Equivariant monads and equivariant lifts versus a 2-category of distributive laws

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Fix a monoidal category $\mathcal{C}$. The 2-category of monads in the 2-category of $\mathcal{C}$-actegories, colax $\mathcal{C}$-equivarant functors, and $\mathcal{C}$-equivariant natural transformations of colax functors, may be recast in terms of pairs consisting of a usual monad and a distributive law between the monad and the action of $\mathcal{C}$, morphisms of monads respecting the distributive law, and transformations of monads satisfying some compatibility with the actions and distributive laws involved. The monads in this picture may be generalized to actions of monoidal categories, and actions of PRO-s in particular. If $\mathcal{C}$ is a PRO as well, then in special cases one gets various distributive laws of a given classical type, for example between a comonad and an endofunctor or between a monad and a comonad. The usual pentagons are in general replaced by multigons, and there are also "mixed" multigons involving two distinct distributive laws. Beck's bijection between the distributive laws and lifts of one monad to the Eilenberg-Moore category of another monad is here extended to an isomorphism of 2-categories. The lifts of maps of above mentioned pairs are colax $\mathcal{C}$-equivariant. We finish with a short treatment of relative distributive laws between two pseudoalgebra structures which are relative with respect to the distributivity of two pseudomonads involved, what gives a hint toward the generalizations.

1. Throughout the paper, $\mathcal{C}$ will be a fixed monoidal category with a monoidal product $\otimes$, a unit object $\mathbf{1}$, the associativity coherence isomorphisms $a_{X,Y,Z} : X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z$, natural in $X,Y,Z\in \mathrm{Ob}\,\mathcal{C}$, the left unit coherence $r : \mathrm{Id}_{\mathcal{C}}\Rightarrow\mathrm{Id}_{\mathcal{C}}\otimes\mathbf{1}$ and the right unit coherence $l : \mathrm{Id}_{\mathcal{C}}\Rightarrow\mathbf{1}\otimes\mathrm{Id}_{\mathcal{C}}$ satisfying for all $A,B,C,D\in\mathrm{Ob}\,\mathcal{C}$ the MacLane pentagon $a_{A,B,C\otimes D}\circ a_{A\otimes B,C,D}\circ(a_{A,B,C}\otimes D) = (A\otimes a_{B,C,D})\circ a_{A,B\otimes C,D}$ and unit triangle coherence relations $a_{\mathbf{1},A,B}\circ l_{A\otimes B} = l_A\otimes B$ and $r_{A\otimes B} = a_{A,B,\mathbf{1}}\circ(A\otimes r_B)$. A left coherent action of $\mathcal{C}$ on a category $\mathcal{N}$ is a coherent monoidal functor $\mathcal{L} : \mathcal{C}\to\mathrm{End}\,\mathcal{N}$ where $\mathrm{End}\,\mathcal{C}$ is strict monoidal with respect to the composition of endofunctors. Equivalently, a $\mathcal{C}$-action will be given by a bifunctor $\Diamond : \mathcal{C}\times\mathcal{N}\to\mathcal{N}$, natural isomorphisms $\Psi : (\_\otimes\_)\Diamond\_\Rightarrow\_\Diamond(\_\Diamond\_)$ and $u : \mathrm{Id}_{\mathcal{N}}\Rightarrow\mathbf{1}\Diamond\mathrm{Id}_{\mathcal{N}}$ satisfying for all $Q,Q',Q''\in\mathrm{Ob}\,\mathcal{C}$ and $N\in\mathrm{Ob}\,\mathcal{N}$ the action pentagon coherence $\Psi_{Q,Q',Q''\Diamond N}\circ\Psi_{Q\otimes Q',Q'',N}\circ(a_{Q,Q',Q''}\Diamond N) = (Q\Diamond\Psi_{Q',Q'',N})\circ\Psi_{Q,Q'\otimes Q'',N}$ and unit action coherences $u_{Q\Diamond N}\circ\Psi_{\mathbf{1},Q,N} = l_Q\Diamond N$ and $(Q\Diamond u_N)\circ\Psi_{Q,\mathbf{1},N} = r_Q\Diamond\boldsymbol{N}$. A $\mathcal{C}$-**actegory** is a category $\mathcal{N}$ equipped with a coherent action $\Diamond,\Psi,u$ of $\mathcal{C}$.

2. A **colax $\mathcal{C}$-equivariant functor of $\mathcal{C}$-actegories** $(F,\zeta) : (\mathcal{M},\Diamond^{\mathcal{M}},\Psi^{\mathcal{M}},u^{\mathcal{M}})\to(\mathcal{N},\Diamond^{\mathcal{N}},\Psi^{\mathcal{N}},u^{\mathcal{N}})$ is a usual functor $F : \mathcal{M}\to\mathcal{N}$ with a binatural transformation of bifunctors $\zeta : F(\_\Diamond^{\mathcal{M}}\_)\Rightarrow \_\tilde{\Diamond}^{\mathcal{N}} F(\_) : \mathcal{C}\times\mathcal{M}\Rightarrow\mathcal{N}$, so that

$$\begin{array}{ccc} \boldsymbol{F}(\mathbf{1}\Diamond^{\mathcal{M}} M) & \xrightarrow{\zeta_{\mathbf{1},M}} & \mathbf{1}\Diamond^{\mathcal{N}} F(M) \\ & {\scriptstyle F(u^{\mathcal{M}}_M)}\searrow \quad \swarrow{\scriptstyle u^{\mathcal{N}}_{F(M)}} & \\ & F(M) & \end{array} \tag{1}$$

$$\begin{array}{ccccc} F((A\otimes B)\Diamond^{\mathcal{M}} M) & & \xrightarrow{\zeta_{(A\otimes B)\Diamond M}} & & (A\otimes B)\Diamond^{\mathcal{N}} F(M) \\ \downarrow{\scriptstyle F(\Psi^{\mathcal{M}}_{A,B,M})} & & & & \downarrow{\scriptstyle \Psi^{\mathcal{N}}_{A,B,F(M)}} \\ F(A\Diamond^{\mathcal{M}}(B\Diamond^{\mathcal{N}} M)) & \xrightarrow{\zeta_{A,B\Diamond M}} & A\Diamond^{\mathcal{N}} F(B\Diamond^{\mathcal{M}} M) & \xrightarrow{A\Diamond\zeta_{B,N}} & A\Diamond^{\mathcal{N}}(B\Diamond^{\mathcal{N}} F(M)) \end{array} \tag{2}$$

$\mathcal{C}$-actegories and colax $\mathcal{C}$-equivariant functors make a category $\mathcal{C}-\mathbf{act}^c_1$: given $(F,\zeta^F) : \mathcal{N}\to\mathcal{P}$ and $(G,\zeta^G) : \mathcal{M}\to\mathcal{N}$ their composition is $(F,\zeta^F)\circ(G,\zeta^G) := (F\circ G, \zeta^{F\circ G}) : \mathcal{M}\to\mathcal{P}$ where $\zeta^{F\circ G}_{C,M} := \zeta^F_{C,GM}\circ F(\zeta_{C,M}) : F(G(C\Diamond^{\mathcal{M}} M))\Rightarrow C\Diamond^{\mathcal{P}} F(G(M))$.

3. A **$\mathcal{C}$-equivariant natural transformation of colax $\mathcal{C}$-equivariant functors** $\alpha : (F,\zeta^F)\Rightarrow(H,\zeta^H) : \mathcal{M}\to\mathcal{N}$ is a natural transformation of underlying ordinary functors $\alpha : F\Rightarrow G$ such that for all $C\in\mathcal{C}, M\in M$ the following square commutes:

$$\begin{array}{ccc} F(C\Diamond^{\mathcal{M}} M) & \xrightarrow{\zeta^F_{C,M}} & C\Diamond^{\mathcal{N}} FM \\ \downarrow{\scriptstyle \alpha_{C\Diamond M}} & & \downarrow{\scriptstyle C\Diamond\alpha_M} \\ G(C\Diamond^{\mathcal{M}} M) & \xrightarrow{\zeta^G_{C,M}} & C\Diamond^{\mathcal{N}} GM \end{array} \tag{3}$$

The usual transformation of usual functors obtained as a vertical or a horizontal composition of $\mathcal{C}$-equivariant natural transformations of colax $\mathcal{C}$-functors is $\mathcal{C}$-equivariant. Thus we obtain a strict 2-category $\mathcal{C}-\mathbf{act}^c$ which has all cartesian products, namely the usual products in **Cat** equipped with the diagonal $\mathcal{C}$-action, e.g. for binary products of $\mathcal{C}$-actegories $C\Diamond(M,N) = (C\Diamond M, C\Diamond N)$, and for $\mathcal{C}$-functors $(F,\zeta^F)\times(G,\zeta^G) = (F\times G, \zeta^F\times\zeta^G)$.

4. Let G be an endofunctor on a category $\mathcal{M}$. For a given monad $\mathbf{T} = (\mathbf{T},\mu,\eta)$ with a multiplication $\mu : TT\Rightarrow T$ and unit $\nu : \mathrm{Id}\Rightarrow T$ a **distributive law** between G and T is a natural transformation $l : GT\Rightarrow TG$ such that

$$\begin{array}{ccccc} GTT & \xrightarrow{lT} & TGT & \xrightarrow{Tl} & TTG \\ \downarrow{\scriptstyle G\mu} & & & & \downarrow{\scriptstyle \mu G} \\ GT & & \xrightarrow{l} & & TG \end{array} \tag{D1}$$

commutes and $l\circ G\eta = \eta G : G\Rightarrow TG$. A **lift** of an endofunctor (resp. (co)monad) G to a category C equipped with a functor U to $\mathcal{M}$ is an endofunctor (resp. (co)monad) $\tilde{G}$ such that $U\tilde{G} = GU$ (and obvious additonal conditions for the (co)monad case). The basic motivating fact for this definition states that the distributive laws between G and T are in a canonical bijection with the lifts of endofunctor G to the Eilenberg-Moore category $\mathcal{M}^{\mathbf{T}}$ of modules (M,ν) with respect to the forgetful functor $U : (M,\nu) \mapsto M$ (as usual, $M \in \mathrm{Ob}\,\mathcal{M}$ and $\nu : TM \to M$). Often G is also a (co)monad. Then, two additional axioms are required for l which ensure that $\tilde{G}$ is also a (co)monad. Modulo quoting this very fact, no proof in this paper needs repair when replacing distributive laws and lifts where G is endofunctor, with the version where G is a (co)monad.

5. In every strict 2-category, endo-1-cells of a fixed object and their natural transformations form a strict monoidal category, with the horizontal composition as the tensor product. In particular, $\mathbf{End}_{\mathcal{C}}(\mathcal{M}) := \mathcal{C}{-}\mathbf{act}^c(\mathcal{M},\mathcal{M})$ is a strict monoidal category. If $\tilde{T} = (T,\zeta)$ is an object in $\mathbf{End}(\mathcal{M})$, that is a colax $\mathcal{C}$-equivariant endofunctor, then its tensor square is $\tilde{T}\tilde{T} := (T \circ T, \zeta_T \circ T(\zeta))$. Here $(\zeta_T \circ T(\zeta))_{C,M} := \zeta_{C,TM} \circ T(\zeta_{C,M}) : TT(C\Diamond M) \Rightarrow C\Diamond TTM$. Let now $\tilde{\mathbf{T}} = (\tilde{T},\mu,\eta)$ be a monad in $\mathcal{C}{-}\mathbf{act}^c$. Our next aim is to decipher these data in terms of data in $\mathbf{Cat}$. $\tilde{T} = (T,\zeta)$ is a colax $\mathcal{C}$-equivariant endofunctor hence the two diagrams (1,2) commute with T in place of F. The multiplication $\mu : \tilde{T}\tilde{T} \Rightarrow \tilde{T}$ is a natural transformation $\mu : TT \Rightarrow T$, whose $\mathcal{C}$-equivariance says that (3) commutes for $\alpha = \mu$, $F = TT$, $G = T$, $\zeta^F = \zeta_T \circ T(\zeta)$ and $\zeta^G = \zeta$. From this we obtain the following pentagon

$$\begin{array}{ccccc} TT(C\Diamond M) & \xrightarrow{T(\zeta_{C,M})} & T(C\Diamond TM) & \xrightarrow{\zeta_{C,TM}} & C\Diamond TTM \\ \downarrow{\scriptstyle \mu_{C\Diamond M}} & & & & \downarrow{\scriptstyle C\Diamond \mu_M} \\ T(C\Diamond M) & & \xrightarrow{\qquad\zeta_{C,M}\qquad} & & C\Diamond TM \end{array} \tag{4}$$

The unit $\eta : (\mathrm{Id}_{\mathcal{M}}, \mathrm{Id}_{\mathrm{Id}}) \Rightarrow (T,\zeta)$ is a natural tranformation $\eta : \mathrm{Id} \to T$ and its $\mathcal{C}$-equivariance means that (3) commutes for $F = \mathrm{Id}_{\mathcal{M}}$, $\zeta^F = \mathrm{Id}_{\mathrm{Id}}$, $G = T$ and $\zeta^G = \zeta$ what reduces to the triangle

$$\begin{array}{ccc} & C\Diamond M & \\ {\scriptstyle \eta_{C\Diamond M}}\swarrow & & \searrow{\scriptstyle T(\eta_M)} \\ T(C\Diamond M) & \xrightarrow{\quad\zeta_{C,M}\quad} & C\Diamond TM \end{array} \tag{5}$$

The identities for μ and η (monad associativity $\mu \circ T(\mu) = \mu \circ \mu_T$ and unit axioms) simply say that the underlying endofunctor has a structure of a monad.

Proposition. *A monad $\tilde{\mathbf{T}} = (\tilde{T},\mu,\eta)$ in $\mathcal{C}{-}\mathbf{act}^c$ is the same as a usual monad $\mathbf{T} = (T,\mu,\eta)$ together with a binatural transformation $\zeta : T(\_\Diamond\_) \Rightarrow \_\Diamond T(\_)$, satisfying (1),(2) with $T = F$ and (4),(5), i.e. the* **distributive law** *between $\mathcal{C}$-action and* $\mathbf{T}$.

6. More generally, we may be given two actions of monoidal categories $\mathcal{C}$ and $\mathcal{D}$ on the same category $\mathcal{M}$. The distributive law between these two actions will be a binatural transformation of bifunctors $\mathcal{D}\Diamond(\mathcal{C}\Diamond\mathcal{M}) \Rightarrow \mathcal{C}\Diamond(\mathcal{D}\Diamond\mathcal{M})$ satisfying again some coherences; this general case will be studied elsewhere and, in the case of one left and one right action also in [9]. Let us now recall the classical case.

Let G be an endofunctor on a category $\mathcal{M}$. For a given monad $\mathbf{T} = (\mathbf{T},\mu,\eta)$ with a multiplication $\mu : TT \Rightarrow T$ and unit $\nu : \mathrm{Id} \Rightarrow T$ a **distributive law** between G and T is a natural transformation $l : GT\Rightarrow TG$ such that

$$\begin{array}{ccccc} GTT & \xrightarrow{lT} & TGT & \xrightarrow{Tl} & TTG \\ \downarrow{\scriptstyle G\mu} & & & & \downarrow{\scriptstyle \mu G} \\ GT & & \xrightarrow{\quad l \quad} & & TG \end{array} \tag{D1}$$

commutes and $l \circ G\eta = \eta G : G\Rightarrow TG$. A **lift** of an endofunctor (resp. (co)monad) G to a category C equipped with a functor U to $\mathcal{M}$ is an endofunctor (resp. (co)monad) $\tilde{G}$ such that $U\tilde{G} = GU$ (and obvious additonal conditions for the (co)monad case). The basic motivating fact for this definition states that the distributive laws between G and T are in a canonical bijection with the lifts of endofunctor G to the Eilenberg-Moore category $\mathcal{M}^{\mathbf{T}}$ of modules (M, ν) with respect to the forgetful functor $U : (M, \nu) \mapsto M$ (as usual, $M \in \mathrm{Ob}\,\mathcal{M}$ and $\nu : TM \to M$). Often G is also a (co)monad. Then, two additional axioms are required for l which ensure that $\tilde{G}$ is also a (co)monad. The generalizations of these additional axioms for the case of PRO are also studied below. We start with the easier partial case of monads.

7. A **map of monads** in a fixed category $\mathcal{M}$ is a natural transformation $\alpha : T \Rightarrow T'$ for which $\alpha \circ \mu_T = \mu_{T'}$ and $\mu \circ T\eta = \mu \circ \eta T = \mathrm{id} : T \Rightarrow T$. Every map of monads α induces a functor of Eilenberg-Moore categories $H^\alpha : \mathcal{M}^{\mathbf{T}'} \to \mathcal{M}^{\mathbf{T}}$ by the formula $H^\alpha(M, \nu') = (M, \nu' \circ \alpha_M)$. Conversely, if a functor $H : \mathcal{M}^{\mathbf{T}'} \to \mathcal{M}^{\mathbf{T}}$ is such that $UH = U'$, where U, U' are forgetful and F, F' are free T-algebra functors, then H induces a natural transformation $\alpha^H : T \Rightarrow T'$ given by the composition

$$T \stackrel{T\eta'}{\longrightarrow} TT' = UFU'F' = UFUHF' \stackrel{U\epsilon HF'}{\longrightarrow} UHF' = U'F' = T'. \tag{6}$$

These two rules are mutual inverses.

8. More generally, given monad $\mathbf{S}$ in category $\mathcal{M}$ and monad $\mathbf{T}$ in category $\mathcal{N}$, a **map of monads** $(K, \alpha) : \mathbf{T} \to \mathbf{S}$ is a pair of a functor $K : \mathcal{M} \to \mathcal{N}$ and natural transformation $\alpha : TK\Rightarrow KS : \mathcal{M} \to \mathcal{N}$ such that

$$\begin{array}{ccccc} TTK & \xrightarrow{T\alpha} & TKS & \xrightarrow{\alpha S} & KSS \\ \downarrow{\scriptstyle \mu^T K} & & & & \downarrow{\scriptstyle K\mu^S} \\ TK & & \xrightarrow{\quad \alpha \quad} & & KS \end{array}$$

commutes and $\alpha \circ \eta^T K = K\eta^S : K \to KS$. In $\mathcal{C}-\mathbf{act}^c$, the monads are now pairs $\tilde{\mathbf{S}} = (\mathbf{S}, l^S)$,$\tilde{\mathbf{T}} = (\mathbf{T}, l^T)$ and K is replaced by a colax $\mathcal{C}$-equivariant functor $(K, \zeta^K) : \mathcal{M}\Rightarrow\mathcal{N}$, i.e. $\zeta^K_{C,M} : K(C\Diamond^{\mathcal{M}} M)\Rightarrow C\Diamond^{\mathcal{N}} KM$ form a binatural transformation of functors satisfying the coherences of types (1),(2).

9. A map of monads (K, α) is a **map of pairs** $(K, \alpha) : (\mathbf{T}, l^T) \to (\mathbf{S}, l^S)$ if the following hexagon commutes

$$\begin{array}{ccccc} TKG^{\mathcal{M}} & \xrightarrow{T\zeta^K} & TG^{\mathcal{N}}K & \xrightarrow{l^T K} & G^{\mathcal{N}}TK \\ {\scriptstyle \alpha G^{\mathcal{M}}}\downarrow & & & & \downarrow{\scriptstyle G^{\mathcal{N}}\alpha} \\ KSG^{\mathcal{M}} & \xrightarrow{Kl^S} & KG^{\mathcal{M}}S & \xrightarrow{\zeta^K S} & G^{\mathcal{N}}KS \end{array} \tag{7}$$

where $G^{\mathcal{M}} = (C\Diamond^{\mathcal{M}}\_) \in \mathrm{End}_{\mathcal{C}}(\mathcal{M})$ etc. (for all C).

10. If $(K, \alpha) : \mathbf{T} \to \mathbf{S}$ and $(L, \beta) : \mathbf{V} \to \mathbf{T}$ are two maps of monads, then their composition is $(K, \alpha) \circ (L, \beta) := (L \circ K, L\alpha \circ \beta K) : \mathbf{V} \to \mathbf{S}$ which is again a map of monads as it follows by simple pasting:

$$\begin{array}{ccccccc} VV(LK) & & \xrightarrow{V(L\alpha\circ\beta K)} & VLKS & \xrightarrow{(L\alpha\circ\beta K)S} & & LKSS \\ & \searrow^{V\beta K} & \nearrow_{VL\alpha} & & \searrow^{\beta KS} & \nearrow_{L\alpha S} & \\ {\scriptstyle \mu^V LK}\downarrow & VLTK & \xrightarrow{\beta TK} & LTTK & \xrightarrow{LT\alpha} & LTKS & \downarrow{\scriptstyle LK\mu^S} \\ & & & \downarrow{\scriptstyle L\mu^T K} & & & \\ V(LK) & & \xrightarrow{\beta K} & LTK & \xrightarrow{L\alpha} & & (LK)S \end{array}$$

11. For the equivariant case, there is nothing more here to show, as this makes sense in any 2-category. The composition of maps of pairs is in detail

$$(L, \zeta^L, \alpha) \circ (K, \zeta^K, \beta) = (L \circ K, \zeta^L K \circ L\zeta^K, L\alpha \circ \beta K).$$

The diagram expressing the fact that

$$L\alpha \circ \beta K : (VLK, l^V LK \circ V\zeta^L K \circ VL\zeta^K) \Rightarrow (LKS, \zeta^L KS \circ L\zeta^K S \circ LKl^S)$$

is $\mathcal{C}$-equivariant may be obtained as follows:

$$\begin{array}{ccccccc} VLKG^{\mathcal{M}} & \xrightarrow{VL\zeta^K} & VLG^{\mathcal{N}}K & \xrightarrow{V\zeta^L K} & VG^{\mathcal{P}}LK & \xrightarrow{l^V LK} & G^{\mathcal{P}}VLK \\ {\scriptstyle \beta KG^{\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle \beta G^{\mathcal{N}}K} & & & & \downarrow{\scriptstyle G^{\mathcal{P}}\beta K} \\ LTKG^{\mathcal{M}} & \xrightarrow{LT\zeta^K} & LTG^{\mathcal{N}}K & \xrightarrow{Ll^T K} & LG^{\mathcal{N}}TK & \xrightarrow{\zeta^L TK} & G^{\mathcal{P}}LTK \\ {\scriptstyle L\alpha G^{\mathcal{M}}}\downarrow & & & & {\scriptstyle LG^{\mathcal{N}}\alpha}\downarrow & & \downarrow{\scriptstyle G^{\mathcal{P}}L\alpha} \\ LKSG^{\mathcal{M}} & \xrightarrow[LKl^S]{} & LKG^{\mathcal{M}}S & \xrightarrow[L\zeta^K S]{} & LG^{\mathcal{N}}KS & \xrightarrow[\zeta^L KS]{} & G^{\mathcal{P}}LKS \end{array}$$

12. A transformation of maps of (usual) monads $\sigma : (K, \alpha) \Rightarrow (L, \beta) : \mathbf{T} \to \mathbf{S}$ is a natural transformation $\sigma : K \Rightarrow L$ such that

$$\begin{array}{ccc} TK & \xrightarrow{T\sigma} & TL \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ KS & \xrightarrow{\sigma S} & LS \end{array} \tag{8}$$

commutes. For monads in $\mathcal{C}-\mathbf{act}^c$, te requirements ar the same, but of course the components need to be $\mathcal{C}$-equivariant. Thus the transformation of maps of pairs

$$\sigma : (K, \zeta^K, \alpha) \Rightarrow (L, \zeta^L, \beta) : (\mathbf{T}, l^T) \to (\mathbf{S}, l^S)$$

is a usual transformation $\sigma : K \Rightarrow L$ satisfying the same Eq. (8), but viewed as a transformation of pairs $\sigma : (K, \zeta^K) \Rightarrow (L, \zeta^L) : (\mathbf{T}, l^T) \to (\mathbf{S}, l^S)$, is required to be $\mathcal{C}$-equivariant, i.e. the square

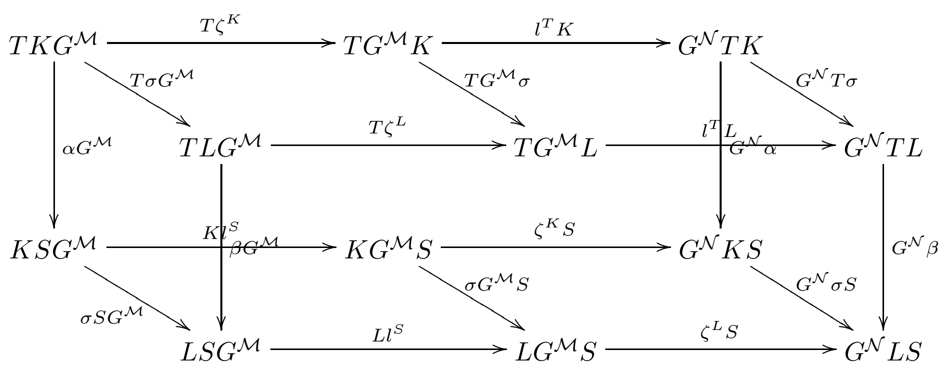
$$\begin{array}{ccc} K(C\Diamond^{\mathcal{M}} M) & \xrightarrow{\zeta^K_{C,M}} & C\Diamond^{\mathcal{N}} KM \\ \downarrow{\scriptstyle \sigma_{C\Diamond M}} & & \downarrow{\scriptstyle C\Diamond \sigma_M} \\ L(C\Diamond^{\mathcal{M}} M) & \xrightarrow[\zeta^L_{C,M}]{} & C\Diamond^{\mathcal{N}} LM \end{array} \tag{9}$$

commutes for all C in $\mathcal{C}$ and M in $\mathcal{M}$.

13. (The cube for transformations of maps of pairs) Denoting again, $G^{\mathcal{M}} = C\Diamond^M M_{\_}$ etc. we have the commutative diagram

$$\begin{array}{ccccc} TKG^{\mathcal{M}} & \xrightarrow{T\zeta^K} & TG^{\mathcal{N}}K & \xrightarrow{l^T K} & G^{\mathcal{N}}TK \\ \downarrow{\scriptstyle T\sigma G^{\mathcal{M}}} & & \downarrow{\scriptstyle TG^{\mathcal{N}}\sigma} & & \downarrow{\scriptstyle G^{\mathcal{N}}T\sigma} \\ TLG^{\mathcal{M}} & \xrightarrow[T\zeta^L]{} & TG^{\mathcal{N}}L & \xrightarrow[l^T L]{} & G^{\mathcal{N}}TL \end{array}$$

which is actually the upper face of the cube



(10)

where the bottom face is analogous commutative diagram involving L instead of K, where the left and right faces commute because σ is a transformation of usual monads, and the front and back hexagons commute because β and α are maps of pairs, cf. diagram (7). Hence the cube commutes.

14. Theorem. (Mixed heptagon formula, given a map of distributive laws) *Let l^S, l^T be two distributive laws between a $\mathcal{C}$-actions and monads,* $\mathbf{S}, \mathbf{T}$ *in $\mathcal{C}$-actegories*

$\mathcal{M},\mathcal{N}$ respectively, and $(K,\zeta^K,\alpha) : (\mathbf{T}, l^T)\Rightarrow(\mathbf{S}, l^S)$ a map of pairs as above. Then for each C in $\mathcal{C}$, $G^{\mathcal{M}} := C\Diamond^{\mathcal{M}}\_ \in \mathrm{End}_{\mathcal{C}}(\mathcal{M})$, the following diagram commutes

$$\begin{array}{ccccccc}
TKSG^{\mathcal{M}} & \xrightarrow{TKl^S} & TKG^{\mathcal{M}}S & \xrightarrow{T\zeta^K S} & TG^{\mathcal{N}}KS & \xrightarrow{l^T K} & G^{\mathcal{N}}TKS \\
{\scriptstyle (K\mu^S\circ\alpha S)G^{\mathcal{M}}}\big\downarrow & & & & & & \big\downarrow{\scriptstyle G^{\mathcal{N}}(K\mu^S\circ\alpha S)} \\
KSG^{\mathcal{M}} & & \xrightarrow{\quad Kl^S \quad} & & KG^{\mathcal{M}}S & \xrightarrow{T\zeta^K} & G^{\mathcal{N}}KS
\end{array}$$

Proof. This is obtained by the pasting of the following diagram

$$\begin{array}{ccccccc}
TKSG^{\mathcal{M}} & \xrightarrow{TKl^S} & TKG^{\mathcal{M}}S & \xrightarrow{T\zeta^K S} & TG^{\mathcal{N}}KS & \xrightarrow{l^T K} & G^{\mathcal{N}}TKS \\
{\scriptstyle \alpha SG^{\mathcal{M}}}\big\downarrow & & {\scriptstyle \alpha G^{\mathcal{M}}S}\big\downarrow & & & & \big\downarrow{\scriptstyle G^{\mathcal{N}}\alpha S} \\
KSSG^{\mathcal{M}} & \xrightarrow{KSl^S} & KSG^{\mathcal{M}}S & \xrightarrow{Kl^S S} & KG^{\mathcal{M}}SS & \xrightarrow{\zeta^K SS} & G^{\mathcal{N}}KSS \\
{\scriptstyle K\mu^S G^{\mathcal{M}}}\big\downarrow & & & & \big\downarrow{\scriptstyle KG^{\mathcal{M}}\mu^S} & & \big\downarrow{\scriptstyle G^{\mathcal{N}}K\mu^S} \\
KSG^{\mathcal{M}} & & \xrightarrow{\quad Kl^S \quad} & & KG^{\mathcal{M}}S & \xrightarrow{\zeta^K S} & G^{\mathcal{N}}KS
\end{array}$$

where the upper left corner is commutative by naturality of α, the upper right by the pair property of α, the left lower corner is the pentagon for the distributive law l^S and the right lower corner comes from the naturality of μ^S. Q.E.D.

15. Recall that a PRO is a strict monoidal category, whose object part is the set of natural numbers (including 0) and the tensor product of objects is the addition of natural numbers (and the unit object is 0). Different PRO-s differ by the morphisms, and the tensor product on morphisms is still usually denoted by + but typically it is not commutative. A (strict) representation of PRO $\mathcal{D}$ in a monoidal category $\mathcal{E}$ is a strict monoidal functor $\mathcal{D} \to \mathcal{E}$. There is an obvious way to define PRO-s by morphism generators (under composition and "addition") and relations.

We saw above that an endocell in $\mathcal{C}-\mathbf{act}^c$ is an endofunctor T together with a "distributive law" between $\mathcal{C}$ and T what is a binatural transformation l^T satisfying two commutative diagrams (1), (2) with $F = T$ and $l^T = \zeta^F$. Given a representation $\boldsymbol{T}^\bullet : \mathcal{D} \to \mathcal{E}$ we denote by $\boldsymbol{T}^n := T(n)$ and simply $\boldsymbol{T} := \boldsymbol{T}(1)$

Theorem. *A (strict) representation of a PRO $\tilde{\mathbf{T}}^\bullet : \mathcal{D} \to \mathrm{End}_{\mathcal{C}}(\mathcal{M})$ is the same as a pair $(\mathbf{T}, l)$ where $\mathbf{T}^\bullet : \mathcal{D} \to \mathrm{End}(\mathcal{M})$ is a representation and $l = l^T$ is a binatural transformation*

$$l^T : T(\_\Diamond\_)\Rightarrow\_\Diamond T(\_), \quad l^T_{C,M} : T(C\Diamond M)\Rightarrow C\Diamond T(M)$$

satisfying (1), (2) and such that for every $\alpha : n \to m$ the (n+m+2)-gon

$$\begin{array}{ccccccc}
T^n(C\Diamond M) & \xrightarrow{T^{n-1}l_{C,M}} & T^{n-1}(C\Diamond TM) & \xrightarrow{T^{n-2}l_{C,TM}} & \ldots T(C\Diamond T^{n-1}M) & \xrightarrow{l_{C,T^{n-1}M}} & C\Diamond T^n M \\
{\scriptstyle \alpha_{C\Diamond M}}\big\downarrow & & & & & & \big\downarrow{\scriptstyle C\Diamond\alpha_M} \\
T^m(C\Diamond M) & \xrightarrow[T^{m-1}l_{C,M}]{} & T^{m-1}(C\Diamond TM) & \xrightarrow[T^{m-2}l_{C,TM}]{} & \ldots T(C\Diamond T^{m-1}M) & \xrightarrow[l_{C,T^{m-1}M}]{} & C\Diamond T^m M
\end{array}$$

commutes.

The last condition simply says that $\alpha : T^n \Rightarrow T^m$ is in fact a $\mathcal{C}$-equivariant transformation $\alpha : (T^n, l^{T^n}) \Rightarrow (T^m, l^{T^m})$ of colax $\mathcal{C}$-equivariant endofunctors, where $l^{T^n} := lT^{n-1} \circ \ldots \circ T^{n-2} lT \circ T^{n-1} l$. This gives as many new diagrams as there are many primitive natural transformations in the game. For example a nonunital comonad has a coproduct δ hence the distributive laws between $\mathcal{C}$-action and a nonunital comonad satisfy one more axiom, what amounts to 3 diagrams total. More precisely, one has a structure of a PRO on natural numbers where δ etc. are the maps between n and m instead of T^n and T^m and we are dealing in fact with a strict monoidal functor from this PRO to the category of endofunctors of $\mathcal{M}$ (called also a strict representation of this PRO). Now I claim that a strict representation in $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$ is simply a pair of a representation in $\mathrm{End}(\mathcal{M})$ and a distributive law in the generalized sense, satisfying $n+2$ relations if the PRO is generated by n morphisms.

16. Now specialize $\mathcal{C}$ to the image of a representation $G_\bullet : \mathcal{C}_0 \to \mathrm{End}(\mathcal{M})$ of (another) PRO $\mathcal{C}_0$ in $\mathrm{End}(\mathcal{M})$. The generating object is $G_1 = G_\bullet(1)$. $\mathcal{C}$ is itself not necessarily a PRO even in this case, as there may be a nonzero kernel of $G_\bullet$ on the level of objects, but this presents no difficulty in the following. This is a strict monoidal subcategory of $\mathrm{End}(\mathcal{M})$. Thus we have now two PRO-s in the game. First of all in this case the diagrams (1), (2) may be skipped all together! Namely $\Psi, u, l_{\mathbf{1},M}$ are all identities, hence (1) is a tautology, while (2) for general $A = G^n$, $B = G^m$ says simply

$$l_{G^{n+m},M} = G^n(l_{G^m,M}) \circ l_{G^n,G^m M}. \tag{11}$$

and in particular

$$l_{G^n,M} = G^{n-1}(l_{G,M}) \circ l_{G^{n-1},GM}. \tag{12}$$

what can be iterated to obtain

$$l_{G^n,M} = G^{n-1}(l_{G,M}) \circ G^{n-2}(l_{G,GM}) \circ \ldots \circ G(l_{G,G^{n-2}M}) \circ l_{G,G^{n-1}M}. \tag{13}$$

Thus **every $l_{G^n,M}$ can be in the case when Ψ-s are strict described in terms of $l_{G,G^s M}$ for varying** $s \leq n$. In particular, it is enough to consider the distributive laws with one naturality

$$l : TG \Rightarrow GT, \quad l_M := l_{G,M}.$$

We denote by $l^{(n)}_M := l_{G^n,M}$. This way we have

$$l^{(n)} = G^{n-1} l \circ G^{n-2} lG \circ \ldots \circ GlG^{n-2} \circ lG^{n-1}. \tag{14}$$

The naturality of $l_{C,M}$ in first argument, for $\delta : G^n \to G^m \in \mathrm{Mor}\,\mathcal{C} = G_\bullet(\mathcal{C}_0)$ says that $(n+m+2)$-gon

$$\begin{array}{ccccccc}
TG^n & \xrightarrow{lG^{n-1}} & GTG^{n-1} & \xrightarrow{GlG^{n-2}} \ldots & G^{n-1}TG & \xrightarrow{G^{n-1}l} & G^nT \\
\downarrow{\scriptstyle T\delta} & & & & & & \downarrow{\scriptstyle \delta T} \\
TG^m & \xrightarrow{lG^{m-1}} & GTG^{m-1} & \xrightarrow{GlG^{m-2}} \ldots & G^{m-1}TG & \xrightarrow{G^{m-1}l} & G^mT
\end{array} \tag{15}$$

commutes.

From now on, whenever we discuss the distributive law between two representations of PRO-s we will consider just the transformation l with one naturality.

For example, let $\mathcal{C}$ be the PRO for counital coalgebras. Its set of morphisms is generated by a morphism $\delta : 1 \to 2$, satisfying the coassociativity $(\delta + \mathrm{id})\delta = (\mathrm{id} + \delta)\delta$ and a morphism $\epsilon : 1 \to 0$ satisfying $(\epsilon + \mathrm{id})\delta = (\mathrm{id} + \epsilon)\delta = \mathrm{id}$. An action of this PRO is, of course, a counital comonad. Then, (15) becomes a pentagon for δ and a triangle for ϵ.

More generally, if we have two endofunctors first with a structure arising from a representation of one PRO and another with a structure arising from another PRO, with k and p relations respectively, then we get in total $k+p$ additional diagrams for l (there are no conditions on l except to be a transformation $TG \to GT$ otherwise). The sizes of diagrams are always $n+m+2$ where n and m are the domain and codomain of a morphism in one or another PRO in question.

17. If $G = T$ is an underlying functor or a comonad $\boldsymbol{G}$ and the distributive law $l : GG \Rightarrow GG$ satisfies the quantum Yang-Baxter equation $Gl \circ lG \circ Gl = lG \circ Gl \circ lG$ we say that l is a strong braiding on the comonad $\boldsymbol{G}$. Then formula (14) defines a distributive law between G and G^n where the latter is inductively equipped with a composite comonad structure using $l^{(p)}$. for $p < n$. These results are discussed in our earlier article [7].

18. Suppose $\tilde{\boldsymbol{S}}_\bullet : \mathcal{P} \to \mathrm{End}_{\mathcal{C}}(\mathcal{M})$, $\tilde{\boldsymbol{T}}_\bullet : \mathcal{P} \to \mathrm{End}_{\mathcal{C}}(\mathcal{N})$ are representations of a fixed PRO $\mathcal{P}$. As before, $\tilde{\boldsymbol{S}} = (\boldsymbol{S}_\bullet, l^S)$ and $\tilde{\boldsymbol{T}} = (\boldsymbol{T}_\bullet, l^T)$. A (colax) **map of pairs** $(K, \zeta^K, \alpha) : (\mathbf{T}_\bullet, l^T) \to (\mathbf{S}_\bullet, l^S)$ is a colax $\mathcal{C}$-equivariant functor $(K, \zeta^K) : \mathcal{M} \to \mathcal{N}$ together with a binatural transformation $\alpha : TK \Rightarrow KS$ such that hexagon (7) commutes and such that for every morphism $\tau : n \to p$ in $\mathcal{P}$ with $\tau^T := \boldsymbol{T}_\bullet(\tau)$ the following diagram also commutes:

$$\begin{array}{ccccccc} T^n K & \xrightarrow{T^{n-1}\alpha} & T^{n-1}KS & \xrightarrow{T^{n-2}\alpha S} \dots & TKS^{n-1} & \xrightarrow{\alpha S^{n-1}} & KS^n \\ \downarrow{\scriptstyle \tau^T K} & & & & & & \downarrow{\scriptstyle K\tau^S} \\ T^m K & \xrightarrow{T^{p-1}\alpha} & T^{p-1}KS & \xrightarrow{T^{p-2}\alpha S} \dots & TKS^{p-1} & \xrightarrow{\alpha S^{p-1}} & KS^p \end{array} \qquad (16)$$

A map of pairs may be thought of as a colax $\mathcal{C}$-equivariant intertwiner from $\tilde{\boldsymbol{S}}_\bullet$ to $\tilde{\boldsymbol{T}}_\bullet$.

19. Generalizing the notation from (14) for any natural transformation $\alpha : TK \Rightarrow KS$ define $\alpha^{(n)} := T^{n-1}\alpha \circ T^{n-2}\alpha S \circ \dots \circ \alpha S^{n-1} : T^n K \to KS^n$. Let $L : \mathcal{N} \to \mathcal{R}$ be a functor, $\mathbf{V}_\bullet : \mathcal{P} \to \mathbf{End}_{\mathcal{C}}(\mathcal{R})$ a $\mathcal{C}$-equivariant representation of $\mathcal{P}$, and $(L, \zeta^L, \beta) : (\mathbf{V}, l^V) \to (\mathbf{T}, l^T)$ a map of pairs.

Lemma. $L\alpha^{(n)} \circ \beta^{(n)} K = (L\alpha \circ \beta K)^{(n)}$.

This follows by easy induction. Using this one easily proves that the analogue of the multigon (16) for $L\alpha \circ \beta K$ is commutative. This together with **10** gives

Proposition. *The rule*

$$(L, \zeta^L, \beta) \circ (K, \zeta^K, \alpha) := (L \circ K, \zeta^L K \circ L\zeta^K, L\alpha \circ \beta K) : (\mathbf{V}, l^V) \to (\mathbf{S}, l^S)$$

gives a (associative) composition of maps of pairs.

20. Theorem. (Mixed heptagon for maps of endofunctor $\mathcal{C}$-equivariant

representations of a PRO) *For every $(\tau : n \to p) \in \mathrm{Mor}\,\mathcal{P}$, the following diagram*

$$\begin{array}{ccccccc}
TKS^{n-1}G^{\mathcal{M}} & \xrightarrow{TKl^{S(n-1)}} & TKG^{\mathcal{M}}S^{n-1} & \xrightarrow{T\zeta S^{n-1}} & TG^{\mathcal{N}}KS^{n-1} & \xrightarrow{l^{T}KS^{n-1}} & G^{\mathcal{N}}TKS^{n-1} \\
\Big\downarrow{\scriptstyle (K\tau^{S}\circ\alpha S^{n-1})G^{\mathcal{M}}} & & & & & & \Big\downarrow{\scriptstyle G(K\tau^{S}\circ\alpha S^{n-1})} \\
KS^{p}G^{\mathcal{M}} & \xrightarrow{Kl^{S(p-1)}} & KG^{\mathcal{M}}S^{p} & & \xrightarrow{\qquad\quad \zeta S^{p} \quad\qquad} & & G^{\mathcal{N}}KS^{p}
\end{array}$$

commutes, where $l^{S(n-1)} = S^{n-2}l^{S} \circ \ldots \circ l^{S}S^{n-2}$ and $\zeta = \zeta^{K}$.

This proof is completely analogous to the proof of **14**, hence it is left to the reader. We call these identities "mixed" because unlike the diagrams for l^T and l^S separately, they involve both l^T and l^S.

21. Finally, the notion of transformation of maps of pairs $\sigma : (K, \zeta^K, \alpha) \Rightarrow (L, \zeta^l, \beta) : (\boldsymbol{T}_\bullet, l^T) \to (\boldsymbol{S}_\bullet, l^S)$ is identical as in the case of monads in **12** (as it does not involve morphisms in $\mathcal{P}$): require the commutativity of (8) and (9).

22. Theorem. *The $\mathcal{C}$-equivariant endofunctor representations of PRO $\mathcal{P}$ in varying $\mathcal{C}$-actegories are objects of a 2-category $\mathrm{Rep}_{\mathcal{C}-\mathbf{act}^c}(\mathcal{P})$ where 1-cells are (colax) maps of pairs in the sense of* **18** *and 2-cells are transformations of maps of pairs in the sense of* **21**. *We also consider the 2-subcategory $\mathrm{Rep}_{\mathcal{C}-\mathbf{act}^p}(\mathcal{P}) \subset \mathrm{Rep}_{\mathcal{C}-\mathbf{act}^c}(\mathcal{P})$ where the 1-cells are those maps (K, ζ^K) of pairs whose coherences ζ^K are invertible.*

The details are left to the reader.

23. (The category $\mathfrak{Distr}(\mathcal{M}, G)$ of distributive laws between an endofunctor (resp. a (co)monad) G and varying monads in a fixed category $\mathcal{M}$.) Objects of $\mathfrak{Distr}(\mathcal{M}, G)$ are pairs $(\mathbf{T}, l)$ where $\mathbf{T}$ is a monad in $\mathcal{M}$ and l is a distributive law from G to T. Morphisms $(\mathbf{T}, l) \to (\mathbf{T}, l')$ are the monad morphisms $\alpha : \mathbf{T} \to \mathbf{T}'$ such that there is the following commuting square of natural transformations of endofunctors:

$$\begin{array}{ccc}
TG & \xrightarrow{l} & GT \\
\Big\downarrow{\scriptstyle \alpha G} & & \Big\downarrow{\scriptstyle G\alpha} \\
T'G & \xrightarrow{l'} & GT'
\end{array} \tag{17}$$

It is clear that if we $\mathcal{C}$ is the PRO with only trivial morphisms, and $G = \boldsymbol{G}(\mathbf{1})$ for a representation $\boldsymbol{G} : \mathcal{C} \to \mathbf{End}(\mathcal{P})$ then $\mathfrak{Distr}(\mathcal{M}, G)$ is simply the full sub-1-category of (the decategorification of) $\mathrm{Rep}_{\mathcal{C}-\mathbf{act}^c}(\mathcal{P})$ whose 0-cells are equivariant representations $\boldsymbol{T}$ of the PRO $\mathcal{P}$ for monoids (i.e. monads) in $\mathcal{M}$.

24. The original theorem of Beck.

(i) Let $l : TG \Rightarrow GT$ be a distributive law from an endofunctor (resp. monad) G to a monad $\mathbf{T} = (T, \mu, \eta)$. Then the rule

$$\tilde{G} : (M, \nu) \mapsto (GM, \nu_l) = (GM, G(\nu) \circ l_M), \;\; \nu_l : TGM \stackrel{l_M}{\to} GTM \stackrel{G(\nu)}{\to} GM, \tag{18}$$

defines an endofunctor on $\mathcal{M}^{\mathbf{T}}$ lifting G to an endofunctor (resp. monad).

(ii) Conversely, if $U : \mathcal{M}^{\mathbf{T}} \to \mathcal{M}$ is the forgetful functors (forgetting the monad action: $(M, \nu) \mapsto M$), and $\tilde{G} : \mathcal{M}^{\mathbf{T}} \Rightarrow \mathcal{M}^{\mathbf{T}}$ and endofunctor such that $U\tilde{G} = GU$ then for any object M in $\mathcal{M}$, the composition

$$TGM \xrightarrow{TG(\eta_M)} TGTM \xrightarrow{U(\epsilon_{\tilde{G}FM})} GTM \tag{19}$$

defines the M-component of a distributive law $TG \Rightarrow GT$.

(iii) These two rules are mutual inverses.

25. Proposition. *Condition (17) ensures that the induced functor H^α among the Eilenberg-Moore categories will be G-equivariant.*

Proof. Let $\tilde{G}$ and $\tilde{G}'$ be the lifts of G in $\mathcal{M}^{\mathbf{T}}$ and $\mathcal{M}^{\mathbf{T}'}$ respectively.

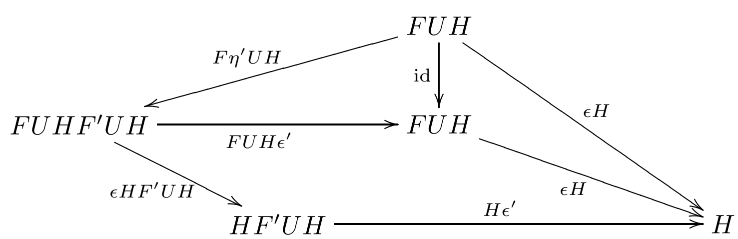
For all (M, ν') in $\mathcal{M}^{\mathbf{T}}$,

$$\begin{array}{rcl}
H^\alpha(\tilde{G}'(M, \nu')) & = & H^\alpha(GM, G(\nu') \circ l_M) \\
& = & (GM, G(\nu') \circ l_M \circ \alpha_{GM}) \\
& \stackrel{(17)}{=} & (GM, G(\nu') \circ G(\alpha_M) \circ l'_M) \\
& = & (GM, G(\nu' \circ \alpha_M) \circ l_M) \\
& = & \tilde{G}(M, \nu' \circ \alpha_M) \\
& = & \tilde{G}H^\alpha(M, \nu').
\end{array}$$

26. Lemma. *Given any functor $H : \mathcal{M}^{\mathbf{T}'} \to \mathcal{M}^{\mathbf{T}}$ satisfying $UH = U'$ the following identity holds*

$$H\epsilon' \circ \epsilon HF'UH \circ F\eta'UH = \epsilon H \tag{20}$$

Proof. This follows from the naturality square $\epsilon H \circ FUH\epsilon' = H\epsilon' \circ \epsilon HF'U'$ and the adjunction triangle $UH\epsilon \circ \eta'UH = \mathrm{id}_{UH}$ for $F' \vdash U' = UH$:

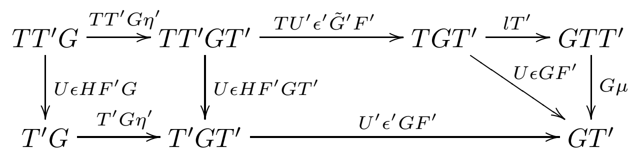


27. Theorem. (**Mixed pentagon formula, given a functor** H) *Let l, l' be two distributive laws from an endofunctor G on $\mathcal{M}$ to the monads $\mathbf{T}, \mathbf{T}'$ respectively. If $H : \mathcal{M}^{\mathbf{T}'} \to \mathcal{M}^{\mathbf{T}}$ is a functor such that $UH = U'$ and $\tilde{G}H = \tilde{G}'$ then*

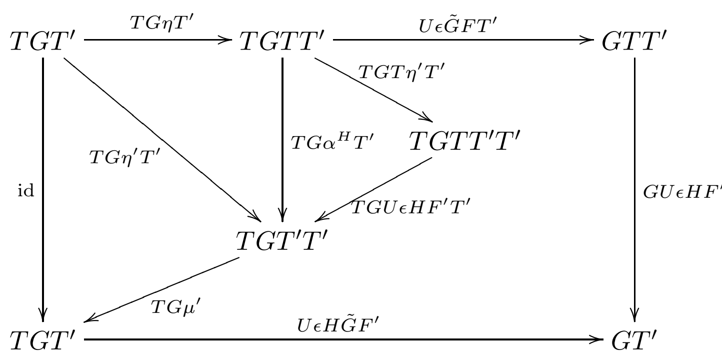
$$\begin{array}{ccccc}
TT'G & \xrightarrow{Tl'} & TGT' & \xrightarrow{lT'} & GTT' \\
\downarrow{\scriptstyle U\epsilon HF'G} & & & & \downarrow{\scriptstyle GU\epsilon HF'} \\
T'G & & \xrightarrow{\quad l' \quad} & & GT'.
\end{array} \tag{D1M}$$

Notice that two different distributive laws (twice l' and once l) appear in the formula and that this formula (D1M) reduces to (D1) if $\mathbf{T} = \mathbf{T}'$ and $l = l'$ (then $H = \mathrm{id}$ and $\mu = U\epsilon F$).

Proof. By (19), axiom (D1M) will follow by the commutativity of



Here the left-hand square and the middle rectangle commute by the naturality of $U\epsilon$. The corner triangle on the left will be expanded further to prove its commutativity:



The upper horizontal line is expanded using (19) and noticing $TGTT' = UFGUFT' = UFU\tilde{G}FT'$; what is sent by $U\epsilon\tilde{G}FT'$ into $U\tilde{G}FT' = GUFT' = GTT'$. The commutativity of the 3 triangles on the left is evident: for the leftmost follows from $\mu' \circ \eta' T = \mathrm{id}$ and the functoriality of TG; for the next triangle by the unit axiom for α^H, i.e. $\eta' = \alpha^H \circ \eta$; and for the third triangle by the definition (6) of α^H. To prove that the right-hand hexagon also commutes it is sufficient to prove that the 3 sides on the right compose to $UFUX$ where $X = \tilde{G}\epsilon HF'$. Indeed, $UX = GU\epsilon HF'$ is the top-down morphism on the right and the hexagon readily reduces to a naturality rectangle for $U\epsilon$. The 3 arrows on the right in fact compose to

$$\begin{aligned} TG(\mu' \circ U\epsilon HF'T' \circ T\eta'T') &= UFG(UH\epsilon'F' \circ U\epsilon HF'U'F' \circ UF\eta'U'F') \\ &= UFU\tilde{G}(H\epsilon' \circ \epsilon HF'U' \circ F\eta'U')F' \\ &= UFU\tilde{G}(H\epsilon' \circ \epsilon HF'U' \circ F\eta'U')F' \end{aligned}$$

The RHS is evidently equal to $UFUX$ as required if the expression in the brackets equals ϵH. This is exactly the content of the previous Lemma **26**, i.e. formula (20).

This finishes the proof of the "mixed pentagon formula".

28. Corollary. *If $H : \mathcal{M}^{\mathbf{T}'} \to \mathcal{M}^{\mathbf{T}}$ is a functor satisfying $UH = U'$, equivariant in the sense $H\tilde{G} = \tilde{G}'H$, then $\alpha = \alpha^H$ is a morphism in $\mathfrak{Distr}(\mathcal{M}, G)$, i.e. it satisfies (17).*

Proof. The required commutativity of (17), by the definition $\alpha^H := U\epsilon HF' \circ T\eta'$, reduces to the commutativity of the external part of the diagram

$$
\begin{array}{ccccc}
 & TG & \xrightarrow{\quad l \quad} & GT & \\
 \swarrow{\scriptstyle T\eta' G} & & \searrow{\scriptstyle TG\eta'} & & \searrow{\scriptstyle GT\eta'} \\
TT'G & \xrightarrow{\quad Tl' \quad} & TGT' & \xrightarrow{\quad lT' \quad} & GTT' \\
\downarrow{\scriptstyle U\epsilon HF'G} & & & & \downarrow{\scriptstyle GU\epsilon HF'} \\
GT' & & \xrightarrow{\qquad\qquad l' \qquad\qquad} & & GT'
\end{array}
$$

The commutativity of the left-top triangle is the unit axiom for the distributive law l, the right-top rectangle is commutative by the naturality of l, and the bottom is the pentagon (D1M) from **27**. Q.E.D.

29. Theorem. (Mixed pentagon formula, given a map α of distributive laws) *Let l, l' be two distributive laws from an endofunctor G to to monad $\mathbf{T}, \mathbf{T}'$ respectively and $\alpha : (\mathbf{T}, l) \Rightarrow (\mathbf{T}', l')$ a morphism in $\mathfrak{Distr}(\mathcal{M}, G)$. Then the following diagram commmutes*

$$
\begin{array}{ccccc}
TT'G & \xrightarrow{Tl'} & TGT' & \xrightarrow{lT'} & GTT' \\
\downarrow{\scriptstyle (\mu' \circ \alpha T')G} & & & & \downarrow{\scriptstyle G(\mu' \circ \alpha T')} \\
T'G & & \xrightarrow{\qquad l' \qquad} & & GT'.
\end{array}
\tag{D1Ma}
$$

Proof. This simple proof is due M. Jibladze (personal communication). Recall that $\mu = U\epsilon F$. Then the following diagram is commutative:

$$
\begin{array}{ccccc}
TT'G & \xrightarrow{Tl'} & TGT' & \xrightarrow{lT'} & GTT' \\
\downarrow{\scriptstyle \alpha T'G} & & \downarrow{\scriptstyle \alpha GT'} & & \downarrow{\scriptstyle G\alpha T'} \\
T'T'G & \xrightarrow{T'l'} & T'GT' & \xrightarrow{l'T'} & GT'T' \\
\downarrow{\scriptstyle \mu' G} & & & & \downarrow{\scriptstyle G\mu'} \\
T'G & & \xrightarrow{\qquad l' \qquad} & & GT'
\end{array}
$$

Indeed, the lower pentagon is a part of the statement that l' is a distributive law. The left upper corner square is a naturality square for α. Finally the right upper corner is expressing the condition that α is a map in $\mathfrak{Distr}(\mathcal{M}, G)$ (composed by T'). The external part of this diagram evidently gives (D1Ma). Q.E.D.

30. Proposition. If $H = H^\alpha$ in (D1M), or equivalently, by 2, $\alpha = \alpha^H$, then the vertical arrows in (D1M) are identical to the corresponding compositions of vertical arrows in (D1Ma).

Proof. It is sufficient to show $U\epsilon HF' = \mu' \circ \alpha^H T'$ as this implies the assertion both for the left-hand and right-hand vertical arrows. In fact we show the stronger assertion that $U\epsilon H = \mu' \circ \alpha^H U'$. Setting $\mu' = U'\epsilon' F' = UH\epsilon' F'$ and $\alpha^H = U\epsilon HF' \circ UF\eta'$ we reduce the required identity to $U\epsilon H = UH\epsilon' F' \circ U\epsilon HF'U' \circ UF\eta' U'$. By the functoriality of U, the assertion follows from Lemma **26**, that is formula (20). Q.E.D.

31. Theorem. *Given an endofunctor (resp. comonad) G in a category $\mathcal{M}$, the category $\mathfrak{Distr}(\mathcal{M}, G)$ is canonically isomorphic to the category of Eilenberg-Moore categories of varying monads equipped with a lift of G, and functors commuting with the forgetful functors and intertwining the lifts of G.*

Proof. **(i)** (**Bijection for objects**) By the definition, Eilenberg-Moore categories of $\mathbf{T}$-modules $\mathcal{M}^{\mathbf{T}}$ are trivially in 1-1 correspondence with the monads $\mathbf{T}$, and for a fixed monad the distributive laws are in bijection with lifts by Beck's theorem **24**.

(ii) (**Bijection of** Hom**-sets**) Given a pair of monads $\mathbf{T}$, $\mathbf{T}'$, it is also classical that morphism of monads are in 1-1 correspondence $\alpha \mapsto H^\alpha$ with the functors of Eilenberg-Moore categories commuting with the forgetful functor. So to show the bijection for morphisms there is only one nontrivial thing to prove: the property that a map α of monads is actually a morphism in $\mathfrak{Distr}(\mathcal{M}, G)$ corresponds exactly to the fact that H^α is intertwining the corresponding lifts of G. But all the hard work there has been already done: Proposition **25**, states this in one direction, and Corollary **28** does the converse.

(iii) (**$\alpha \mapsto H^\alpha$ is a contravariant functor**) This is certainly known, but we do not know the reference. First of all, the identity functor $H = \mathrm{id}$ gives $\alpha^{\mathrm{id}} = \mathrm{id}$ as it is clear by the adjunction triangle $\epsilon F \circ F\eta$. In the situation

$$\mathcal{M}^{\mathbf{T}''} \xrightarrow{H'} \mathcal{M}^{\mathbf{T}'} \xrightarrow{H} \mathcal{M}^{\mathbf{T}}$$

with $UH = U'$, $U'H' = U''$, we need to show that $\alpha^{H'} \circ \alpha^H = \alpha^{H \circ H'}$. The LHS is the composition

$$UF \xrightarrow{UF\eta'} UFU'F' \xrightarrow{U\epsilon HF'} UHF' \xrightarrow{U'F'\eta''} U'F'U''F'' \xrightarrow{U'\epsilon' H'F''} U'H'F'' = T''$$

By naturality of $U\epsilon$ we may interchange $U'F'\eta'' \circ U\epsilon HF' = U\epsilon HF'U''F'' \circ UFUHF'\eta''$ and furthermore interchange $U'\epsilon' H'F'' \circ U\epsilon HF'U''F' = U\epsilon HH'F'' \circ UFU'\epsilon' H'F''$: $UFU'F'U''F'' \to U'H'F'' = T''$. Thus we obtain that LHS equals

$$UF \xrightarrow{UF\eta'} UFU'F' \xrightarrow{UFUHF'\eta''} UFU'F'U''F'' \xrightarrow{UFU'\epsilon' H'F''} UFU''F'' \xrightarrow{U\epsilon HH'F''} T''$$

Now the composition of the second and third morphism is $UF\alpha^{H'}$ by the definition, and $\alpha^{H'} \circ \eta' = \eta''$ hence the composition of the first three transformations is $UF\eta''$, therefore all 4 compose to the $U\epsilon HH'F'' \circ UF\eta'' = \alpha^{H \circ H'}$ by the definition. Q.E.D.

32. It is again standard that maps of monads $\alpha : \mathbf{T} \to \mathbf{S}$ are in 1-1 correspondence with the functors $H : \mathcal{M}^{\mathbf{S}} \to \mathcal{N}^{\mathbf{T}}$, such that $U^T H = KU^S$.

$$\begin{array}{ccc} \mathcal{M}^{\mathbf{S}} & \xrightarrow{H} & \mathcal{N}^{\mathbf{T}} \\ {\scriptstyle U^S}\downarrow & & \downarrow{\scriptstyle U^T} \\ \mathcal{M} & \xrightarrow{K} & \mathcal{N} \end{array}$$

We will below need the explicit formulas for this bijection. Given a functor H as above, the corresponding map of monads $\alpha^H : TK \Rightarrow KS$ (cf. Borceux, II 4.5.1) is the composition

$$TK \xrightarrow{TK\eta^S} TKS = U^T F^T K U^S F^S = U^T F^T U^T H F^S \xrightarrow{U^T \epsilon^T H F^S} U^T H F^S = K U^S F^S = KS$$

Conversely, given a morphism of monads α we obtain the lift H^α simply as

$$H^\alpha(M,\nu) := (KM, K(\nu)\circ\alpha_M).$$

All together this is a canonical bijection, clearly extending the formulas in **7**. Moreover, for any transformation of maps of monads $\sigma : (K,\alpha)\Rightarrow(K',\alpha') : \mathbf{T}\Rightarrow\mathbf{S}$ one defines a natural transformation $\tilde\sigma : H^\alpha\Rightarrow H^{\alpha'}$ by $\tilde\sigma = \sigma U^T$, i.e.

$$\tilde\sigma_{(M,\nu)} := \sigma_M : (KM, K(\nu)\circ\alpha_M) \to (K'M, K'(\nu)\circ\alpha_M).$$

We leave for the reader to check that $\tilde\sigma_{(M,\nu)}$ is really a morphism in $\mathcal{N}^{\mathbf{T}}$, i.e.

$$\sigma_M\circ K(\nu)\circ\alpha_M = K'(\nu)\circ\alpha'_M\circ T(\sigma_M).$$

This transformation lifts σ, when considered just as a transformation of functors $\sigma : K\Rightarrow K'$. That means $\tilde\sigma = \sigma U^T$.

Conversely, given any natural transformation $\theta : H^\alpha\Rightarrow H^{\alpha'}$ such that $U^T(\theta_{(M,\nu)}) : KM\to K'M$ does not depend on ν and hence lifts a (unique) transformation of functors $\theta_* : K\Rightarrow K'$, then θ_* is automatically given by formula $(\theta_*)_M = U^T(\theta_{(TM,\mu_M)})$ which is a transformation of maps of monads

$$\theta_* : (K,\alpha)\Rightarrow(K',\alpha') : \mathbf{T}\to\mathbf{S}$$

Now we claim that H is equivariant (intertwines $G^{\mathcal{M}}$ and $G^{\mathcal{N}}$) iff α is a map of pairs, i.e. (7) holds. Of course, if the coherence ζ is non-trivial then one needs to equip also H with a coherence. Moreover, one can consider a certain 2-category of small categories each equipped with an endofunctor G, a monad, say $\mathbf{T}$, and a distributive law; with the maps of pairs as morphisms and certain class of compatible modifications of such morphisms. Then there is a 2-isomorphism with a 2-category of Eilenberg-Moore categories, equipped with lifts, equivariant functors of such and their equivariant natural transformations where everything commutes with the forgetful functors.

33. Proposition. *Condition (7) ensures a 2-cell $H^\alpha G^{\mathcal{M}}\Rightarrow G^{\mathcal{N}}H^\alpha$.*
Proof. For all $(M,\nu)\in\mathcal{M}^{\mathbf{S}}$,

$$\begin{array}{rcl} H^\alpha G^{\mathcal{M}}(M,\nu) &=& H^\alpha(G^{\mathcal{M}}M, G^{\mathcal{M}}(\nu)\circ l^S_M)\\ &=& (KG^{\mathcal{M}}M, KG^{\mathcal{M}}(\nu)\circ K(l^S_M)\circ\alpha_{GM})\\ &=& (KG^{\mathcal{M}}M, KG^{\mathcal{M}}(\nu)\circ G^{\mathcal{N}}(\alpha_M)\circ l^T_{KM})\\ &\Rightarrow& (G^{\mathcal{N}}KM, G^{\mathcal{N}}(K(\nu)\circ\alpha_M)\circ l^T_{KM})\\ &=& G^{\mathcal{N}}(KM, K(\nu)\circ\alpha_M)\\ &=& G^{\mathcal{N}}H^\alpha(M,\nu)\end{array}$$

We used in the middle step the 2-cell $\zeta^K_M : KG^{\mathcal{M}}M\Rightarrow G^{\mathcal{N}}KM$ in the first component and composing with it in the second component.

34. Theorem. *The natural transformation $\tilde\sigma : H^\alpha\Rightarrow H^{\alpha'}$ induced from a transformation of monads σ is equivariant iff $\sigma : \alpha\Rightarrow\alpha'$ is a transformation of maps of pairs.*

35. Theorem. *If $\mathcal{P}$ is the PRO for monoids then 2-category $\mathrm{Rep}_{\mathcal{C}-\mathbf{act}^c}(\mathcal{P})$ is isomorphic to the following 2-category: the objects are triples $(\mathcal{M},\boldsymbol{T},U^T:\mathcal{M}^T\to\mathcal{M})$ where*

$\boldsymbol{T}$ is a monad in a $\mathcal{C}$-actegory $\mathcal{M}$, $\mathcal{M}^{\boldsymbol{T}}$ is the Eilenberg-Moore category of $\boldsymbol{T}$ equipped with a $\mathcal{C}$-action making U^T a strict monoidal functor; 1-cells are colax $\mathcal{C}$-equivariant functors of Eilenberg-Moore categories $\mathcal{M}^{\boldsymbol{T}} \to \mathcal{N}^{\boldsymbol{S}}$ commuting with the forgetful functor and 2-cells the natural transformations of colax $\mathcal{C}$-equivariant functors.

36. In his classical article [10] R. Street has considered monads and Eilenberg-Moore objects in general 2-categories. The fact that the Beck's bijection between lifts and distributive laws extends to an isomorphism of 2-categories, may be viewed, after applying our correspondence between the 2-category of distributive laws and the 2-category of equivariant monads, as the correspondence between the Eilenberg-Moore objects and monads inside the 2-category $\mathcal{C}-\mathbf{act}^c$. For this one needs to apply a result on the existence of Eilenberg-Moore objects in this setup. S. Lack has proved a general result of this type, namely existence of certain lax limits whose combinations include the Eilenberg-Moore objects, in the 2-category of pseudoalgebras over a 2-monad. In our case the 2-monad is a strictification of the pseudomonad $\mathcal{C}\times$ on **Cat**, whose structure is induced from the monoidal category structure on $\mathcal{C}$, and whose pseudocoalgebras are coherent $\mathcal{C}$-actions. In a way this is more general than our approach as it allows other 2-monads: on the other hand our case is more general as the monads are generalized to actions of PRO-s and more general $\mathcal{D}$-actions. Some subtleties of the latter case are discussed in [9]. Our approach also emphasizes on explicit formulas for all the correspondences and isomorphisms instead of equivalences at certain places.

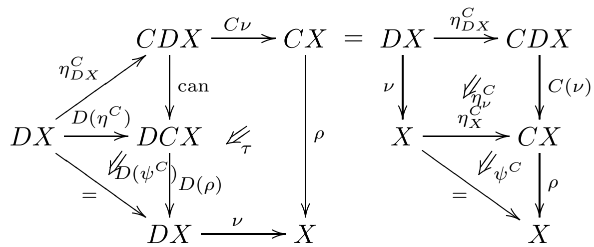
37. (Relative distributive laws) Recall that a pseudomonad in a Gray-category $\mathcal{K}$ is an object H in $\mathcal{K}$ and a pseudomonoid in the Gray-monoid $\mathcal{K}(H,H)$ (for Gray-pseudomonoids see e.g. [3]). Thus a pseudomonad is a tuple $\boldsymbol{D} = (D, \mu, \eta, \alpha^l, \alpha^r, \alpha^\mu)$ where $D : X \to X$ is a 1-cell in $\mathcal{K}$, $\mu : DD \to D$ and $\eta : D \to DD$ are 2-cells in $\mathcal{K}$ and the coherence for right unit $\alpha^r : \mu \circ D\eta \Rightarrow \mathrm{id}_D$, the coherence for left unit $\alpha^l : \mu \circ \eta D \Rightarrow \mathrm{id}_D$ and the coherence for associativity $\alpha^\mu : \mu \circ (D\mu) \Rightarrow \mu \circ (\mu D)$ are invertible 2-cells in $\mathcal{K}$ satisfying 2 standard coherence identities. Suppose we are given pseudomonads $\boldsymbol{C}$ and $\boldsymbol{D}$ in $\mathcal{K}$, and a *fixed* 1-cell X in $\mathcal{K}(H', H)$, for some object H' in $\mathcal{K}$. Suppose that X is both the $\boldsymbol{C}$-pseudoalgebra (X, ρ, ψ^C, ξ^C) and $\boldsymbol{D}$-pseudoalgebra (X, ν, ψ^D, ξ^D), one may ask what makes the $\boldsymbol{D}$-pseudoalgebra structure (say colax-) $\boldsymbol{C}$-equivariant in the sense that the defining 1-cell $\nu : DX \to X$ and the invertible 2-cells

$$\psi^D : \nu \circ \eta_X \Rightarrow \mathrm{Id}_X, \qquad \chi^D : \nu \circ (D\nu) \Rightarrow \nu \circ \mu_X,$$

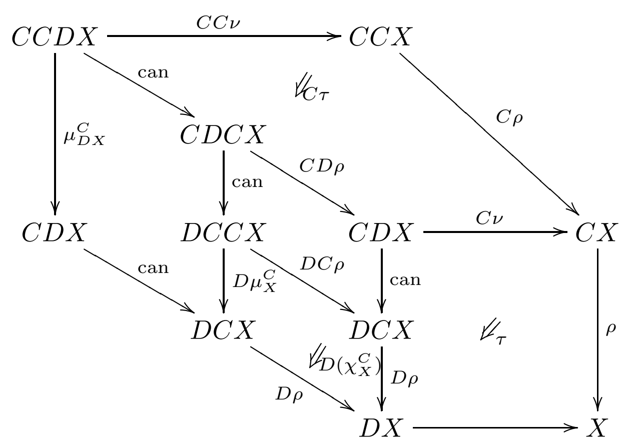
are equipped with a structure of 1-cell and 2-cells in the 2-category of $\boldsymbol{C}$-pseudoalgebras, colax morphisms of pseudoalgebras, and their natural transformations. For this to make sense we need also a $\boldsymbol{C}$-structure on DX what may need another distributive law, but in many cases this part of the data is in fact canonically provided, while the additional structure above is not. For example, if the pseudomonads are the cartesian products with monoidal categories then we can just use the commutativity of the cartesian product to identify DCX and CDX while their actions on concrete X does *not* trivially commute and what we discuss here is precisely the additional distributive structure for the two actions. More generally, we can consider just some "higher" distributive law between the pseudomonads, can : $DC \to CD$ and define the distributive laws for pseudoalgebras relatively to it. For the 1-cell $\nu : DX \to X$ the additional structure is a 2-cell

$$\tau : \rho \circ C\nu \Rightarrow \nu \circ D(\rho) \circ \mathrm{can}$$

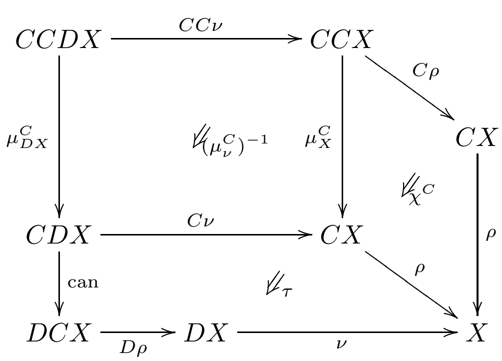
in $\mathcal{K}$, where two coherences hold for τ, namely



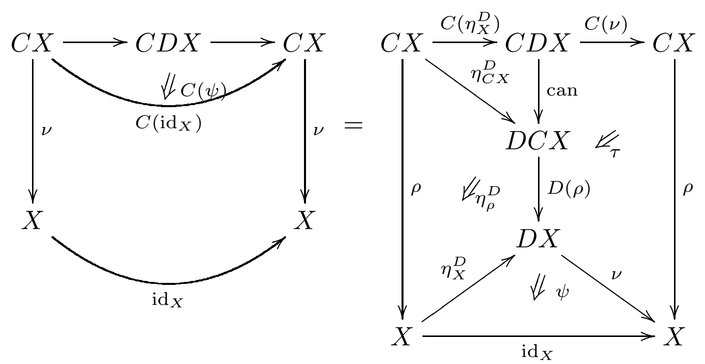
and the pasting



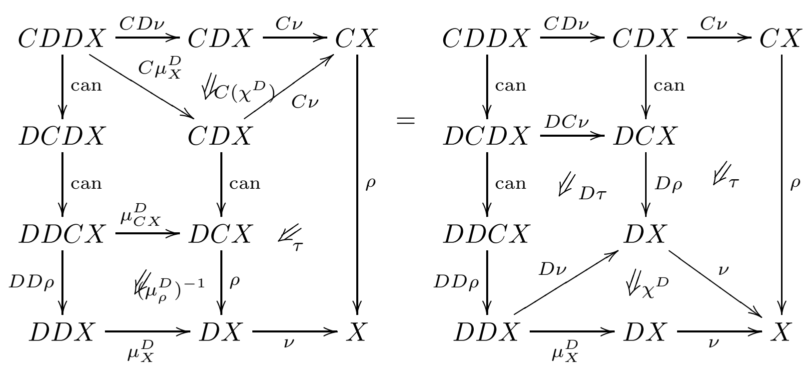
equals the pasting



These coherences say precisely that $(\psi^D, \tau) : (X, \rho, \psi^C, \chi^C) \to (X', \rho', \psi'^C, \chi'^C)$ is a colax morphism of $\boldsymbol{C}$-pseudoalgebras. Notice that if the pseudonaturality of μ^C and η^C is in fact naturality then we exactly get one triangle and one pentagon for the nonidentity 2-cells. For the 2-cells $\psi^D : \nu \circ \eta^D_X \Rightarrow \mathrm{id}_X$ and $\chi^D : \nu \circ D(\nu) \Rightarrow \nu \circ \mu^D_X$ there is no additional structure but rather a *requirement* that they are natural transformations of colax $\boldsymbol{C}$-equivariant morphisms of $\boldsymbol{C}$-pseudoalgebras, what boils down to a bit expanded tin-can diagrams:



If η^D is again natural (in particular $\eta^D_\rho = \mathrm{id}$), this identity boils down to a triangle for natural transformations. The tin can identity for χ^D is as follows



(21)

Again in the 2-categorical situation, when μ^D_ρ is the identity this boils down to a pentagon for natural transformation. The distributive laws between two actions of monoidal categories on a fixed category X are a special case of this construction. Notice that each of the two pentagons and two triangles, is defined using a pasting diagram which contains embedded exactly one pentagon or triangle for the higher distributive law. This is an interesting "recursive" structure. We see that the distributive laws between the pseudoalgebras are defined relative to a higher distributive law can : $CD \to DC$ between their pseudomonads which is in our case "canonical" and invertible, but it may be not so. Moreover, the higher distributive law may be in fact a pseudodistributive law as in [6], and we again, *mutatis mutandis*, define the distributive laws between the pseudoalgebras using essentially the same "relative" pasting diagrams as above, sometimes with nontrivial 2-cells inserted in place of trivial ones. For example, the upper left pentagon in the left-hand diagram in (21) is then filled with a nontrivial 2-cell.

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