

SUPERCOVER PLANE RASTERIZATION

A Rasterization Algorithm for Generating Supercover Plane Inside A Cube

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Abstract: An analysis of a rasterization algorithm for generating supercover planes in 3D voxel space is presented. The derived algorithm is an extension to the classical 2D line rasterization algorithm. Additional voxels needed to form the supercover 3D plane are identified by rasterizing two additional 2D lines per volume slice. A discussion on how to modify the algorithm to rasterize finite supercover 3D plane segments with arbitrary parameters by using integer arithmetic only is given.

1 Introduction

One of the problems in various applications of computer graphics along with line drawing is rasterization of planary surfaces. Our ability to obtain huge volumetric data is ever increasing, especially in biomedical fields where 256^3 is currently the most common volume size. When analyzing such volumetric data-sets one is not always satisfied with the simple line and plane drawing and rasterization algorithms as some interesting and sometimes not desirable effects can be introduced, such as existence of tunnels in surfaces so for example 26-connected line can pass through the 18-connected plane (Cohen-Or and Kaufman, 1997).

Most popular algorithm for line drawing was presented by Bresenham (Bresenham, 1965), and first generic study on plane rasterization was done by Kim (Kim, 1984). A fast plane weaving algorithm for rasterizing 18-connected digital planes was described by Lincke et al. (Lincke and Wüthrich, 1999). In this article a simple 3D supercover plane generation algorithm is presented. The described algorithm is an extension of the weaving algorithm presented by Lincke et al.

The article is organized as follows: in section 2 we present the notation and definitions. In section 3 a review of exact weaving plane algorithm is given. Section 4 presents an idea for weaving the 3D super-

cover plane and explains details about the rasterization algorithm. We conclude the article in section 5.

2 Preliminaries

Let \mathbb{R} be the set of real numbers and let \mathbb{Z} be the set of integers. When digitizing three dimensional space a grid usually used is the cubic grid. As we are interested in finite volumes only a subset of \mathbb{Z}^3 is needed. So our area of interest is a subset of 3D Euclidean space \mathbb{R}^3 that consist of all points whose coordinates are integers and are within chosen intervals.

Square brackets denote both the rounding operator and a set of all residues module q when the index is written, so $[x]$ is the rounding operator, but $[p]_q$ is the set of all residues of p modulo q .

A *voxel* in \mathbb{R}^3 corresponding to a discrete point $(x, y, z) \in \mathbb{Z}^3$ is defined by the continuous unit cube with the center at (x, y, z) . Let us denote the cube associated with one voxel as $\mathbb{V}(x, y, z) = [x - \frac{1}{2}, x + \frac{1}{2}] \times [y - \frac{1}{2}, y + \frac{1}{2}] \times [z - \frac{1}{2}, z + \frac{1}{2}]$. The voxel is sometimes called *spel* (spatial element), but note that a spel is not necessarily a voxel. A face of a cube associated with a voxel is just a face, but if two voxels are in relation then corresponding shared face is called a *surfel* (surface element). Let X be any set and ρ be a

binary relation on X . If $(p, q) \in \rho$ then p is ρ -adjacent of q . If ρ is a symmetric relation then for any $p, q \in X$ $(p, q) \in \rho$ if and only if $(q, p) \in \rho$. We say that p and q are ρ -adjacent (Herman, 1998).

For $X = \mathbb{Z}^3$ we define three symmetric binary relations on \mathbb{Z}^3 that correspond to 26, 18 and 6 voxel connectivity. For any two points $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ of \mathbb{Z}^3 we say that they are 18 connected or $(p, q) \in \delta_3$ if and only if they share a face or an edge. The simplest plane rasterization in 3D is 18-connected, so adjacent voxels in a digital plane will share a face or an edge. Such plane is not tunnel-free and 26-connected line can pass through such plane. For some applications where this is not acceptable one must find a plane that is tunnel-free. A supercover plane is a likely candidate as it has some additional desirable properties and is also a tunnel-free structure. We will adopt the definitions by Andrès (Andrès, 2003):

Definition 2.1 (Supercover) A supercover $\mathbb{S}(X)$ of a continuous object X is the set of all the discrete points $p \in \mathbb{Z}^n$ and associated voxels such that $\mathbb{V}(p) \cap X \neq \emptyset$.

One of the drawbacks of the supercover objects is existence of bubbles.

Definition 2.2 (Bubble) A k -bubble is the supercover of an Euclidean point that has exactly k half-integer coordinates.

3 An Exact Weaving of Digital Plane

An attractive method to produce digital planes is by using weaving techniques (Lincke and Wüthrich, 1999). Basic idea is to decompose the rasterization of a surface into two orthogonal curve rasterizations. For the planes both curves are lines and by copying one along the other plane is obtained. The line being copied is usually called *master* and the line used for determining the positions of the master is called *base*. As all lines in the plane along the base are copies of the master and thus have the same chain code we only need to compute the rasterizations of the master and base lines. Copying the master line then completes the plane weaving.

We usually denote a line with the letter L . As we are interested in lines with the same slope and different intercepts we only need to know the value of the intercept.

Definition 3.1 (Straight line) For any $k \in \mathbb{Z}$ and a pair p and q of relatively prime numbers, $q \neq 0$, the straight line L_k is a set of points $L_k = \{(x, y) \in \mathbf{R}^2 : y = \frac{p}{q}x + \frac{k}{q}\}$.

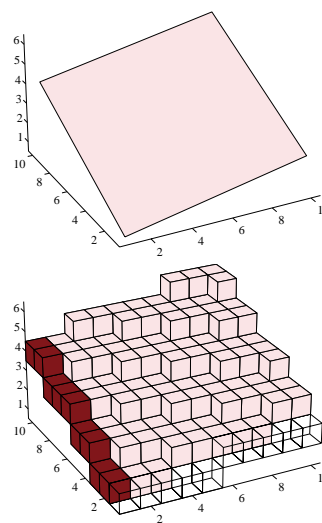


Figure 1: A continuous and digital representation of the plane defined by $x + 2y - 5z = 0$. For the digital representation voxels belonging to the *master* are shown as wire-frame and *base* is shown using darker shading

However, when copying the master line one must notice that simple copying of the master along the base will not produce nearest neighbor rasterization as defined in (Wüthrich, 1998). To obtain a proper rasterization one must also consider shift in the line chain code that is introduced as the intercept changes. Lincke et al. (Lincke and Wüthrich, 1999) have presented an exact weaving rasterization algorithm for digital planes. Main result is the theorem stating how to compute a shift of any line at given position:

Theorem 3.1 (Line shift) Let L_e be a straight line given by $y(x) = \frac{p}{q}x + e$ where p and q are relatively prime numbers, $p, q \in \mathbb{Z}$, $p \leq q$, $q \neq 0$ and e is an arbitrary real intercept. The shift s of L_e at position $i \in \mathbb{Z}$ is given by $[s]_q = [rp^*]_q$ with $r = [pi + qe]_q$ if q is odd and $r = [pi + qe + \frac{1}{2}]_q$ if q is even.

The weaving algorithm now copies the master line along the base, but the chain code is shifted by s . Rasterization of the plane $x + 2y - 5z = 0$ is shown in figure 1.

The plane weaving algorithm produces an exact rasterization of the plane $Ax + By + Cz + D = 0$ with rational coefficients A, B, C and D . Produced rasterization is 18-connected set that does not contain all the voxels a plane intersects. In figure 2 an 18-connected plane $x + 2y - 5z = 0$ is superimposed over continuous plane. One can immediately notice that continuous plane is not contained within the 18-connected representation.

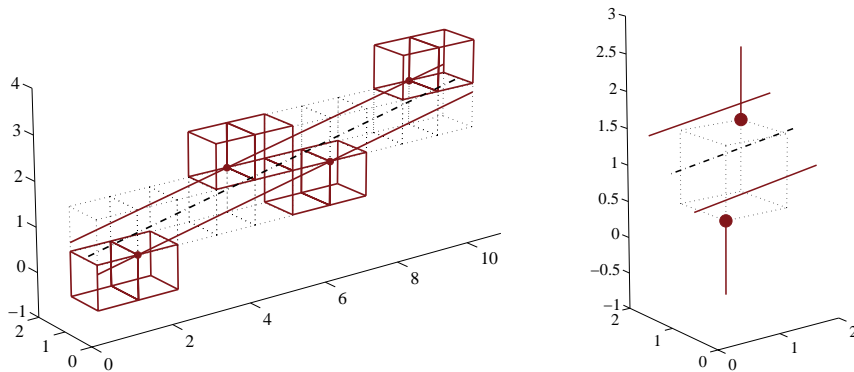


Figure 3: A middle line with upper and lower lines we need to rasterize to obtain the supercover plane rasterization. On the right a single voxel belonging to the master line is shown. For each such voxel we must determine where the intersection between upper (or lower) line and neighboring voxel borders is

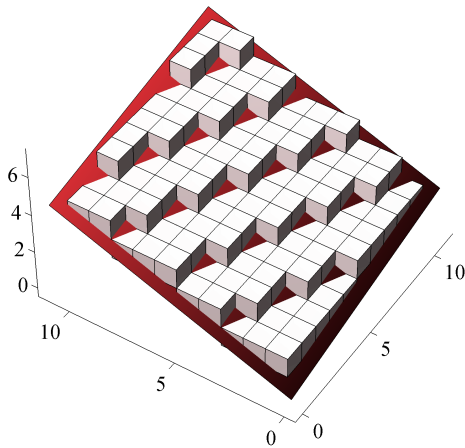


Figure 2: A continuous plane $x + 2y - 5z = 0$ is not contained within 18-connected digital representation

4 Extension to Supercover Planes

Consider a continuous plane P given by $Ax + By + Cz + D = 0$ where $A, B, C, D \in \mathbb{Z}$. Let also a be an absolute value of A etc., so $a = |A|$, $b = |B|$, $c = |C|$ and $d = |D|$. Without loss of generality we can assume $0 < a, b \leq c$. The implicit formula for such plane is $z = -\frac{1}{c}(Ax + By + D)$ with the corresponding 18-connected rasterization $\text{Dig}(P) = \{(x, y, z) : x, y \in \mathbb{Z}, z = \lfloor -\frac{1}{c}(Ax + By + C) \rfloor\}$. The supercover rasterization $\mathbb{S}(P)$ of the plane P is slightly different, and is defined by $\mathbb{S}(P) = \{(x, y, z) \in \mathbb{Z}^3 : -\lfloor \frac{a+b+c}{2} + D \rfloor < Ax + By + Cz \leq \lfloor \frac{a+b+c}{2} - D \rfloor + 1\}$. When weaving an arbitrary plane we simply copy the master line. To each copy we can assign one slice of the space \mathbb{Z}^3 .

Definition 4.1 (Slice) A slice $S_i(k)$, $k \in \mathbb{Z}$, in discrete space \mathbb{Z}^n is a subset of spels with i -th coordinate fixed,

$$S_i(k) = \{\mathbf{x} \in \mathbb{Z}^n : x_i = k\}.$$

We will call two slices $S_i(k)$ and $S_i(l)$ adjacent iff $|k - l| = 1$.

Weaving algorithms compute 2D rasterization of a line in one slice which is then replicated for all other slices we are interested in. When we want to obtain a supercover rasterization of a plane segment we must trace two additional lines per slice along with the master line as is shown in figure 3. The master line (dot-dash line) corresponds to the line $L_m : z = -\frac{A}{c}x - \frac{1}{c}(By_0 + D)$, so the intercept is $-\frac{1}{c}(By_0 + D)$ and $m = -\frac{A}{c}$. The lower line L_l and the upper line L_u are passing through planes with half-integer coordinates in y . Again, without loss of generality we can assume the slope of the plane along y dimension is such that z increases as we move from $y_0 - \frac{1}{2}$ to $y_0 + \frac{1}{2}$ (so $B > 0$). Upper and lower lines are now $L_u : z = -\frac{A}{c}x - \frac{1}{c}(By_0 + \frac{B}{2} + D)$ and $L_l : z = -\frac{A}{c}x - \frac{1}{c}(By_0 - \frac{B}{2} + D)$.

Lemma 4.1 (Continuity) For plane P and two adjacent slices $S_i(k)$ and $S_i(l)$ either L_l from $S_i(k)$ and L_u from $S_i(l)$ or L_u from $S_i(l)$ and L_l from $S_i(k)$ are the same.

The result is obvious and follows immediately from $|n - m| = 1$. Together with the following theorem by Lincke et al. (Lincke and Wüthrich, 1999) it provides the basis for weaving supercover planes.

Theorem 4.2 (Line Equivalence) Let L_k be the 2D line $y = \frac{p}{q}x + \frac{k}{q}$ where p and q are relatively prime. For all $k \in \mathbb{Z}$ the set of straight lines \mathbb{L}_k having the same rasterization (up to shift) is an equivalence class and it contains all lines defined by $y = \frac{p}{q}x + e$, $e \in \mathbb{R}$ with $\frac{k}{q} - \frac{1}{2q} < e \leq \frac{k}{q} + \frac{1}{2q}$ if q is odd and $\frac{k-1}{q} < e \leq \frac{k}{q}$ if q is even.

In our example the intercept e for upper and lower line is $e = -\frac{1}{C}(By_0 \pm \frac{B}{2} + kB)$. We must show that for upper and lower lines in two adjacent slices we obtain the same line cover when shift is introduced—when we copy the master slice lower (or upper) line rasterization in one slice must be equal to the rasterization of the upper (or lower) line in the adjacent slice.

Lemma 4.3 (Equal shifts) *When copying the three lines in master slice $S_i(0)$ to slice $S_i(j)$ shifts s_m , s_u and s_l are at constant shift distance.*

We must compute shifts for three lines L_m , L_l and L_u . All three lines have the same slope, but the intercepts are different. By theorem 3.1 for odd C we have $[s_m]_q = [[pj + qm]p^*]_q = [[Aj + Bj + D]A^*]_c$, $[s_u]_q = [[pj + qu]p^*]_q = [[Aj + Bj + \frac{B}{2} + D]A^*]_c$ and $[s_l]_q = [[pj + ql]p^*]_q = [[Aj + Bj - \frac{B}{2} + D]A^*]_c$. As $Aj + Bj + D$ is a whole number we have $[s_u]_c = [(Aj + Bj + D)A^* + [\frac{B}{2}]A^*]_c = [s_m]_c + [[\frac{B}{2}]A^*]_c$ and $[s_l]_c = [s_m]_c - [[\frac{B}{2}]A^*]_c$. The distance between the shifts is the same, so we do not need to copy three lines separately, but we can copy the whole slice.

Without loss of generality let us compare the lower line L_l and upper line L_u in one slice when $0 < a \leq b \leq c$. Note that both lines have the same slope. Corresponding intercepts are $e_u = -\frac{1}{C}(By_0 + \frac{B}{2} + D)$ and $e_l = -\frac{1}{C}(By_0 - \frac{B}{2} + D)$. As $q = -C$ for odd C and upper line L_u we have $-\frac{k_u}{C} + \frac{1}{2C} < -e_u \leq -\frac{k_u}{C} - \frac{1}{2C}$. Doing the same for L_l we obtain two inequalities $k_u - 1 < \frac{1}{2}(2By_0 - B + 2D - 1) \leq k_u$ and $k_l - 1 < \frac{1}{2}(2By_0 - B + 2D - 1) \leq k_l$. By examining obtained inequalities for odd and even B , and then for even C we can compute the difference between the intercepts. We obtain $k_u - k_l = B$ or $u - l = B$.

Lemma 4.4 (Switching) *When copying the upper, master and lower lines for two adjacent slices $S_i(k)$ and $S_i(l)$ covered voxels faces selected by upper line from one slice and lower line from another will be the same.*

Let us first compute the shift distance between two master lines in two adjacent slices $S_i(k)$ and $S_i(l)$, $l = k + 1$. We have $[s_{m,k}]_c = [[Ai + Bk + D]A^*]_c$ and $[s_{m,l}]_c = [[Ai + B(k + 1) + D]A^*]_c = [s_{m,k}]_c + [BA^*]_c$, so two slices are shifted by $[BA^*]_c$. Note that as the shift between any two adjacent slices is the same we can simply shift-and-copy the chain codes from the previous slice. Now as $[s_{l,k+1}]_c$ and $[s_{u,k}]_c$ are shifted by $\pm[[\frac{B}{2}]A^*]_c$ shifts for upper and lower line are also the same.

By combining those results we can state that intersection of voxel faces selected by the master and lower (or upper) line from slice $S_i(k)$ and voxel faces selected by the master and upper (or lower) line from

slice $S_i(l)$ will form the supercover of the line shared by two slices. So when weaving supercover planes we could find two 2D supercover rasterizations of upper and lower lines (they must contain the master line), for example by using modified Bresenham algorithm presented in (Dedu, 2002). However, we can trace original line and only check whether upper or lower lines have non-empty intersection with upper or lower adjacent voxel as shown in figure 3.

4.1 Computing the cover for one slice

How can we compute the cover for one slice only? Our plane P is given by $Ax + By + Cz + D = 0$ with $0 < a \leq b \leq c$. If we start the line at coordinates (x_0, y_0, z) we can compute the shifts $[s_m]_c$, $[s_l]_c$ and $[s_k]_c$ and then we can copy the chain codes as done in (Lincke and Wüthrich, 1999) for naive planes. Now we have several possibilities when weaving a plane: a) we can compute the shift for each slice as done by Lincke et al. (Lincke and Wüthrich, 1999), or b) we can compute the shift difference between two adjacent slices and simply correct the shift from previous slice (p^* required for the shift computation can be computed when rasterizing the master slice). Alternatively we can compute the starting values for error variables and rasterize each slice separately.

Let us compute the starting error for single slice. The real value of z coordinate is $-\frac{A}{C}x_0 - \frac{1}{C}(By_0 + D)$, and $[z]$ is the closest integer value. Now the error variable is difference $z - [z]$ scaled to $2C$, so

$$e_z = 2C(z - [z]) = 2(-Ax_0 - By_0 - C[z] + D).$$

When computing the cover for one slice we also need to know the error variables for upper and lower lines. When $0 < a \leq b \leq c$ the error from the slice defined by $[z]$ for the lower line is $e_l = e_z - a - b$, and for the upper line is $e_u = e_z + a + b$. If either of e_l or e_u falls outside of the voxel the lower and upper lines will start at $[z] + 1$ or $[z] - 1$ respectively. Now we can trace those three lines simultaneously to obtain the cover for one slice.

The upper line and the lower line must be supercover lines. In the previous section we have shown that the chain codes for upper line and lower line are shifted $\pm[[\frac{B}{2}]A^*]_c$ when compared to the master, however we must note that the computed shift is for the simple chain code. As we want to compute the supercover of the both upper and lower lines unfortunately the rounding operator must have different definitions for those lines. The rounding operator $[x]$ is defined by $k - \frac{1}{2} < x \leq k + \frac{1}{2}$. The problem occurs when either of upper and lower lines passes exactly through the point with half-integer coordinates. For the upper line

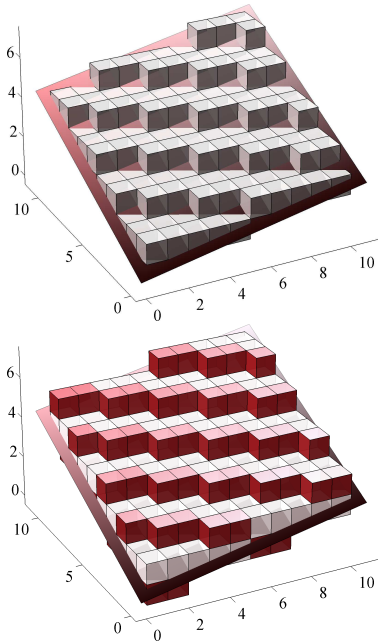


Figure 4: A 18-connected digital representation of the plane defined by $x + 2y - 5z = 0$ and its supercover

as is shown in figure 3 we must select upper voxel when the line passes through back upper right voxel vertex (shown as a circle), and for the lower line we must select the lower voxel when the line passes through front lower left vertex (also shown as a circle). So the rounding operator is different for lower and upper lines.

By using similar reasoning as done in the previous section we can show that the shift for the supercover case will differ at most by one when compared to the shift $\pm \lceil \frac{B}{2} A^* \rceil_c$. The additional shift by one is consistent for all the slices and will not affect the cover shared between two slices. By copying and shifting the slice we can obtain the plane.

One rendering of a plane is shown $x + 2y - 5z = 0$ in figure 4. Note that one should expect the regular chain codes for upper and lower lines to be shifted by $\pm \lceil \frac{B}{2} A^* \rceil_c = \pm 1$, but as the lines are supercover the shift is two.

4.2 Restriction to a Finite Volume

Usually we are interested in computing a plane within a finite volume—usually a cube or a parallelepiped, so the plane-generating algorithm should be able to draw planes with arbitrary parameters within the subvolume.

Chain code of the digital 2D line $y = \frac{p}{q}n + e$ with $p, q \in \mathbb{N}$ and $0 < p \leq q$ is periodic. If p and q are

relatively prime the period is q (Pham, 1987). If we restrict the continuous plane $P : Ax + By + Cz + D = 0$ with arbitrary parameters (so $A, B, C, D \in \mathbb{R}$) to a cube we want to find another plane having the same rasterization, but with the coefficients being whole numbers. Without the loss of generality we can only consider the planes where $0 < a, b \leq c$ such that intersection with the parallelepiped is not an empty set. As the C is the largest coefficient by absolute value when rasterizing a plane we must compute only the z coordinates, $[z] = \lceil -\frac{1}{C}(Ai + Bj + D) \rceil$, so a digitized plane would be $\text{Dig}(P) = \{(i, j, k) : i, j \in [i_1, i_2] \times [j_1, j_2], k = \lceil -\frac{1}{C}(Ai + Bj + D) \rceil\}$. For digitized coordinates we require the following inequalities to have same solutions in k :

$$k - \frac{1}{2} \leq -\frac{1}{\hat{C}}(\hat{A}i + \hat{B}j + \hat{D}) \leq k + \frac{1}{2} \quad (1)$$

$$k - \frac{1}{2} \leq -\frac{1}{C}(Ai + Bj + D) \leq k + \frac{1}{2} \quad (2)$$

Here the \hat{A} , \hat{B} , \hat{C} and \hat{D} are whole numbers representing a plane that has same rasterization as the plane P . For the midpoint line drawing algorithm usually we must double all the values, so we can expect something similar here. Let us assume that $\hat{A} = \lceil \alpha A \rceil$, $\hat{B} = \lceil \alpha B \rceil$ etc., where α is positive real constant. We can rewrite the (1) and (2) so $\lceil -\frac{1}{\lceil \alpha C \rceil}(\lceil \alpha A \rceil i + \lceil \alpha B \rceil j + \lceil \alpha D \rceil) \rceil = \lceil -\frac{1}{C}(Ai + Bj + D) \rceil$ and consequently $|\lceil -\frac{1}{\lceil \alpha C \rceil}(\lceil \alpha A \rceil i + \lceil \alpha B \rceil j + \lceil \alpha D \rceil) + \frac{1}{C}(Ai + Bj + D) \rceil < \frac{1}{2}$. Now we have

$$\begin{aligned} \lceil \alpha A \rceil < 2 \left(\lceil \alpha A \rceil i + \lceil \alpha B \rceil j \right. \\ \left. - \frac{Ai + Bj + D}{C} \lceil \alpha C \rceil + \lceil \alpha D \rceil \right) < \lceil \alpha C \rceil, \end{aligned} \quad (3)$$

and as $0 < a, b \leq c$. As $\hat{C} = \lceil \alpha C \rceil$ is integer we can put $\alpha = \frac{n}{\hat{C}}$, $n \in \mathbb{N}$, so $\hat{C} = n$. Now (3) transforms to

$$\begin{aligned} 2 \left| \left(\left\lceil n \left\{ \frac{A}{C} \right\} \right\rceil - n \left\{ \frac{A}{C} \right\} \right) i + \left(\left\lceil n \left\{ \frac{B}{C} \right\} \right\rceil - \right. \right. \\ \left. \left. n \left\{ \frac{B}{C} \right\} \right) j + \left(\left\lceil n \left\{ \frac{D}{C} \right\} \right\rceil - n \left\{ \frac{D}{C} \right\} \right) \right| < |n|. \end{aligned} \quad (4)$$

As $(\lceil n \{ \frac{A}{C} \} \rceil - n \{ \frac{A}{C} \})$ and other similar constructs are always between $-\frac{1}{2}$ and $\frac{1}{2}$ we can find the worst case $|\hat{C}| = |n| > \max |i| + \max |j| + 1$. In fact when rasterizing arbitrary plane within finite volume we only need to check the size of the finite volume. Consequently, we can scale the coefficients so the largest one has the absolute value greater than $\max(|i_1|, |i_2|) + \max(|j_1|, |j_2|) + 1$.

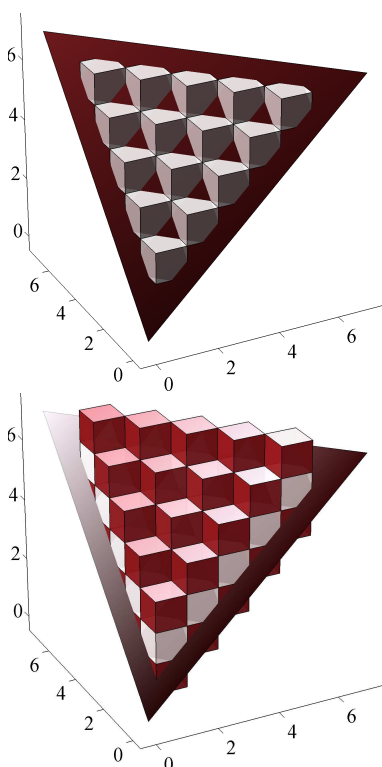


Figure 5: A 18-connected digital representation of the plane defined by $5x + 5y - 5z = 0$ and its supercover

5 Conclusion

A supercover plane algorithm that uses only integer arithmetic was presented. Two variants are possible, one that simply traces a line for each slice and a weaving algorithm. Additionally it was shown that if we want to draw a finite segment of a plane we only need to scale and round the plane coefficients.

If the square plane segment of side lengths n and m , $n < m$ has to be generated the complexity of first approach is $O(mn)$. The weaving approach needs to generate one line segment of the length q and then it is copied n times, so we can expect the complexity of $O(nq)$. Due to the large variety of available hardware performance analysis and code profiling was not done as it would probably be application and hardware specific, however we are currently working on this problem.

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