# A generalized elastica-type approach to the analysis of large displacements of spring-strips

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**Abstract:** An elastica-type analytical solution to the problem of large deflections of slightly curved spring-strips, fixed at one end and loaded at the other with couples and with forces of various directions, is obtained in this work. The main methods of calculation of elliptic integrals are studied, and the limits of their applicability are established as functions of the required degrees of accuracy and of the loading conditions of the spring-strips.

The results obtained with the proposed method are then compared to particular cases already developed in the literature with different approaches. It is shown that in all the cases considered the method permits accurate results to be obtained.

**Keywords:** large deflections, elastica, elliptic integrals

## NOTATION

| | |
|---|---|
| $a, f$ | global Cartesian coordinates of the points of the elastic curve of the spring-strip |
| $C^2$ | $= P/(EI)$ |
| $C_i$ | constant of integration |
| $e$ | relative error |
| $E$ | Young's modulus |
| $E(k)$ | $= E(k, \pi/2)$, $E(k, \varphi_i)$; complete and incomplete elliptic integral of the second kind: $E(k) = \int_0^{\pi/2} \sqrt{(1 - k^2 \sin^2 \varphi)}\, d\varphi$, $E(k, \varphi_i) = \int_0^{\varphi_i} \sqrt{(1 - k^2 \sin^2 \varphi)}\, d\varphi$ |
| $I$ | second moment of area of the cross-section of the spring-strip |
| $k$ (or $k^*$) | parameter of integration of the elliptic integrals |
| $K(k)$ | $= F(k, \pi/2)$, $F(k, \varphi_i)$; complete and incomplete elliptic integral of the first kind: $K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{(1 - k^2 \sin^2 \varphi)}}$, $F(k, \varphi_i) = \int_0^{\varphi_i} \frac{d\varphi}{\sqrt{(1 - k^2 \sin^2 \varphi)}}$ |
| $L$ | length of the spring-strip |
| $M$ | couple acting at the free end of the spring-strip |
| $n$ | integer number |
| $P$ | force acting at the free end of the spring-strip |
| $P_{cr}$ | critical load |
| $r$ | radius of curvature of the spring-strip in the unloaded condition |
| $R$ | radius of curvature of the spring-strip in the loaded condition |
| $s$ | length of a section of the spring-strip |
| $x, y$ | Cartesian coordinates of the points of the elastic curve linked to the inclination of force $P$ |
| $\beta$ | slope angle of force $P$ with respect to axis $a$ |
| $\theta_i$ | slope angle of the elastic curve with respect to force $P$ |
| $\varphi_i$ (or $\varphi_i^*$) | amplitude of the incomplete elliptic integral |

## 1 INTRODUCTION

Positioning devices that make use of large deflections of spring-strips (flexural pivots, parallel spring translators,

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etc.) have been widely used in precision mechanics (**1–4**), especially in the fields of aerospace products (**5**), scientific instrumentation (**6**) and, more recently, integrated circuit technology (**7**) and microelectromechanical systems (MEMs) (**8, 9**). In the most common cases the mechanical behaviour of these devices can be evaluated (**10, 11**) by considering the geometrical non-linearities of slender cantilever beams loaded at the free end with torques and with forces of various directions (Fig. 1); an initial curvature $1/r$ must also be considered as the manufacturing process seldom permits a perfectly straight shape to be obtained. Other second-order effects such as those induced by the anticlastic curvature and by the constraint compliance are generally negligible in the usual cases of slender springs with large height–width ratios (**10, 12**) which are fixed at the edges as suggested in reference (**1**).

The problem of large deflections of a straight cantilever beam loaded at the free end with an axial force is known in literature as the 'elastica' (**13**) and its solution procedure is given in reference (**14**). In the case of cantilever beams loaded with lateral or inclined forces, different solution procedures can be traced in the literature (**11, 15–21**). However, in the latter case the calculation is simplified to a great extent if the method proposed in reference (**20**) is used, since this procedure again permits the 'elastica' approach to be obtained. The procedure given in reference (**20**) has been used in reference (**22**), where some numerical aspects of the calculation have been considered, and in reference (**23**), where the above analytical procedure is applied to the case of various equilibrium configurations.

The situation of slightly curved cantilever beams, or beams loaded with couples, is analysed in references (**24**) and in (**25**) using a rather complicated graphic-analytical method. On the other hand, in references (**26**), (**27**) and (**28**) this problem is solved by means of elliptic integrals. Although correct, these solution procedures are obtained using an analytical approach different from that of the 'elastica'. Hence, the aim of this work is to extend the 'elastica' method to the case of slightly curved cantilever beams loaded at the free end with torques and with variously inclined forces.

From the theoretical point of view, the availability of a general analytical solution capable of solving all the above-mentioned load conditions permits an easier understanding of the influence of the design parameters on the mechanical behaviour of the spring-strips. Moreover, this approach is advantageous from the computational viewpoint since, due to the implicit form of the governing equations and to the presence of elliptic integrals, the numerical solution must follow an iterative approach to be carefully tuned in order to avoid numerical instabilities. Hence, a section of this paper will be dedicated to a thorough analysis of the main numerical aspects of the solution, with special emphasis on the procedure employed for the calculation of elliptic integrals. In fact, even if several approximate solutions are available in the literature, the problem of their applicability limits in large displacement problems has never been considered.

## 2 THEORETICAL FORMULATION

The general case shown in Fig. 1, i.e. that of a cantilever beam loaded by an inclined force $P$ and a couple $M$ (positive if it decreases the curvature) at the free end, is developed in this work. As is done in the 'elastica' case, a curvilinear coordinate system whose orientation is determined by the inclination of the force is introduced; the usual hypothesis of a slender and inextensible beam is adopted in this case too. On the other hand, the geometry of a variable cross-section is not considered since it is of little practical interest.

The differential equation of the deflection curve is thus given by

$$M_A = M - Py = EI\left(\frac{d\theta}{ds} - \frac{1}{r}\right) \tag{1}$$

where the exact expression for the curvature is used.

It must be observed that, according to the original formulation (**14**) and to the initial hypothesis of a slender rod, in this expression the influence of the transverse shear deformation is not taken into account; in the case of stubby beams this simplification does not

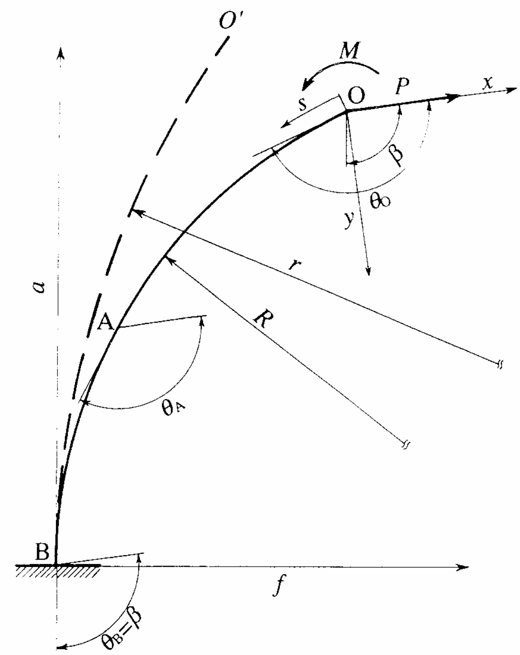


**Fig. 1** Undeflected (-----) and deflected (——) shapes of a curved cantilever beam loaded with a torque and an inclined end load

hold any more and therefore the approach suggested by reference (**12**) must be followed.

Due to the torque $M$ and to the initial curvature $1/r$, the problem considered in this work is not symmetric and therefore the direction of force $P$ has a physical meaning. It must in fact be observed that, as the orientation of the coordinate system is determined by the inclination of the load, a change of the force's direction (not its sign) will also imply a change of the value of the slope angle to its complement.

Thus, deriving expression (1) with respect to $s$ and bearing in mind that $dy/ds = \sin\theta$,

$$\frac{d^2\theta}{ds^2} = -C^2 \sin\theta \tag{2}$$

Hence, multiplying equation (2) by $d\theta/ds$ and integrating, the following relation is obtained:

$$\frac{1}{2}\left(\frac{d\theta}{ds}\right)^2 = C^2 \cos\theta + C_i \tag{3}$$

The constant of integration $C_i$ is found by applying expressions (1) and (3) to the free end of the spring-strip so that equation (3) becomes

$$\frac{1}{2}\left(\frac{d\theta}{ds}\right)^2 = C^2(\cos\theta - \cos\theta_0) + \frac{1}{2}\left(\frac{M}{EI} + \frac{1}{r}\right)^2 \tag{4}$$

Thus, separating the variables, taking the appropriate sign and integrating between 0 and $s$:

$$s = \int_{\theta_A}^{\theta_0} \frac{d\theta}{\sqrt{2}\,C\sqrt{(\cos\theta - \cos\alpha)}} \tag{5}$$

in which

$$\cos\alpha = \cos\theta_0 - \frac{1}{2C^2}\left(\frac{M}{EI} + \frac{1}{r}\right)^2 \tag{6}$$

The usual substitutions of the 'elastica' method are now introduced:

$$k = \sin\left(\frac{\alpha}{2}\right), \qquad \sin\varphi = \frac{\sin(\theta/2)}{k} \tag{7}$$

Hence

$$Cs = F(k, \varphi_0) - F(k, \varphi_A) \tag{8}$$

If the loading conditions and the mechanical characteristics of the spring-strip are known, the parameter $C$ can easily be determined, but the values of $s$, $k$, $\varphi_0$ and $\varphi_A$ are still unknown. Equation (8) has thus to be applied at the fixed end of the strip (point B in Fig. 1) and the following expression is obtained:

$$CL = F(k, \varphi_0) - F(k, \varphi_B) \tag{9}$$

Variables $k$, $\varphi_0$ and $\varphi_B$ are all functions of the angle $\theta_0$ which is therefore the only unknown of the problem. Nevertheless, the presence of elliptic integrals makes the problem transcendental so that its solution requires an iterative procedure.

It must also be observed that, similar to what was done in the case of cantilever beams subjected only to inclined end loads in reference (**23**), the utilization of expressions (8) and (9) allows different equilibrium configurations of the spring-strips to be determined also in the more general case of Fig. 1. This can be achieved simply by adding or subtracting multiples of $\pi$ from the integration limits of the integrating function:

$$Cs = F(k, \pm\varphi_0 \pm n\pi) - F(k, \pm\varphi_A \pm n\pi) \tag{10}$$

$$CL = F(k, \pm\varphi_0 \pm n\pi) - F(k, \pm\varphi_B \pm n\pi) \tag{11}$$

Since $dy/ds = \sin\theta$ and $dx/ds = \cos\theta$, it is now possible to obtain the expressions suitable for the calculation of the $x$ and $y$ coordinates of the elastic curve of the strip:

$$\frac{x_A}{s} = 2\frac{E(k, \varphi_0) - E(k, \varphi_A)}{F(k, \varphi_0) - F(k, \varphi_A)} - 1 \tag{12}$$

$$\frac{x_B}{L} = 2\frac{E(k, \varphi_0) - E(k, \varphi_B)}{F(k, \varphi_0) - F(k, \varphi_B)} - 1 \tag{13}$$

$$\frac{y_A}{s} = \frac{2k(\cos\varphi_A - \cos\varphi_0)}{F(k, \varphi_0) - F(k, \varphi_A)} \tag{14}$$

$$\frac{y_B}{L} = \frac{2k(\cos\varphi_B - \cos\varphi_0)}{F(k, \varphi_0) - F(k, \varphi_B)} \tag{15}$$

It can be observed that these expressions are formally equal to those of 'elastica'; in that case, however, $\cos\alpha$ was equal to $\cos\theta_0$ so that the values of $k$ could have only been less than or equal to 1 [see equations (6) and (7)]. In the general case considered in this work the absolute value of the right-hand side of equation (6) is greater than 1 if

$$\frac{1}{2C^2}\left(\frac{M}{EI} + \frac{1}{r}\right)^2 \geqslant 1 + \cos\theta_0 \tag{16}$$

In this case $k$ must be obtained directly. In fact, with simple trigonometric substitutions from equations (6) and (7) it is possible to obtain

$$k = \sqrt{\left\{\frac{1 - \cos\theta_0 + [1/(2C^2)](M/EI + 1/r)^2}{2}\right\}} \tag{17}$$

From equation (17) it follows that $k$ is always real, but it is greater than 1 if condition (16) holds true. In this case, as the elliptic integrals are defined only for values of $k$ smaller than 1, it is necessary to use the approach suggested in reference (**26**) and the following substitutions have to be introduced:

$$k^* = \frac{1}{k}, \qquad k\sin\varphi = \sin\varphi^* \tag{18}$$

The new expressions of equations (8), (9), (12), (13),

(14) and (15) can be easily obtained, remembering that

$$E(k, \varphi_i) = \frac{1}{k^*} E(k^*, \varphi_i^*) - \frac{1-k^{*2}}{k^*} F(k^*, \varphi_i^*) \quad (19)$$

$$F(k, \varphi_i) = k^* F(k^*, \varphi_i^*) \quad (20)$$

From simple trigonometric arrangements it is now finally possible to obtain the $a$ and $f$ coordinates of point A of the elastic curve of the spring-strip:

$$\frac{a_A}{L} = \frac{x_B}{L} \cos\beta + \frac{y_B}{L} \sin\beta - \frac{s}{L}\left(\frac{x_A}{s}\cos\beta + \frac{y_A}{s}\sin\beta\right) \quad (21)$$

$$\frac{f_A}{L} = -\frac{x_B}{L}\sin\beta + \frac{y_B}{L}\cos\beta + \frac{s}{L}\left(\frac{x_A}{s}\sin\beta - \frac{y_A}{s}\cos\beta\right) \quad (22)$$

## 3 NUMERICAL CALCULATION

When the loading conditions of the spring-strips are known, an iterative procedure with successive approximations must be used to calculate from expression (9) the value of $\theta_0$ and hence to determine the coordinates of the spring-strip free end in the deflected position (Fig. 2); the convergence of the procedure can be speeded up as suggested in reference (**29**). Expression (8) is then used for the calculation of the elastic line. However, in this case the variable of integration must be $\varphi_A$ and not $\theta_A$, since numerical instabilities can occur when $k \cong \sin\theta_A/2$ [see equation (7)]. Subsequently $\theta_A$ is determined so that, with $\theta_0$ known, it is possible to calculate, using expressions (12), (13), (14), (15), (21) and (22), the $x$, $y$, $a$ and $f$ coordinates of the elastic curve.

The iterative procedure makes it necessary to calculate repetitively the elliptic integrals of the first kind which appear in expressions (8) and (9). It is thus particularly suitable to use approximate expressions for the calculation of these integrals.

It is known (**30**) that only in cases when $k = 0$, $k = 1$ or $\varphi_i = 0$ are the values of the elliptic integrals easily established. In the general case, various approaches are suggested in the literature, which make use of interpolating procedures or other approximate solutions for the calculation of elliptic integrals. An accurate analysis of the error induced by these methods has been performed by comparing the results obtained with the methods suggested in the literature (**22, 30–32**) with those achieved using a numerical method of integration (**33**).

The calculation of complete elliptic integrals of the first kind is necessary in the iterative procedure when $\varphi_i = \pi/2$. In this case three approximate solutions are generally employed: the method of the hypergeometric function, the method of polynomial approximations and the method of the arithmetic–geometric mean. It has been verified that the method of the hypergeometric function (also called infinite series method) (**30, 32**) introduces an unbounded error when $k$ approaches 1, so that in this particular case an adequate number of series elements has to be accounted for. The method of polynomial approximations (**30**) produces a very small but still appreciable error. Only the arithmetic–geometric mean method (**22, 30, 31**) permits exact values of the complete elliptic integrals of the first kind to be obtained in the whole interval of integration and therefore it has to be preferred to the other approximate solutions.

In the general case, in the iterative procedure it is necessary to calculate the incomplete elliptic integrals of the first kind. In this case the trigonometric series method (**32**) introduces an error which can be up to 35 per cent for $k$ approaching 1 and $\varphi_i$ approaching $\pi/2$. In the same interval of integration the descending Landen transformation method (**30**) introduces an error of 5 per cent (see Fig. 3a). The arithmetic–geometric mean method (**22, 30, 31**) introduces an error up to 20 per cent in the proximity of $k = 1$ and $\varphi_i = \pi/2$ (see Fig. 4). Only the ascending Landen transformation method permits the exact values of the incomplete elliptic integrals of the first kind to be obtained, as for all values of $k$ and $\varphi_i$ an error lower than $1 \times 10^{-13}$ per cent is met.

The aim of optimizing the iterative procedure is thus achieved when the arithmetic–geometric mean method is used for the calculation of the complete elliptic integrals of the first kind and the ascending Landen transformation method for the calculation of the incomplete elliptic integrals of the first kind (see the Appendix).

Concerning the complete elliptic integrals of the second kind, in this case the arithmetic–geometric mean method also permits an exact calculation. In the case of incomplete elliptic integrals of the second kind it has been verified that none of the methods suggested in the literature guarantees exact results in the whole interval of integration to be obtained. In fact, similar to the case of the incomplete integrals of the first kind, the trigonometric series method (**32**) introduces an error that can reach 35 per cent. The descending Landen transformation method requires the calculation of $F(k, \varphi_i)$ (**30**). This can be performed using the ascending Landen transformation method, but in this case (see Fig. 3b) a significant error is introduced for $k$ close to 1 and $\varphi_i$ close to $\pi/2$. Figure 5 shows the error induced by the arithmetic–geometric mean method. In case (a) the evaluation of $F(k, \varphi_i)$ required by the method was done using the arithmetic–geometric mean, while in case (b) the ascending Landen transformation method was applied; in both cases $K(k)$ and $E(k)$ were evaluated using the arithmetic–geometric mean method. From these results it is obvious that a numerical method of integration has to be used; however, this fact does not affect significantly the calculation since the elliptic integrals of the second kind have to be computed only

 

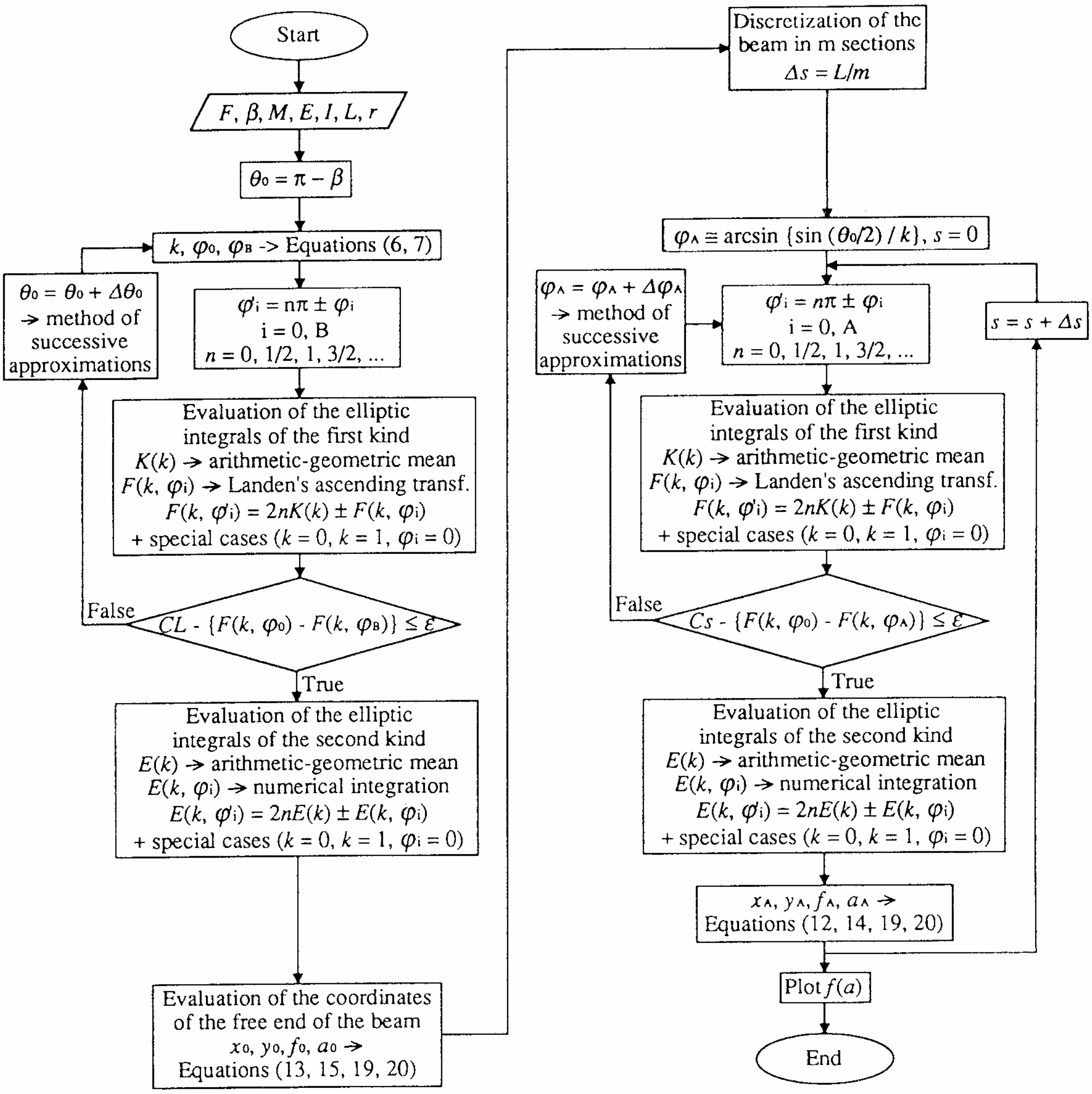


**Fig. 2** Numerical calculation procedure scheme

when the convergence of the iteration has been reached (see Fig. 2).

## 4 EXAMPLES

The method developed here has been compared with some significant cases extracted from the literature which adopt different solution strategies.

### 4.1 Straight cantilever beam with lateral loads

The case of straight beams loaded with lateral loads was developed in references (**15**) and (**16**) and was enhanced from the computational point of view by Mattiasson (**22**) with the aim of providing a reference for the validation of finite element method (FEM) models. This simple case aims at illustrating the sensitivity of the solution to errors in the evaluation of the elliptic integrals. Figure 6 shows the elastic line evaluated with the method proposed in this work, as well as that evaluated with the procedure suggested in reference (**22**), for different values of the lateral load. The marked difference between the two methods can be easily explained considering that in reference (**22**) the arithmetic–geometric mean method is suggested for the calculation of the incomplete elliptic integrals. As pointed out previously, the arithmetic–geometric mean method introduces a significant error in the calculation of the incomplete integrals (refer to Fig. 4); for small loads this fact causes an error in the determination of the elastic curve at the free end of the beam (Fig. 6). On the other

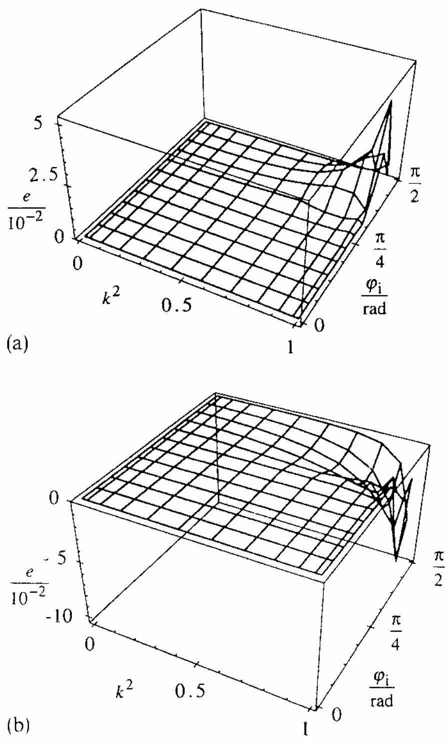


**Fig. 3** Error introduced by the descending Landen transformation method in the calculation of the incomplete integral of (a) the first and (b) the second kinds

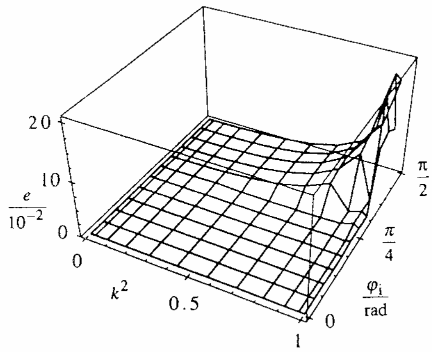


**Fig. 4** Error introduced by the arithmetic–geometric mean method in the calculation of the incomplete elliptic integrals of the first kind

hand, for larger loads the maximum error occurs close to the fixed end of the beam. This behaviour is easily explainable if the graph of the integrating function *Cs* is considered. Figure 7 shows the integrating function *Cs* versus the iteration angle $\varphi_A$ (see Fig. 2) evaluated with the procedure herein proposed and with that proposed in reference (**22**); the case of a high load $[(CL)^2 = 10]$ is considered. Particularly evident is the fact that, especially for values of $\varphi_A$ approaching $\pi/2$ (in the proximity of the free end of the beam), the error in the calculation of *Cs* causes an appreciable increase of the slope of the curve that can give rise to problems in the convergence of the iterative procedure.

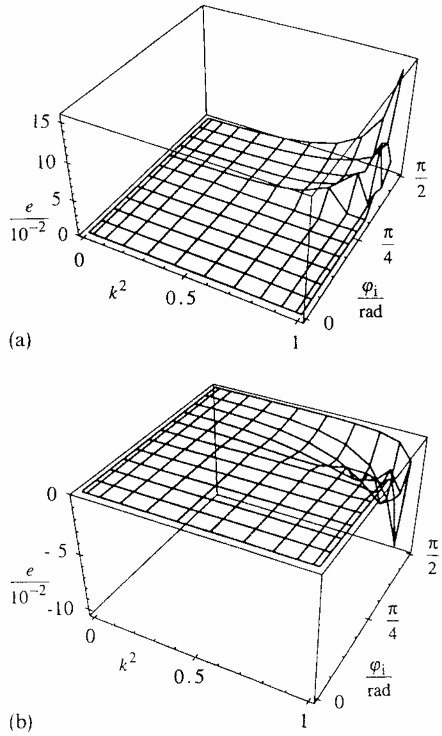


**Fig. 5** Error introduced by the arithmetic–geometric mean method in the calculation of $E(k, \varphi_i)$ if $F(k, \varphi_i)$ is evaluated (a) with the arithmetic–geometric mean method and (b) with the Landen ascending transformation method

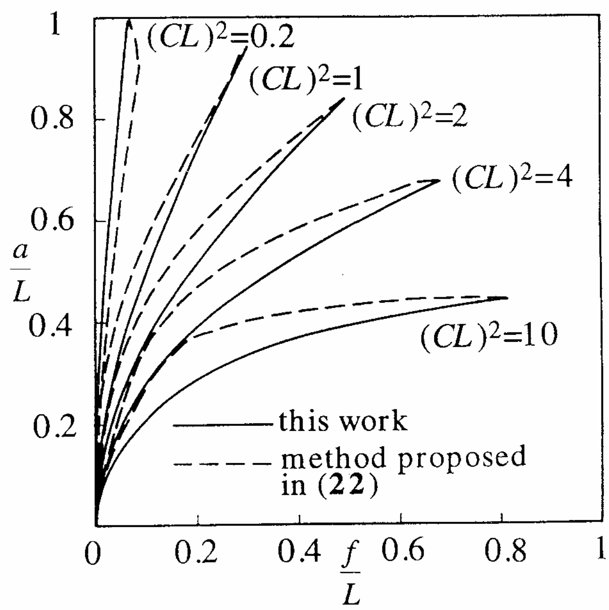


**Fig. 6** Straight cantilever beam with an axial end load

This example proves clearly that, especially when a high degree of accuracy has to be achieved, an error

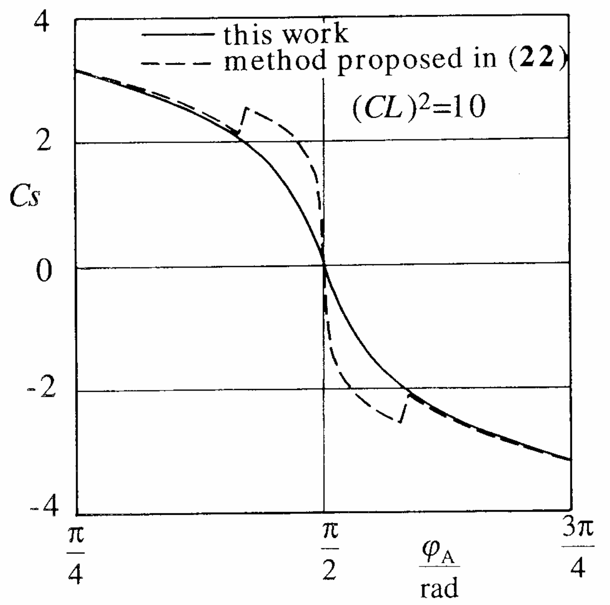


**Fig. 7** Graphic representation of the integrating function $Cs$

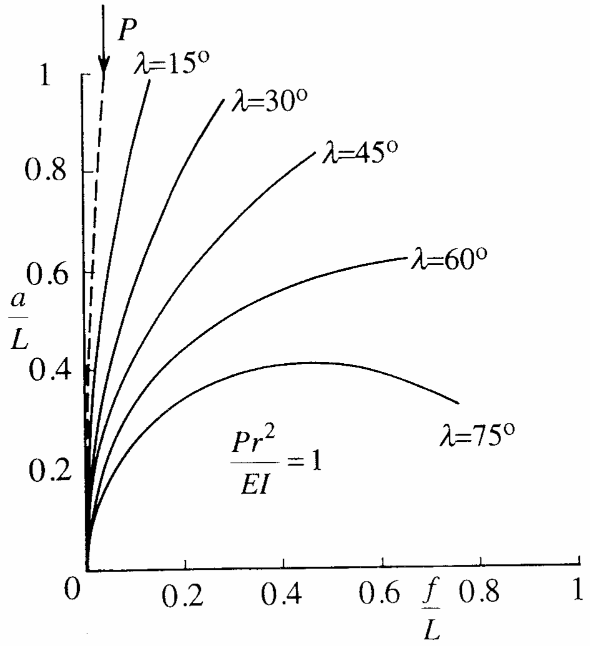


**Fig. 8** Curved cantilever beam with an axial end load

sensitivity test has to be performed, at least in some cases.

### 4.2 Curved cantilever beam with axial loads

The case of initially curved beams loaded by concentrated forces at the edges is considered in references (**25**), (**27**) and (**28**). In reference (**25**) the assumption that the elastic line can be approximated by a number of circular arcs tangent to each other at the point of intersection is made; consequently a numerical solution based on a discrete model of the beam is proposed. This approach is not analytical and therefore is of little interest nowadays when a more general case can be considered simply by adopting a FEM model.

An analytical solution of the problem is suggested in references (**27**) and (**28**), but in these works an elliptical-type approach different from that of 'elastica' is followed.

Figure 8 shows the elastic lines obtained with the procedure described in this work in the case of a spring-strip loaded by a concentrated end load and with different initial curvatures. The values of the parameters are chosen according to the numerical example developed in reference (**27**) where the absolute displacement of the free end was given for values of $\lambda = L/r$ varying from 0° to 80°. The results obtained in this work are in close agreement with those given in reference (**27**). Moreover, in this work the complete elastic line is evaluated for five cases (15°, 30°, 45°, 60°, 70°) in the same range of angular values.

### 4.3 Other cases

The problem of curved flat springs forced to bend in the direction opposite to the initial curvature and then put between two fixed supports has been proposed in reference (**24**). In that work it was shown that various possible shapes of the buckled spring must have at least one point of inflection and therefore, with the aid of some graphical considerations, the problem was interpreted as that of two or more straight beams fixed at one end and having an inclined load at the free end. An elliptical-type solution was thus obtained and the coordinates of the free edge and of the inflection point were calculated for two different equilibrium configurations.

By simply using $n = 2$ in equations (10) and (11) the same problem can be easily solved following the elastica-type approach suggested in this work. Figure 9 shows the complete elastic line for a selection of increasing loads; the values of the beam geometric parameters are chosen according to the numerical case proposed in reference (**24**). However, in that case only the 'direct' problem was solved as the axial loads were obtained by imposing the values of $\theta_0$, whereas, obviously, in this case the 'inverse' problem is considered.

Other special cases were also tested, as that of the pinned–fixed square diamond frame described in reference (**34**) and that of the square frame loaded at the mid-points by a pair of forces applied at opposite sides (**35**). The equilibrium configurations of cantilever beams loaded by an axial force recently obtained by Navee and Elling (**23**) were also checked. In all these cases the results of this work were in perfect agreement with those obtained in the literature, therefore confirming the

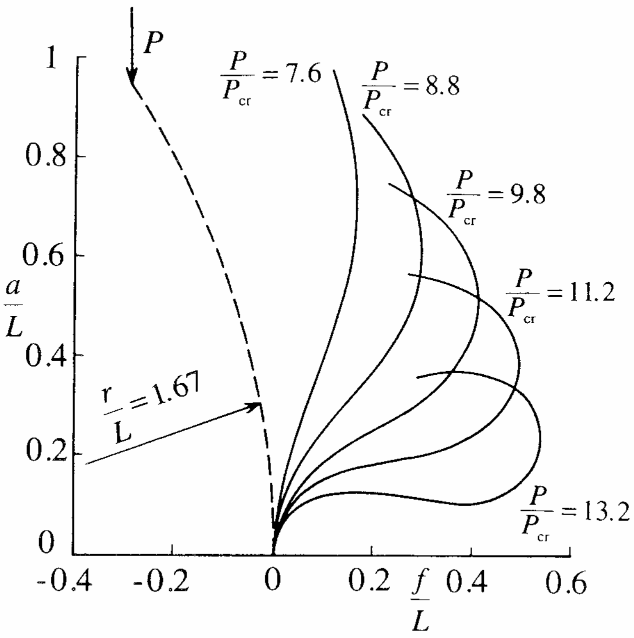


**Fig. 9** Curved beam with an axial load (second equilibrium configuration)

large versatility of the proposed generalized elastica-type approach.

## 5 CONCLUSIONS

In this work the elliptical solution originally developed to study large displacements of straight beams deflected by axial end loads (**12**), and then applied to the case of inclined end loads (**20**), has been extended to the more general case of curved beams loaded at the free end with couples and with inclined loads. The governing equations obtained have an implicit form and therefore an iterative approach has to be followed. With the example of a cantilever beam loaded by a lateral load it has been shown that in some cases the solution is very sensitive even to small errors in the calculation of elliptic integrals; therefore a careful analysis of the error introduced by the approximated methods proposed in the literature has been performed and the most suitable methods have been detected. The results obtained are useful not only in large displacement problems but also in those mechanical problems in which the solution is expressed in terms of elliptic integrals [e.g. the Hertzian contact problem (**36**)].

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## REFERENCES

1 **Jones, R. V.** Some uses of elasticity in instrument design. *J. Sci. Instrum.*, 1962, **39**, 193–203.
2 **Weinstein, W. D.** Flexure-pivot bearings—part 1. *Mach. Des.*, 10 June 1965, 150–157.
3 **Weinstein, W. D.** Flexure-pivot bearings—part 2. *Mach. Des.*, 8 July 1965, 136–145.
4 **Trilynski, W.** *Fine Mechanisms and Precision Instruments—Principles of Design*, 1971 (Pergamon Press, Oxford).
5 **Anderson, E. H., Moore, D. M., Fanson, J. L.** and **Ealey, M. A.** Development of an active truss element for control of precision structures. *Opt. Engng*, 1990, **29**, 1333–1341.
6 **Reiss, R. S.** Opto-mechanical instrument design. *Proc. SPIE*, 1987, **817**, 154–170.
7 **Slocum, A.** *Precision Machine Design*, 1992 (Prentice-Hall, Englewood Cliffs, New Jersey).
8 **Fang, W.** and **Wickert, J. A.** Post buckling of micromachined beams. *J. Micromech.*, 1994, **4**, 116–122.
9 **Liu, C., Tsao, T., Tai, Y. C., Leu, T. S., Ho, C. M., Tang, W. L.** and **Miu, D.** Out of plane permalloy actuators for delta-wing control. Proceedings of IEEE Conference on *Micro Electro Mechanical Systems*, Amsterdam, 1995, pp. 7–12.
10 **Haringx, J. A.** The cross-spring pivot as a constructional element. *Appl. Sci. Res., Part A*, 1949, **1**, 313–332.
11 **Plainevaux, J. E.** Guidage par suspension sur lames elastiques. *Nuovo Cimento*, 1953, **X**, 1451–1460.
12 **Theocharis, P. S.** and **Panayotounakos, D. E.** Exact solution of the non-linear differential equation concerning the elastic line of straight rod due to terminal loading. *Int. J. Non-linear Mechanics*, 1982, **17**(5/6), 395–402.
13 **Timoshenko, S. T.** *History of Strength of Materials*, 1953 (McGraw-Hill, New York).
14 **Timoshenko, S. T.** and **Gere, J. M.** *Theory of Elastic Stability*, 2nd edition, 1961 (McGraw-Hill, New York).
15 **Barten, H. J.** On the deflection of a cantilever beam. *Q. Appl. Math.*, 1945, **II**, 168–171; **III**, 275–276.
16 **Bisshopp, K. E.** and **Drucker, D. C.** Large deflection of cantilever beams. *Q. Appl. Math.*, 1945, **III**, 272–275.
17 **Conway, H. D.** The large deflection of simply supported beams. *Phil. Mag.*, 1947, **38**, 905–911.
18 **Scott, E. J.** and **Carver, D. R.** On the nonlinear differential equation for beam deflection. *Trans. ASME, J. Appl. Mech.*, 1955, **22**(77), 245–248.
19 **Gospodnetic, D.** Deflection curve of a simply supported beam. *Trans. ASME, J. Appl. Mech.*, 1956, **23**(78), 675–676.
20 **Frisch-Fay, R.** A new approach to the analysis of the deflection of thin cantilevers. *Trans. ASME, J. Appl. Mech.*, 1961, **28**(83), 87–90.
21 **Manuel, F. S.** and **Lee, S. L.** Flexible bars subjected to arbitrary discrete loads and boundary conditions. *J. Franklin Inst.*, 1968, **285**, 452–474.
22 **Mattiasson, K.** Numerical results from large deflection beam and frame problems analysed by means of elliptic integrals. *Int. J. Numer. Meth. Engng*, 1981, **16**, 145–153.
23 **Navee, S.** and **Elling, R. E.** Equilibrium configurations of cantilever beams subjected to inclined end loads. *Trans. ASME, J. Appl. Mech.*, 1992, **59**, 572–579.

 

24 **Hymans, F.** Flat springs with large deflections. *Trans. ASME, J. Appl. Mech.*, 1946, **13**(68), A-223–A-230.

25 **Seames, A. E.** and **Conway, H. D.** A numerical procedure for calculating the large deflections of straight and curved beams. *Trans. ASME, J. Appl. Mech.*, 1957, **79**, 289–294.

26 **van Wijngaarden, A.** Large distortions of circular rings and straight rods—parts I and II. *Proc. Kon. Ned. Akad. v. Wetensch. (Proc. R. Acad.)*, 1949, **49**, 648–664.

27 **Conway, H. D.** The nonlinear bending of thin circular rods. *Trans. ASME, J. Appl. Mech.*, 1956, **23**(78), 7–10.

28 **Mitchell, T. P.** The nonlinear bending of thin rods. *Trans. ASME, J. Appl. Mech.*, 1959, **26**(81), 40–43.

29 **Dorn, W. S.** and **McCracken, D. D.** *Numerical Methods with Fortran IV Case Studies*, 1972 (John Wiley, New York).

30 **Abramowitz, M.** and **Stegun, I. A.** *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 1970 (Dover Publications, New York).

31 **King, L. V.** *On the Direct Numerical Calculation of Elliptic Functions and Integrals*, 1924 (Cambridge University Press, Cambridge).

32 **Gradshteyn, I. S.** and **Ryzhik, I. M.** *Table of Integrals, Series and Products*, 1980 (Academic Press, San Diego).

33 **Press, W. H., Flannery, B. P., Tenkolsky, S. A.** and **Vetterling, W. T.** *Numerical Recipes—The Art of Scientific Computing*, 1986 (Cambridge University Press, Cambridge).

34 **Jenkins, J. A., Seitz, T. B.** and **Przemieniecki, J. S.** Large deflections of diamond-shaped frames. *Int. J. Solids Structs*, 1966, **2**, 591–603.

35 **Kerr, C. N.** Large deflections of a square frame. *Q. J. Mech. and Appl. Math.*, 1964, **XVII**(1), 23–38.

36 **Dyson, A., Evans, H. P.** and **Snidle, R. W.** A simple, accurate method for calculation of stresses and deformations in elliptical Hertzian contacts. *Proc. Instn Mech. Engrs*, 1992, **206**, 139–141.


## APPENDIX

### Calculation of the complete elliptic integrals using the arithmetic–geometric mean method

The arithmetic–geometric mean scheme is

$$\begin{aligned} a_0 &= 1 \\ a_1 &= \tfrac{1}{2}(a_0 + b_0) \\ a_2 &= \tfrac{1}{2}(a_1 + b_1) \\ &\cdots\cdots \\ a_N &= \tfrac{1}{2}(a_{N-1} + b_{N-1}) \end{aligned} \tag{23}$$

$$\begin{aligned} b_0 &= \sqrt{(1-k^2)} \\ b_1 &= (a_0 b_0)^{1/2} \\ b_2 &= (a_1 b_1)^{1/2} \\ &\cdots\cdots \\ b_N &= (a_{N-1} b_{N-1})^{1/2} \end{aligned} \tag{24}$$

$$\begin{aligned} c_0 &= k \\ c_1 &= \tfrac{1}{2}(a_0 - b_0) \\ c_2 &= \tfrac{1}{2}(a_1 - b_1) \\ &\cdots\cdots \\ c_N &= \tfrac{1}{2}(a_{N-1} - b_{N-1}) \end{aligned} \tag{25}$$

The process is repeated until $a_N \approx b_N$, i.e. $c_N \approx 0$. Then

$$K(k) = \frac{\pi}{2a_N} \tag{26}$$

$$E(k) = K(k) - \frac{K(k)}{2}(c_0^2 + 2c_1^2 + 2^2 c_2^2 + \cdots + 2^N c_N^2) \tag{27}$$

### Calculation of the incomplete elliptic integrals of the first kind using the ascending Landen transformations

In this case a recurrence formula is used:

$$\left[1 + \sin\left(\frac{\alpha_n}{2}\right)\right]\left[1 + \cos\left(\frac{\alpha_{n+1}}{2}\right)\right] = 2 \qquad \left(\frac{\alpha_{n+1}}{2} > \frac{\alpha_n}{2}\right) \tag{28}$$

$$\sin(2\varphi_{n+1} - \varphi_n) = \sin\left(\frac{\alpha_n}{2}\right)\sin\varphi_n \qquad (\varphi_{n+1} < \varphi_n) \tag{29}$$

and the expression for the calculation of the incomplete elliptic integral of the first kind is given by

$$F(k, \varphi_i) = \left[\frac{1}{\sin\alpha/2}\prod_{m=1}^{\infty}\sin\left(\frac{\alpha_m}{2}\right)\right]^{1/2} \times \ln\left[\tan\left(\frac{\pi}{4} + \frac{1}{2}\lim_{n\to\infty}\varphi_n\right)\right] \tag{30}$$