac{1}{2}$ and $a = \pm\frac{\sqrt{-3}}{2}$. By squaring the equation

$$\pm s = \pm cr \pm st$$

and using (1) we obtain $c(b - c \pm 2rs) = 0$, so $c = b \pm 2rs$. Squaring the last equation, we get

$$b^2 + bc + c^2 = \pm 2\sqrt{-3}(b + c) + 4.$$

$$\left(c + \frac{1+\sqrt{-3}}{2}b\right)\left(c - \left(\frac{1+\sqrt{-3}}{2}\right)^2 b\right) = \pm 2\sqrt{-3}(b + c) + 4. \tag{19}$$

Observe that one of the factors on the left side of (19) has to be constant. Assume first that $e := c + \frac{1+\sqrt{-3}}{2}b$ is a constant. From (19), we get

$$e^2 - \sqrt{-3}be = \pm 2\sqrt{-3}e + 4 \pm b(\sqrt{-3} + 3),$$

where on the both sides we have polynomials of the form $g_1 b + g_2$, with constants g_1 and g_2. By comparing the coefficients, we obtain $e = \pm(-1 + \sqrt{-3})$, where the signs $\pm$ are the same as the signs of a. Hence, $e = \pm 2(\frac{1+\sqrt{-3}}{2})^2$. Denote $u^2 := \frac{1+\sqrt{-3}}{2}$ and notice that $u^4 - u^2 + 1 = 0$. Now, we have $c = -u^2 b \pm 2u^4$ and, by (1), $t = bu^4 \pm 1$. Applying [6, Lemma 1] to the polynomial Diophantine triple $\{b, c, d\}$, we obtain $d = b + c \pm 2t$. Let $d = b + c + 2t$, hence $d = bu^4 \pm 2u^2$. From this, it follows that $ad + 1 = r^2u^4$. Moreover, $bd + 1 = (bu^2 \pm 1)^2$ and $cd + 1 = -u^6(b \mp u(2u^2 - 1))^2$. Now we conclude that for every nonconstant polynomial $p \in \mathbb{K}[X]$ there exists the polynomial Diophantine quadruple

$$\mathcal{D}_p = \left\{\frac{\sqrt{-3}}{2}, -\frac{2\sqrt{-3}}{3}(p^2 - 1), \frac{-3+\sqrt{-3}}{3}p^2 + \frac{2\sqrt{-3}}{3}, \frac{3+\sqrt{-3}}{3}p^2 + \frac{2\sqrt{-3}}{3}\right\}.$$

When $d = b + c - 2t$, in analogous way we obtain $ad + 1 = -3u^4r^2 + u^2(3 + \sqrt{-3}) - 2 - 2\sqrt{-3}$, a contradiction. Assume now that $e := c - (\frac{1+\sqrt{-3}}{2})^2 b$ is a constant. From (19), it follows that

$$e^2 + \sqrt{-3}be = \pm 2\sqrt{-3}e + 4 \pm (-3 + \sqrt{-3})b.$$

From the above relation, we obtain $e = \pm(1 + \sqrt{-3})$. Hence, $e = \pm 2u^2$. We get $c = bu^4 \pm 2u^2$ and $t = bu^2 \pm 1$. If $d = b + c + 2t$, then $ad + 1 = 3u^2r^2 + u^2(-3 +$

$\sqrt{-3})+1+\sqrt{-3}$, a contradiction. But, for $d=b+c-2t=-bu^2\pm 2u^4$, we obtain a polynomial Diophantine quadruple $\mathcal{D}_p$ again. Notice that $\mathcal{D}_p$ is not a regular Diophantine quadruple because we have $\deg(d_+)=2\gamma$, $\deg(d_-)=0$ and $\deg(d)=\gamma$.

3) Assume that $(m,n)=(2,1)$. Then $z=V_2=W_1$. By (4) and (5) we have

$$z=Z_0+2c(aZ_0+sX_0)=tZ_1+cY_1. \tag{20}$$

By Lemma 3, if $Z_1=\pm1$, then $Z_0=\pm t$. Now, by (2), $X_0=\pm r$ and, by (3), $Y_1=\pm1$. Hence, $z=\pm t\pm c$ and $\deg(z)\le\gamma$. From (4) and [9, Lemma 5], it follows that $\deg(V_2)\ge\gamma+\frac{\alpha+\beta}{2}>\gamma$, so we obtain a contradiction.

If $Z_1\ne\pm1$, then by Lemma 3, one of the polynomials $tZ_1\pm cY_1$ has degree less than γ and is equal to Z_0. If

$$Z_0=tZ_1+cY_1,$$

then from (20) it follows that $\deg(d)<\gamma$. If

$$Z_0=tZ_1-cY_1,$$

then from (20) we obtain that $aZ_0+sX_0=Y_1$. Combining the last two equations, we get $Z_0=\dfrac{tZ_1-scX_0}{s^2}$ and $Y_1=\dfrac{sX_0+atZ_1}{s^2}$. Inserting this into (2) and (3), we obtain

$$Z_1^2(b-a)=s^2(b-a).$$

We get that $Z_1=\pm s$ and, by (3), $Y_1=\pm r$. Hence, $z=W_1=\pm st\pm cr$. Analogously to the case $(m,n)=(1,1)$, we conclude that

$$d=a+b+c+2abc\pm 2rst.$$

From Lemma 2, it follows that either $d=d_+$ or $\deg(d)<\gamma$.

4) Let $(m,n)=(1,2)$. Then $z=V_1=W_2$. From (4) and (5) we have

$$z=sZ_0+cX_0=Z_1+2c(bZ_1+tY_1).$$

Analogously to the case $(m,n)=(2,1)$, if $Z_0=\pm1$, then $X_0=\pm1$ and we obtain a contradiction.

If $Z_0\ne\pm1$ and X_0 is a constant then, by Lemma 3, $Z_0=\pm s$ and we have

$$z=\pm ac\pm1+cX_0.$$

Hence, $\deg(z)\le\gamma$. As in the case $(m,n)=(2,1)$, we obtain the contradiction $\deg(z)>\gamma$.

If $Z_0\ne\pm1$ and X_0 is not a constant, analogously to the case $(m,n)=(2,1)$, we obtain that either $d=d_+$ or $\deg(d)<\gamma$.

5) Assume now that $(m,n)=(2,2)$. Then $z=V_2=W_2$. From (4) and (5) we have

$$z=Z_0+2c(aZ_0+sX_0)=Z_1+2c(bZ_1+tY_1). \tag{21}$$

From Lemma 3, it follows that $Z_0=Z_1$. Inserting this into (21), we get

$$sX_0-tY_1=(b-a)Z_0. \tag{22}$$

Combining (2) and (3) we obtain

$$(b-a)Z_0^2 = b - a + cY_1^2 - cX_0^2. \tag{23}$$

Now, from (22) and (23) we have

$$\begin{aligned}(b-a)^2 + (b-a)(cY_1^2 - cX_0^2) &= (sX_0 - tY_1)^2 \\ &= acX_0^2 + X_0^2 - 2stX_0Y_1 + bcY_1^2 + Y_1^2 \\ &= t^2X_0^2 - 2stX_0Y_1 + s^2Y_1^2 + (b-a)(cY_1^2 - cX_0^2).\end{aligned}$$

Therefore, we conclude that

$$tX_0 - sY_1 = \pm(b-a). \tag{24}$$

Furthermore, from (2) and (3) we obtain

$$s^2(bX_0^2 + a - b) = as^2Y_1^2, \tag{25}$$

and from (24) it follows that

$$as^2Y_1^2 = a(tX_0 \mp (b-a))^2. \tag{26}$$

Hence, from (25) and (26) we have

$$(ac+1)(bX_0^2 + a - b) = a(bcX_0^2 + X_0^2 \mp 2tX_0(b-a) + (b-a)^2),$$

from which we conclude that

$$\begin{aligned}(b-a)(X_0^2 \pm 2atX_0 + a^2t^2) &= (b-a)(ab+1)(ac+1) \\ (b-a)(X_0 \pm at)^2 &= (b-a)r^2s^2.\end{aligned}$$

Therefore, $X_0 = \pm rs \mp at$ and, by (2), $Z_0 = \pm st \mp cr$. Now, we obtain $V_2 = \pm st \mp cr + 2c(\pm ast \mp acr \pm rs^2 \mp ats) = \pm st \pm cr$ and

$$d = a + b + c + 2abc \pm 2rst.$$

From Lemma 2, we conclude that either $d = d_+$ or $\deg(d) < \gamma$.

6) Let $(m,n) = (3,1)$. Then $z = V_3 = W_1$. From (4) and (5) we have

$$z = sZ_0 + c(4asZ_0 + 3X_0) + 4ac^2X_0 = tZ_1 + cY_1. \tag{27}$$

Now from (6)-(9), it follows that $\gamma \le \beta - 4\alpha \le \beta$, so $\beta = \gamma$ and $\alpha = 0$. Assume first that X_0 and Y_1 are not constants. By Lemma 3, we have four possibilities to consider. Let us start with

$$sZ_0 + cX_0 = tZ_1 + cY_1, \tag{28}$$

where both sides have degrees less than γ. Now, from (27), we conclude that $\deg(W_1) < \gamma$. Hence, $\deg(V_3) < \gamma$. Combining (27) and (28) we obtain

$$V_3 = tZ_1 + cY_1 + c(4asZ_0 + 2X_0 + 4acX_0).$$

Therefore, $\deg(V_3) \ge \gamma$, unless $4asZ_0 + 2X_0 + 4acX_0 = 0$. So, we have $4a(sZ_0 + cX_0) = -2X_0$ and $\deg(sZ_0 + cX_0) \le \dfrac{\gamma}{4}$. But, now from (17), it follows that $\deg(sZ_0 - cX_0) \ge \frac{7}{4}\gamma$, which is a contradiction. If

$$sZ_0 - cX_0 = tZ_1 + cY_1,$$

we obtain the same contradiction as for (28). If we have

$$sZ_0 + cX_0 = tZ_1 - cY_1, \tag{29}$$

where both sides have degrees less than γ, then from (27) we obtain that

$$-2Y_1 + 2X_0 = -4a(sZ_0 + cX_0).$$

Hence, $\deg(sZ_0 + cX_0) \leq \dfrac{\gamma}{2}$. But, now from (17) it follows that $\deg(sZ_0 - cX_0) \geq \frac{3}{2}\gamma$, again a contradiction. The last possibility

$$sZ_0 - cX_0 = tZ_1 - cY_1$$

yields a contradiction analogously as for (29).

Let $(Z_0, Z_1) = (\pm 1, \pm cr \pm st)$. From Lemma 3, it follows that either $Z_1 = \pm 1$ or $Z_1 = \pm s$. If $Z_1 = \pm 1$, we get from (3) that $Y_1 = \pm 1$ so by (27) we have $z = \pm t \pm c$ and $\deg(z) \leq \gamma$. If $\deg(z) < \gamma$, then $\deg(d) < \gamma$. Consider the case $\deg(z) = \gamma$. By (2), $X_0 = \pm 1$ and from (27) it follows that $\deg(z) = \deg(V_3) = 2\gamma$, a contradiction. If $Z_1 = \pm s$, then by (3), $Y_1 = \pm r$. Now $z = W_1 = \pm st \pm cr$, where $\deg(\pm st \pm cr) = \gamma + \frac{\alpha+\beta}{2}$ or $\deg(\pm st \pm cr) \leq \gamma - \frac{\alpha+\beta}{2}$. From the equation $\deg(V_3) = \deg(W_1)$, we get a contradiction for both possibilities.

Assume now that $(Z_0, Z_1) = (\pm cr \pm st, \pm 1)$. By (3), $Y_1 = \pm 1$ and from (27) we see that $z = \pm t \pm c$. Hence, $\deg(z) \leq \gamma$. If $\deg(z) < \gamma$, then $\deg(d) < \gamma$. Consider the case $\deg(z) = \gamma$. From [9, Lemma 5, iv], we obtain that $\deg(V_3) = \gamma + \deg(V_1)$ so $\deg(V_1) = 0$, which is a contradiction by [9, Lemma 5, iii].

7) Let $(m, n) = (2, 3)$. Then $z = V_2 = W_3$. From (4) and (5) we have

$$z = Z_0 + 2c(aZ_0 + sX_0) = tZ_1 + c(4btZ_1 + 3Y_1) + 4bc^2Y_1. \tag{30}$$

From Lemma 3, it follows that if $Z_1 = \pm 1$, then $Z_0 = \pm t$. By (2), $X_0 = \pm r$, and by (3), $Y_1 = \pm 1$. Comparing the degrees, from $Z_0 = \pm t$ we obtain that $2\beta + \alpha \leq \gamma$, from which it follows

$$\beta < \gamma.$$

Now, from (30) we have

$$V_2 = \pm t + 2c(\pm at \pm sr),$$
$$W_3 = \pm t + c(\pm 4bt \pm 3) \pm 4bc^2.$$

We conclude that $\deg(W_3) = \deg(4bc^2) = \beta + 2\gamma$, so $\deg(V_2) = \beta + 2\gamma$. Therefore, $\deg(\pm at \pm sr) = \beta + \gamma$. Now we have $\beta + \gamma \leq \max(\deg(\pm at), \deg(\pm rs)) = \alpha + \frac{\beta+\gamma}{2}$ so $\gamma \leq 2\alpha - \beta$. But, from $2\beta + \alpha \leq \gamma$, we get $\beta < \alpha$, a contradiction.

If $Z_1 \neq \pm 1$, from Lemma 3, it follows that

$$Z_0 = tZ_1 \pm cY_1,$$

where the degree of the right-hand side is $< \gamma$. First we assume that

$$Z_0 = tZ_1 - cY_1.$$

Inserting this into (30), we obtain

$$aZ_0 + sX_0 = 2b(tZ_1 + cY_1) + 2Y_1. \tag{31}$$

Since one of the polynomials $tZ_1 \pm cY_1$ has the degree equal to $\gamma + \deg(Y_1)$ and the degree of Z_0 is less than γ, this must be the polynomial $tZ_1 + cY_1$. Now from (31) we conclude that $\deg(aZ_0 + sX_0) = \beta + \gamma + \deg(Y_1)$. But, $\deg(aZ_0 + sX_0) \leq \max(\deg(aZ_0), \deg(sX_0)) \leq \frac{3\alpha+3\gamma}{4}$, so we have $\gamma \leq -3\beta$, a contradiction. Let

$$Z_0 = tZ_1 + cY_1. \tag{32}$$

If we put (32) into (30), we obtain

$$aZ_0 + sX_0 = Y_1 + 2bZ_0. \tag{33}$$

From (32), it follows that $Z_0 = W_1$. By [9, Lemma 5] we conclude that $Z_0 \neq \pm 1$ and $\deg(Z_0) \geq \frac{\gamma}{2}$, $\deg(X_0) \geq \frac{\alpha}{2}$. If $\alpha = 0$ and X_0 is a constant, from Lemma 4 we have $X_0^2 = a^2 + 1$ and $Z_0 = \pm s$. Hence, $\deg(tZ_1 + cY_1) = \frac{\gamma}{2}$. From (6)-(9), it follows that $4\beta \leq \gamma + 3\alpha$ so $\beta < \gamma$. From (16), it now follows that $\gamma - \deg(Y_1) = \frac{\gamma}{2}$, so $\frac{\gamma}{2} = \deg(Y_1) \leq \frac{\beta+\gamma}{4}$. This yields the contradiction $\gamma \leq \beta$. Hence, X_0 cannot be a constant. Since

$$(aZ_0 + sX_0)(aZ_0 - sX_0) = a^2 - ac - X_0^2, \tag{34}$$

from (1) and (2) it follows that one of the polynomials $aZ_0 \pm sX_0$ has degree equal to $\frac{\alpha+\gamma}{2} + \deg(X_0)$, and the other has degree less than that. By [9, Lemma 5], we have that $\deg(V_2) = \beta + \gamma + \deg(Z_0)$. Now, from (30) we conclude that

$$\deg(aZ_0 + sX_0) = \beta + \deg(Z_0). \tag{35}$$

Hence, the polynomial $aZ_0 + sX_0$ must have the degree equal to $\frac{\alpha+\gamma}{2} + \deg(X_0)$. Now, from (35) and (2), we conclude that $\alpha = \beta$. If we transform (33) into

$$sX_0 - aZ_0 = Y_1 + 2(b-a)Z_0, \tag{36}$$

we conclude that

$$\deg(Y_1 + 2Z_0(b-a)) \leq \frac{\alpha+\gamma}{2} - \deg(X_0). \tag{37}$$

If $\deg(b-a) > 0$, then from (37) we obtain that $\deg(X_0) < \frac{\alpha}{2}$, a contradiction. If $\deg(b-a) = 0$, then from (37) we conclude that $\deg(Z_0) = \frac{\gamma}{2}$. Hence, $\deg(X_0) = \frac{\alpha}{2}$. We now transform (36) to the equation

$$sX_0 - Y_1 - bZ_0 = (b-a)Z_0, \tag{38}$$

where the degree of the right-hand side is $\frac{\gamma}{2}$. By (32) and (1), we have $sX_0 - Y_1 - bZ_0 = sX_0 - t(bZ_1 + tY_1)$, so

$$\deg(sX_0 - t(bZ_1 + tY_1)) = \frac{\gamma}{2}. \tag{39}$$

Let $0 < \alpha = \beta < \gamma$. From (1) and (3), we conclude that one of the polynomials $bZ_1 \pm tY_1$ has the degree equal to $\beta + \deg(Z_1)$. Since

$$(bZ_1 + tY_1)(bZ_1 - tY_1) = b^2 - bc - Y_1^2, \tag{40}$$

the degree of the other polynomial is equal to $\gamma - \deg(Z_1)$. But, for neither one of these possibilities the equation (39) holds. Now, let $0 < \alpha = \beta = \gamma$. We have $\deg(X_0) = \deg(Z_0) = \frac{\gamma}{2}$ and $\deg(Z_1) = \deg(Y_1) = \frac{\gamma}{2}$. Moreover, by [6, Lemma 2], it follows that $c = a + b \pm 2r$. Now, let $b - a$ be a constant k. From (1), we have $a^2 + ka + 1 = r^2$. If we denote the leading coefficients of the polynomials a and r by a_1 and r_1, it follows that $a_1 = \pm r_1$. Moreover, we can transform the previous equation into $(a \pm 1)^2 \mp 2a + ka = r^2$, from which we conclude that

$$a(k \mp 2) = (r - a \mp 1)(r + a \pm 1).$$

Hence, it follows that either $a|(r - a \mp 1)$ or $a|(r + a \pm 1)$. If $a|(r - a \mp 1)$, then $a|(r \mp 1)$, so $(a+k)|(r \pm 1)$. We have $r \mp 1 = m_1 a$, $r \pm 1 = m_2(a+k)$, where $m_1, m_2 \in \mathbb{K} \setminus \{0\}$. Considering the leading coefficients in these to equations, we obtain $m_1 = m_2 = \pm 1$ and $k = \pm 2$. The possibility $a|(r + a \pm 1)$ leads to analogous conclusion. Now we have $r = \pm a \pm 1$, $c = 4a \pm 4$, $s = \pm 2a \pm 1$ and $t = \pm 2a \pm 3$. If we insert this into (38), and use (39), it follows that

$$\deg((\pm 2a \pm 3)(X_0 - (bZ_1 + tY_1)) \mp 2X_0) = \frac{\gamma}{2}.$$

If this holds, then $X_0 = bZ_1 + tY_1$. But, then it follows that $\mp 2X_0 = (b-a)Z_0$. Hence, $X_0 = \pm Z_0$ and, by (2), $X_0 = Z_0 = \pm 1$. Therefore, $\gamma = 0$, which is a contradiction.

8) Assume now that $(m,n) = (3,2)$. Then $z = V_3 = W_2$. From (4) and (5) we obtain

$$z = sZ_0 + c(4asZ_0 + 3X_0) + 4ac^2X_0 = Z_1 + 2c(bZ_1 + tY_1). \tag{41}$$

Suppose that $Z_0 = \pm 1$. From Lemma 3, we have that $Z_1 = \pm s$. Then, by (2), $X_0 = \pm 1$ and, by (3), $Y_1 = \pm r$. Let us notice that $\frac{\alpha+\gamma}{2} = \deg(Z_1) \le \frac{3\gamma-\beta}{4}$, so

$$\gamma \ge \beta + 2\alpha.$$

Hence, $\gamma > \alpha$ and $\alpha = 0$ if $\beta = \gamma$. Now from (41) it follows that

$$V_3 = \pm s + c(\pm 4as \pm 3) \pm 4ac^2, \tag{42}$$

$$W_2 = \pm s + 2c(\pm bs \pm tr). \tag{43}$$

We conclude that $\deg(W_2) = \gamma + \deg(\pm bs \pm tr)$ and $\deg(V_3) = \deg(\pm 4ac^2) = 2\gamma + \alpha$. It follows that $\deg(V_3) = \deg(W_2)$, so it must be the case that

$$\deg(\pm bs \pm tr) = \alpha + \gamma. \tag{44}$$

Since

$$(\pm bs + tr)(\pm bs - tr) = b^2 - ab - bc - 1, \tag{45}$$

using (1), we conclude that one of the polynomials $\pm bs \pm tr$ has degree equal to $\beta + \frac{\alpha+\gamma}{2}$ and the other one has degree $\le \frac{\gamma-\alpha}{2}$. Hence, in (44), we must have $\beta + \frac{\alpha+\gamma}{2} = \alpha + \gamma$, from which it follows that

$$\beta = \frac{\alpha+\gamma}{2}, \tag{46}$$

and we conclude that we cannot have $\beta = \gamma$, $\alpha = 0$. So, we get that $\beta < \gamma$. Since $V_3 = W_2$, from (42) and (43) we obtain that

$$2(\pm bs \pm tr) = \pm 4as \pm 4s^2 \mp 1. \tag{47}$$

Notice that, by (47), we also have

$$2(\pm tr \mp bs) \equiv \mp 1 \pmod{s}, \tag{48}$$

where both sides of the congruence relation have degree less than $\deg(s)$. Therefore, in (48), we can replace $\equiv$ with $=$. Moreover, we conclude that $\alpha = \gamma$, which is a contradiction. We are left to check the possibility $\alpha = 0$. Now, by (46), we have that $\gamma = 2\beta$, and by (44),

$$\deg(\pm bs \pm tr) = 2\beta.$$

Also, we have $\deg(\pm bs \mp tr) = \beta$, so $\deg(2(\pm tr \mp bs) \pm 1) = \beta$. From (1) and (45), we conclude that the polynomials $\pm s$ and $2a(\pm tr \mp bs) \pm a$ have the same leading coefficient, so, by (48), it follows that

$$2a(\pm tr \mp bs) = \pm s \mp a. \tag{49}$$

From (47) and (49), we obtain that

$$\pm s = \pm b \mp a \pm \frac{1}{4a}. \tag{50}$$

Now, from (49), using (1) and (50) we obtain that

$$\mp b \mp \frac{1}{4a} \equiv 0 \pmod{r},$$

so we conclude that r^2 divides $b^2 + \frac{b}{2a} + \frac{1}{16a^2} = (ab+1)(\frac{b}{a} - \frac{1}{2a^2}) + \frac{9}{16a^2}$. Hence, r^2 divides the constant $\frac{9}{16a^2}$ and we have $\beta = 0$, a contradiction.

Suppose that $Z_0 \neq \pm 1$. From [9, Lemma 5], it follows that $\deg(Z_0) \geq \frac{\gamma}{2}$ and $\deg(X_0) \geq \frac{\alpha}{2}$. Let us first consider the possibility that $\alpha = 0$ and X_0 is a constant. By Lemma 4, we have that $X_0^2 = a^2 + 1$, $Z_0 = \pm s$ and $Z_1 = \pm 1$ (Z_0 and Z_1 have the same signs). By (3), $Y_1 = \pm 1$, so from (41) we obtain that

$$V_3 = \pm s^2 + c(\pm 4as^2 + 3X_0) + 4ac^2 X_0, \tag{51}$$

$$W_2 = \pm 1 + 2c(\pm b \pm t). \tag{52}$$

We conclude that $\deg(V_3) = \deg(\pm 4acs^2 + 4ac^2X_0) \leq 2\gamma$. Suppose that this degree is less then 2γ. If we denote the leading coefficient of the polynomial c with c_1, we have $\pm 4a^2c_1^2 + 4ac_1^2X_0 = 0$ and $X_0 = \pm a$, a contradiction. Hence, $\deg(V_3) = 2\gamma$ and now from (52) it follows that $\deg(\pm b \pm t) = \gamma$. Therefore, $\beta = \gamma$.

Let us consider the polynomial Diophantine triple $\{a, b, c\}$. Have already noted that if $d_- \neq 0$, then $\deg(d_-) = 0$. Considering (1) and (12), we conclude that in the case $d_- \neq 0$, the only possibility is $d_- = a$. Suppose first that $d_- = 0$ and denote by b_1, c_1, t_1 the leading coefficients of the polynomials b, c, t, respectively. We notice that $b_1 = c_1$ and $t_1 = \pm b_1$. Moreover, from (51) and (52), we obtain that

$$\pm 1 = \pm a^2 + aX_0, \tag{53}$$

where the signs $\pm$ are the same in both sides of the equality. If both signs are positive, it follows that $X_0 = 2a$ and $a^2 = \frac{1}{3}$. Now, from the equation $V_3 = W_2$, using (51) and (52) we get that $15a = \mp 6r$, which is a contradiction because the degree of the left-hand side is equal to 0, and the degree of the right-hand side is equal to $\frac{\gamma}{2}$. If both signs in (53) are negative, we obtain $X_0 = -2a$, $a^2 = \frac{1}{3}$. As above, we obtain a contradiction. If $d_- = a$, from (12) we have that $u = \pm X_0$, $v = \pm r$ and by (13) it follows that $c = b + 2r^2(a \pm X_0)$. Using (1), we obtain that $s^2 = r^2(a \pm X_0)^2$, so $c = b \pm 2rs$. From the above relation, using (1) again, it follows that $s^2 = r^2 \pm 2ars$. Denoting $e := \frac{s}{r}$, we obtain a quadratic equation

$$e^2 \mp 2ae - 1 = 0 \tag{54}$$

whose solutions are constants $e_{1,2} = \pm a \pm X_0$. Using (1), we obtain that $c = e^2 b \pm 2e$ and $t = \pm(eb \pm 1)$. Now, from $V_3 = W_2$, by (51) and (52) it follows that

$$\pm 2b \pm 2t = \pm 5a + 3X_0 + 4ac(X_0 \pm a). \tag{55}$$

Inserting the above expressions for c and t into (55), in both sides of the equation we get polynomials of the form $g_1 b + g_2$, where g_1 and g_2 are constants. Comparing the coefficients, we obtain two polynomial equations in unknowns a, X_0 and e. We take (54) for the third and $X_0^2 = a^2 + m$ for the fourth equation in a system of four equations with unknowns a, X_0, e and m. By changing the signs $\pm$, we obtain 32 different systems of equations. But neither of them gives the solution $m = 1$, so conclude that d_- cannot be equal to a.

Now we are left with the possibility when $Z_0 \neq \pm 1$ and X_0 is not a constant. By Lemma 3, $Z_1 = sZ_0 \pm cX_0$. Assume first that

$$Z_1 = sZ_0 - cX_0.$$

Inserting this into (41), we get

$$bZ_1 + tY_1 = 2X_0 + 2a(sZ_0 + cX_0). \tag{56}$$

By (1) and (2), one of the polynomials $sZ_0 \pm cX_0$ has the degree equal to $\gamma + \deg(X_0)$. Since $\deg(Z_1) < \gamma$, we have $\deg(sZ_0 + cX_0) = \gamma + \deg(X_0)$. From (56), it follows that $\deg(bZ_1 + tY_1) = \alpha + \gamma + \deg(X_0)$. Moreover, $\deg(bZ_1 + tY_1) \leq \max(\deg(bZ_1), \deg(tY_1)) = \frac{3\beta+3\gamma}{4}$, so we have

$$\alpha + \gamma + \deg(X_0) \leq \frac{3\beta + 3\gamma}{4}. \tag{57}$$

From the case $(m, n) = (2, 3)$ we know that one of the polynomials $bZ_1 \pm tY_1$ has degree equal to $\beta + \deg(Z_1)$ and the other one has degree less or equal to $\gamma - \deg(Z_1)$. Suppose first that

$$\alpha + \gamma + \deg(X_0) \leq \gamma - \deg(Z_1). \tag{58}$$

By [9, Lemma 5], $\deg(X_0) \geq \frac{\alpha}{2}$, so from (58) we get that $\deg(Z_1) \leq -\frac{3}{2}\alpha$, which is a contradiction unless $\alpha = 0$ and $\deg(Z_1) = 0$. But then from (17), it follows that $\deg(X_0) = \gamma$, a contradiction. Therefore,

$$\alpha + \gamma + \deg(X_0) = \beta + \deg(Z_1).$$

If $\alpha < \gamma$, by (17), we have that $\deg(Z_1) = \gamma - \deg(X_0)$, so we get that $\deg(X_0) = \frac{\beta-\alpha}{2}$. Now (57) gives us that $\gamma \leq \beta - 2\alpha$, which is true only if $\alpha = 0$ and $\beta = \gamma$. But then $\frac{\gamma}{2} = \deg(X_0) \leq \frac{\alpha+\gamma}{4}$. Thus, $2\gamma \leq \gamma$, which is a contradiction. If $\alpha = \gamma$, from (57) we have $2\gamma \leq \frac{3}{2}\gamma$, a contradiction. It remains to consider the possibility

$$Z_1 = sZ_0 + cX_0.$$

From (41), we obtain

$$bZ_1 + tY_1 = X_0 + 2aZ_1. \tag{59}$$

Notice that $Z_1 = V_1$, so from [9, Lemma 5] it follows that $\deg(V_3) = \alpha + \gamma + \deg(Z_1)$. Therefore, $\deg(W_2) = \gamma + \deg(bZ_1 + tY_1) = \alpha + \gamma + \deg(Z_1)$ and $\deg(bZ_1 + tY_1) = \alpha + \deg(Z_1)$. Recall that the degree of the polynomial $bZ_1 + tY_1$ is either equal to $\beta + \deg(Z_1)$, or is less than or equal to $\gamma - \deg(Z_1)$. If $\alpha < \beta$, then we have $\alpha + \deg(Z_1) \leq \gamma - \deg(Z_1)$, so $\deg(Z_1) \leq \frac{\gamma-\alpha}{2}$. From (17), using the fact that $\deg(sZ_0 - cX_0) = \gamma + \deg(X_0) \leq \frac{\alpha+5\gamma}{4}$, we obtain $2\gamma \leq \frac{7\gamma-\alpha}{4}$, a contradiction. Therefore, $\alpha = \beta$. We now transform (59) into

$$tY_1 - bZ_1 = X_0 + 2Z_1(a - b). \tag{60}$$

The degree of the polynomial $bZ_1 + tY_1$ is equal to $\frac{\beta+\gamma}{2} + \deg(Y_1)$. Thus, from (40) and (60), it follows that

$$\deg(X_0 + 2Z_1(a - b)) \leq \frac{\beta + \gamma}{2} - \deg(Y_1).$$

In the same manner as in the case $(m, n) = (2, 3)$ (see (37)), we obtain a contradiction.

9) Let $(m,n)=(4,2)$. Then $z=V_4=W_2$. Moreover, from (4) and (5), we have

$$\begin{aligned} z &= Z_0+4c(2aZ_0+sX_0)+8ac^2(aZ_0+sX_0) \\ &= Z_1+2c(bZ_1+tY_1). \end{aligned} \tag{61}$$

By Lemma 3, $Z_0=Z_1$, so from (61) it follows that

$$2aZ_0+2(aZ_0+sX_0)+4ac(aZ_0+sX_0)=bZ_1+tY_1,$$

and that

$$\alpha+\gamma+\deg(aZ_0+sX_0)=\deg(bZ_1+tY_1). \tag{62}$$

Recall from the case $(m,n)=(3,2)$ that one of the polynomials $bZ_1\pm tY_1$ has degree equal to $\beta+\deg(Z_1)$, while the other one has degree less or equal to $\gamma-\deg(Z_1)$. From (6)–(9), it now follows that $\gamma>\alpha$. If X_0 is not a constant, from (34) we conclude that $\deg(aZ_0+sX_0)$ cannot be equal to 0. Hence, degree of the left hand side of (62) is larger than γ so the right hand side has degree equal to $\beta+\deg(Z_1)$. Also, from (62), it follows that $\deg(Z_1)>0$ so $Z_1\neq\pm1$. Therefore, by [9, Lemma 5], it follows that $\deg(Z_1)\geq\frac{\gamma}{2}$. If $\deg(aZ_0+sX_0)=\alpha+\deg(Z_0)$, from (62) we get $2\alpha=\beta-\gamma$, which is possible only for $\alpha=0$ and $\beta=\gamma$. But, then $\deg(Z_1)\leq\frac{3\gamma-\beta}{4}=\frac{\gamma}{2}$. Hence, $\deg(Z_1)=\frac{\gamma}{2}$, and then $\deg(X_0)=\frac{\alpha}{2}=0$, which is a contradiction. If $\deg(aZ_0+sX_0)=\gamma-\deg(Z_0)$, from (62), it follows that $\deg(Z_1)=\frac{\alpha-\beta+2\gamma}{2}$. But then we have $\gamma\leq\beta-2\alpha$. Again, this is possible only for $\alpha=0$ and $\beta=\gamma$, so as above we get a contradiction. Hence, X_0 is a constant. By Lemma 4, $X_0^2=a^2+1$ and $Z_0=\pm s$. Then, $Z_1=\pm s$ and from (3) we have that $Y_1=\pm r$. Notice that $\alpha=0$, which implies that $\deg(Z_1)=\frac{\gamma}{2}$. Now, from (62), it follows that $\deg(aZ_0+sX_0)=\beta-\frac{\gamma}{2}$. But, $\deg(aZ_0+sX_0)=\deg(s(\pm a+X_0))=\frac{\gamma}{2}$ so $\beta=\gamma$. As in the case $(m,n)=(3,2)$, we conclude that $d_-=0$ or $d_-=a$. If $d_-=0$, then $c=a+b\pm 2r$. Let b_1, r_1, s_1, t_1 denote the leading coefficients of the polynomials b, r, s, t, respectively, then $c_1=b_1$. Also, $r_1=\pm s_1$ and $t_1=\pm b_1$. If we equate the leading coefficients of the polynomials on the both sides of the equation $V_4=W_2$, from (61) we obtain

$$2a(X_0\pm a)=\pm1, \tag{63}$$

where the signs $\pm$ on both sides of the equation must be the same. If they are both positive, then from (63), we get $X_0=3a$ and $a^2=\frac{1}{8}$. Similarly, for the negative signs, from (63) we get $X_0=-3a$ and $a^2=\frac{1}{8}$. Now, for the positive signs in (63), from (61) we obtain that

$$bs\pm tr=10as+2cs. \tag{64}$$

Eliminating the polynomials whose degree is equal to $\frac{3\beta}{2}$ on both sides of (64), we obtain polynomials with degree equal to β. We get

$$ab_1\pm ab_1=2ab_1\pm 4ab_1,$$

which is a contradiction for all combinations of the signs. For the negative signs in (63), from (61) it follows

$$-bs\pm tr=-10as-2cs.$$

As above, this implies a contradiction. Therefore, d_- cannot be equal to 0. Assume now that $d_-=a$. From the case $(m,n)=(3,2)$, we know that $c=b\pm 2rs$ and $\pm s=r(a\pm X_0)$. We also have relation (54) as well as $c=e^2b\pm 2e$, $t=\pm(eb\pm 1)$.

We insert these equations into (61). First we consider s with the positive sign. After dividing by r, from (61), we get

$$\pm 2eb \pm 1 = \pm 2e(2a + X_0) \pm 4ae(e^2 b \pm 2e)(a + X_0), \tag{65}$$

where both sides of the above equation have the form $g_1 b + g_2$, with g_1 and g_2 constants. Comparing the coefficients from both sides of (65), we obtain the equations $2e = 4ae^3(a + X_0)$ and $\pm 1 = 2e(2a + X_0) \pm 8ae^2(a + X_0)$. These equations together with (54) and the equation $X_0^2 = a^2 + m$ form a system of four equations in unknowns a, X_0, e and m. Considering all possible combinations of the signs $\pm$, we get 8 different systems of equations. But neither of them gives us a solution with $m = 1$, so we conclude that no appropriate a and X_0 exist. For the negative sign with s, the conclusion is completely analogous. Hence, d_- cannot be equal to a.

10) Assume finally that $(m, n) = (5, 2)$ and $z = V_5 = W_2$. In this case, by (4) and (5), we obtain

$$\begin{aligned} z &= sZ_0 + c(12asZ_0 + 5X_0) + 4c^2(5aX_0 + 4a^2 sZ_0) + 16a^2c^3X_0 \\ &= Z_1 + 2c(bZ_1 + tY_1). \end{aligned} \tag{66}$$

From (6)–(9), it follows that $3\gamma \le 3\beta - 8\alpha$, so we have $\alpha = 0$, $\gamma = \beta$. If $Z_0 = \pm 1$, by Lemma 3, $Z_1 = \pm s$ (Z_0 and Z_1 have the same signs). From (2), we obtain $X_0 = \pm 1$ and from (3) $Y_1 = \pm r$. Inserting this into (66), we have

$$V_5 = \pm s + c(\pm 12as \pm 5) + 4c^2(\pm 5a \pm 4a^2 s) \pm 16a^2c^3, \tag{67}$$

$$W_2 = \pm s + 2c(\pm bs \pm tr). \tag{68}$$

From (67), we notice that $\deg(V_5) = \deg(\pm 16a^2c^3) = 3\gamma$, so from (68) it follows that

$$\deg(\pm bs \pm tr) = 2\gamma.$$

But $\deg(\pm bs \pm tr) \le \max(\deg(\pm bs), \deg(\pm tr)) = \frac{3\gamma}{2}$, and we get a contradiction.

Hence, $Z_0 \ne \pm 1$. Suppose first that X_0 is a constant. By Lemma 4, $X_0^2 = a^2 + 1$, $Z_0 = \pm s$ and $Z_1 = \pm 1$ so from (3) we have $Y_1 = \pm r$. Inserting that into (66), we obtain

$$V_5 = \pm s^2 + c(\pm 12as^2 + 5X_0) + 4c^2(5aX_0 \pm 4a^2 s^2) + 16a^2c^3X_0, \tag{69}$$

$$W_2 = \pm 1 + 2c(\pm b \pm t). \tag{70}$$

From (69), we see that $\deg(V_5) = \deg(\pm 16a^2c^2s^2 + 16a^2c^3X_0) \le 3\gamma$ and from (70) we see that $\deg(W_2) = \gamma + \deg(\pm b \pm t) \le 2\gamma$. Let us denote by c_1 the leading coefficient of the polynomial c and suppose that $\deg(V_5) < 3\gamma$. It must be the case that $\pm 16a^2c_1^2ac_1 + 16a^2c_1^3X_0 = 0$, so $X_0 = \pm a$, but this contradicts the fact that $X_0^2 = a^2 + 1$.

Assume now that $Z_0 \ne \pm 1$ and X_0 is not a constant. By Lemma 3, we have

$$Z_1 = sZ_0 \pm cX_0,$$

so it follows that

$$\deg(sZ_0 \pm cX_0) \le \frac{\gamma}{2}. \tag{71}$$

As in the case $(m, n) = (1, 1)$, we can conclude that, if X_0 is not a constant, then one of the polynomials $sZ_0 \pm cX_0$ has the maximal degree $\gamma + \deg(X_0)$. By (17), the degree of the other polynomial must be equal to $\gamma - \deg(X_0)$. But $\gamma - \deg(X_0) \ge \frac{3\gamma}{4}$, so neither of the polynomials $sZ_0 \pm cX_0$ satisfy the inequality (71). □

Now we can prove the following gap principle, which we will use in the proof of Theorem 1.

Lemma 5. *Let $\{a,b,c,d\}$ be a polynomial Diophantine quadruple. Denote by α, β, γ, δ the degrees of a, b, c, d, respectively, and assume that $\alpha \le \beta \le \gamma \le \delta$. Then either $\delta \ge \dfrac{3\beta+5\gamma}{2}$ or $d = d_+$ or $\{a,b,c,d\} = \mathcal{D}_p$, where $p \in \mathbb{K}[X]$ is a nonconstant polynomial.*

Proof. Let $ad+1 = x^2$, $bd+1 = y^2$ and $cd+1 = z^2$. Then there exist integers $m, n \ge 0$ for which $z = V_m = W_n$, where (V_m) and (W_n) are the sequences defined by (4) and (5). By Proposition 1, it follows that if $\{a,b,c,d\}$ is not regular and if it is not equal to $\mathcal{D}_p$ for some nonconstant $p \in \mathbb{K}[X]$, then $m \ge 3$ and $n \ge 3$.

Assume that $n \ge 3$. From $cd+1 = z^2$, we have

$$\deg(d) = 2\deg(W_n) - \gamma. \tag{72}$$

Furthermore, from [9, Lemma 5], it follows that

$$\begin{aligned}\deg(d) &\ge 2(2\deg(t) + \deg(W_1)) - \gamma \\ &= 2(\beta + \gamma + \deg(tZ_1 + cY_1)) - \gamma. \end{aligned} \tag{73}$$

If $\beta < \gamma$, from (16) and the estimate $\deg(tZ_1 \pm cY_1) \le \max(\deg(tZ_1), \deg(cY_1)) = \frac{\beta+5\gamma}{4}$, it follows that $\deg(tZ_1 + cY_1) \ge \frac{3\gamma-\beta}{4}$. Hence, from (73), we obtain

$$\deg(d) \ge \frac{3\beta+5\gamma}{2}.$$

If $\beta = \gamma$, by [9, Lemma 5] it follows that $\deg(W_1) \ge \frac{\gamma}{2}$, so from (73), we have $\deg(d) \ge 4\gamma$. □

4. PROOFS OF THEOREM 1 AND THEOREM 2

Now we can prove our first theorem by combining the gap principle from Lemma 5 with the upper bound from [6, Proposition 1]. This proposition asserts that if $\beta > \alpha$ and $\gamma > 4\beta - \alpha$, then $\delta < 3\gamma$, where α, β, γ, δ are the degrees of a, b, c, d with $\alpha \le \beta \le \gamma \le \delta$, respectively, and $\{a,b,c,d\}$ is a polynomial Diophantine quadruple.

PROOF OF THEOREM 1.
Assume that $\{a_1, a_2, ..., a_8\}$ is a polynomial Diophantine 8-tuple. Denote by α_i the degree of a_i for $i = 1, ..., 8$. Assume that $\alpha_1 \le \alpha_2 \le ... \le \alpha_8$.

Notice that $\{a_5, a_6, a_7, a_8\}$ is a polynomial Diophantine quadruple to which we can apply Lemma 5. Since $\alpha_5 > 0$, it is either regular or $\alpha_8 \ge \dfrac{5\alpha_7 + 3\alpha_6}{2}$. The set $\{a_4, a_6, a_7, a_8\}$ is also a polynomial Diophantine quadruple. Since $\alpha_4 > 0$, Lemma 5 also implies that $a_8 = d_+$ (for the triple $\{a_4, a_6, a_7\}$) or $\alpha_8 \ge \dfrac{5\alpha_7 + 3\alpha_6}{2}$. If $\{a_4, a_6, a_7, a_8\}$ and $\{a_5, a_6, a_7, a_8\}$ are both regular Diophantine quadruples, then for $\{a_6, a_7, a_8\}$ we have $a_4 = d_+$ or $a_4 = d_-$, and $a_5 = d_+$ or $a_5 = d_-$. By Lemma 2, $\deg(d_+) = \alpha_6 + \alpha_7 + \alpha_8$, which is larger than α_4 and α_5. Hence, $a_4 = a_5 = d_-$, which is a contradiction. Therefore, at least one of a_4 and a_5 is different from d_-.

We conclude that at least one of the quadruples $\{a_4, a_6, a_7, a_8\}$ and $\{a_5, a_6, a_7, a_8\}$ is irregular. If we apply Lemma 5 to that irregular quadruple, we get

$$\alpha_8 \geq \frac{5\alpha_7 + 3\alpha_6}{2} \geq \frac{5\alpha_6 + 3\alpha_6}{2} > 3\alpha_6. \tag{74}$$

Consider now the set $\{a_1, a_4, a_6, a_8\}$. If $\alpha_1 = 0$, then $\alpha_4 > \alpha_1$. If $\alpha_1 > 0$, by [6, Lemma 2], it follows that $\alpha_4 \geq \alpha_3 + \alpha_2 > \alpha_1$, so the first condition of [6, Proposition 1] is satisfied. We will show that the second condition is satisfied too. Consider the polynomial Diophantine quadruple $\{a_3, a_4, a_5, a_6\}$. Since $\alpha_3 > 0$, by Lemma 5, it is either regular or $\alpha_6 \geq \frac{5\alpha_5 + 3\alpha_4}{2}$. The set $\{a_2, a_4, a_5, a_6\}$ is also a polynomial Diophantine quadruple. Since $\alpha_2 > 0$, by Lemma 5, we also have that it is either regular or $\alpha_6 \geq \frac{5\alpha_5 + 3\alpha_4}{2}$. As above, we obtain

$$\alpha_6 \geq \frac{5\alpha_5 + 3\alpha_4}{2}. \tag{75}$$

Applying [6, Lemma 2] to the set $\{a_2, a_3, a_4, a_5\}$, from (75), we obtain

$$\alpha_6 \geq 4\alpha_4 + \frac{5}{2}\alpha_3 > 4\alpha_4 \geq 4\alpha_4 - \alpha_1.$$

Therefore, the second condition of [6, Proposition 1] is satisfied too and we conclude that

$$\alpha_8 < 3\alpha_6,$$

which contradicts (74). □

Now we can improve the upper bound from [6, Theorem 3].

PROOF OF THEOREM 2.
The proof is completely analogous to the proof of [6, Theorem 3]. For the upper bound of m, we estimate the Ramsey number $R(8, 6, 4, 5; 2)$ using the recurrence

$$R(n_1, n_2, ..., n_t; 2) \leq 2 - t + \sum_{i=1}^{t} R(n_1, ..., n_{i-1}, n_i - 1, n_{i+1}, ..., n_t; 2).$$

We also use a list of upper bounds for Ramsey numbers $R(n_1, n_2; 2)$ and some other known upper bounds e.g. $R(3, 3, 3, 3; 2) \leq 62$, $R(3, 3, 4; 2) \leq 31$, which are smaller than the upper bounds obtained by the above recurrence. Using all these results which can be found in [16] and some well known properties of Ramsey numbers from [12], we obtain

$$\begin{aligned} m &\leq R(4, 5, 6, 8; 2) \\ &\leq -2 + R(3, 5, 6, 8; 2) + R(4, 4, 6, 8; 2) + R(4, 5, 5, 8; 2) + R(4, 5, 6, 7; 2) \\ &\leq ... \leq 9800216 + 74786R(3, 5) + 119653R(3, 3, 4; 2) + 27420R(3, 3, 3, 3; 2) \\ &\leq ... \leq 16256503 < 2 \cdot 10^7. \end{aligned}$$

□

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