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Which efficient solution in multi objective programming problem should be taken?

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Abstract

In practical problems, which can be stated in the form of multi objective programming problem, we have usually a large set of efficient solutions. So, which solution from this set should be taken, appears as a natural question here. In the paper we propose some sustainable principles and simple numerical method for such choice. The method respects aspirations and priorities of decision makers and enables iterations for possible improvement of the solution. In this way decision makers can clearly understand why a particular solution is obtained and when and how it can be improved. The features of the method are explained by a practical example. An applications to bilevel programming problems are also presented, where the both side mapping, between the set of efficient solutions and the set of all possible priorities, is shown. It is illustrated in detail through several linear and nonlinear examples.

Keywords Multi objective programming problem · Efficient solution · Priority · Aspiration · Bilevel programming problem

1 Introduction and motivation

Multi objective programming problem (MOPP) is one of the most famous problems which has been thoroughly studied in operations research. The essence of the problem is several simultaneous optimizations on the same set (constraint set or *budget*). Several optimal points have to be replaced by only one compromise point. We can find many approaches and a lot of methods for solving MOPP in the literature. Some of these

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methods provide one solution and some a whole set of solutions, so the decision makers choose the most favourable one. Chronological development of the methodology for solving MOPP can be find in Roy (1971), MacCrimmon (1973), Cohon and Marks (1975), Bell et al. (1977), Starr and Zeleny (1977), Hwang and Masud (1979), Ho (1979), Despontion and Spronk (1979), Hwang and Masud (1979), Zionts (1980), Chankong and Haimes (1983), Yu (1985), Steuer (1985), Fandel and Spronk (1985), Lai and Hwang (1996), Figueira et al. (2005), Matejaš and Perić (2014), Perić et al. (2018) and Filatovas et al. (2017).

Using different methods we often get different solutions, which can cause problems in accepting solutions from decision makers (Perić et al. 2018). The problem of obtaining efficient solutions and choosing the preferred solution is especially complicated by multi-level multi-objective programming where each decision maker makes his own decisions independently (Perić et al. 2019). Therefore, to solve such problems, cooperation between decision makers is proposed, as well as the application of multi-objective programming methods that support a multilevel decision-making system and active participation of all decision-makers in the preparation of the preferred efficient solutions (Sinha 2003a, b; Mishra 2007; Pramanik and Roy 2007; Baky 2009, 2010; Osman et al. 2013; Lachwani 2014).

Many of the existing methods involve complex computational procedures, especially for the larger number of objectives, and thus difficult to understand and trust for decision makers. Besides the obtained solution is often unsatisfactory from the practical point of view. This conclusion follows from many practical problems which were stated and solved as MOPP. We show it by the following example.

Example 1 How to supply k users from the given budget (100%) if each of them want to maximize his own amount?

If x_i is the amount given to user $i,\ i = 1, 2, \ldots, k$, then the problem can be stated in the form

$$\begin{aligned} & \max x_1,\ \max x_2, \ldots,\ \max x_k \\ \text{where}\quad & x_1, x_2, \ldots, x_k \geq 0, \quad x_1 + x_2 + \cdots + x_k \leq 100. \end{aligned} \tag{1}$$

It can be considered as MOPP and also as multilevel programming problem (MLPP) if users belong to different levels. We use (1) as a test problem and we solve it by applying some of the most famous existing methods for linear MOPP.

Let $\left(x_1^*, x_2^*, \ldots, x_k^*\right)$ denote the solution to (1).

Weight coefficients method (Hwang and Masud 1979) assigns a particular weight w_i to each user $i,\ \ i = 1, 2, \ldots, k$. The method yields the solution $x_j^* = 100,\ \ x_i^* = 0,\ \ i \neq j$, where j is the index for which $w_j = \max_{1 \leq i \leq k} w_i$ and $w_i < w_j,\ \ i = 1, 2, \ldots, j-1$ hold. We see that one user gets everything (100) while all the others get nothing. In our opinion such solution is completely unacceptable and unsatisfactory for the users and for the analysts. The users who get nothing are implicitly excluded from decision process that is contradiction with the statement (1) where all objectives are maximized at the same time and not only one of them. Moreover, such solution is not acceptable as long as the given budget contains possible solutions $x_i > 0,\ \ i = 1, 2, \ldots, k$.

Goal programming method (Charnes and Cooper 1961; Lee 1972; Ignizio 1976) respects the user goals $g_i, \ i = 1, 2, \ldots, k$ in a way that the solution is as follows.

- If $g_1 > 100$ then $x_1^* = 100, \quad x_i^* = 0, \quad i = 2, 3, \ldots, k$.
- If there exists $j < k$ such that $g_1+g_2+\cdots+g_j \le 100$ and $g_1+g_2+\cdots+g_j+g_{j+1} > 100$ then

$$x_i^* = \begin{cases} g_i & \text{for} \quad i = 1, 2, \ldots, j, \\ 100 - (g_1 + g_2 + \cdots + g_j) & \text{for} \quad i = j + 1, \\ 0 & \text{for} \quad i = j + 2, j + 3, \ldots, k. \end{cases}$$

- If $g_1 + g_2 + \cdots + g_k \le 100$ then $x_i^* = g_i, \quad i = 1, 2, \ldots, k$. We see that, in the case of strict inequality, this solution is inefficient.

Here, the satisfactory solution is obtained only in the case $g_1 + g_2 + \cdots + g_k = 100$, for the same reasons as above.

Lexicographic method (Fishburn 1974) yields the solution $x_1^* = 100, \ x_i^* = 0, \ i = 2, 3, \ldots, k$. Thus, we have again unsatisfactory solution.

STEM method (Benayoun et al. 1971) in the first step distributes the budget equally among the users, $x_i^* = 100/k, \quad i = 1, 2, \ldots, k$. In the further steps of the method we can obtain different distribution depending on additional requirements.

MP method (Matejaš and Perić 2014) respects the user aspirations $d_i, \ i = 1, 2, \ldots, k$ and yields the proportional solution $x_i^* = 100 \cdot d_i/(d_1 + d_2 + \cdots + d_k), \quad i = 1, 2, \ldots, k$.

Methods based on *Stackelberg's model* (Sinha et al. 2018), when the problem is considered as MLPP, yields the solution $x_1^* = 100, \quad x_i^* = 0, \quad i = 2, 3, \ldots, k$.

We see that different methods yield different solutions which are more or less satisfactory or unsatisfactory for the users. Since the set of efficient solutions is large ($x_1 + x_2 + \cdots + x_k = 100$), which solution from this set should be taken, is the main question here. These are the reasons and motivation for our work. In the paper we tried to answer this question and we propose some sustainable principles and simple numerical method for such choice.

In our opinion the method for solving MOPP should meet the following requirements.

- Respect and evaluate the aspirations of decision makers. Namely, each decision maker enters the optimization procedure with certain wants, needs, aims (shortly *aspirations*) which he wants to realize. The method should respect them. At the same time the method should test to what extent the aspirations are appropriate in the frame of the given budget.
- Respect the priorities of decision makers. We know that decision makers in practical situations usually are not equal. They make decisions with different authorities, influences, weights (shortly *priorities*). Such differences have to be incorporated into the solution process and recognized from the computational procedure.
- Enable the iterations (possible improvements) of the solution. If decision makers are convinced that the obtained solution is the best possible choice for all of them in the frame of given budget then such solution is satisfactory one and it represents

an equilibrium state. If it is not the case then the method has to enable iterations of the solution until such state will be reached.

- Simple and understandable computational procedure, so that decision makers can trust the obtained results and understand why such results are obtained.

The method presented in the paper combines all the above criteria. It is an upgraded and modified method from Matejaš and Perić (2014).

2 New approach

Let $S \subset R^n$ be the given budget, k the number of decision makers (DMs) and $F_i : S \to R$ the objective function for DM $i, \quad i = 1, 2, \ldots, k$. We consider the general MOPP,

$$\max_{x \in S} (f_1(x), f_2(x), \ldots, f_k(x)). \tag{2}$$

Usually, S is a bounded closed set which is defined as the intersection of several constraints. As we have already pointed out each DM has an aspiration which he wants to achieve. Let a_i be the aspiration for DM $i, \quad i = 1, 2, \ldots, k$. He want to realize this aspiration, therefore, he seeks such a point $x \in S$ where $f_i(x) \geq a_i$. Next, each DM has a certain priority which is given by the positive real number $p_i, \quad i = 1, 2, \ldots, k$. Now, priority and aspiration together imply the request $f_i(x) \geq p_i a_i$. If there exists $x \in S$ such that this request is met for all $i = 1, 2, \ldots, k$ then we have an ideal situation: each DM can achieve his aspiration according to the priority which he has in the decision-making process. But usually such x does not exist. Then the above request is partially met, it is met to some extent $\lambda, \quad f_i(x) \geq \lambda p_i a_i$. Obviously λ depends on x and i. The basic idea is to find $x = x^* \in S$ where $\lambda = \lambda^*$ is maximal and common for all $i = 1, 2, \ldots, k$. In this way we transform the initial MOPP (2) into single objective programming problem,

$$\max_{(x,\lambda) \in V} \lambda, \quad V = \{(x, \lambda) : \ x \in S, \ \lambda \geq 0, \ f_i(x) \geq \lambda p_i a_i, \ i = 1, 2, \ldots, k\}. \tag{3}$$

Thus, the optimal solution λ^* shows to what maximal extent all DMs could achieve their aspirations within the given budget S, respecting their priorities. Problem (3) can serve as an iteration of the method. Namely, if the obtained solution is not satisfactory for all DMs then they can redefine their aspirations (the priorities are fixed, they are imposed by real situation from which the problem originates) and solve (3) again (perform the next iteration of the method). In this case two questions appear: which aspirations should be changed and how much? We define the *indicators* which will give the first answer and we state some *sustainable principles* how to redefine the aspirations for the second answer.

If (x^*, λ^*) is the optimal solution to problem (3), we define the indicators,

$$\lambda_i = \frac{f_i(x^*)}{p_i a_i}, \quad i = 1, 2, \ldots, k,$$

which show to what extent DMs realize their aspirations in the optimal point. For example, if $k = 4$ and we obtained the following indicators: $\lambda_1 = 0.6,\ \lambda_2 = 0.5,\ \lambda_3 = 0.8,\ \lambda_4 = 1.1$, then the first DM could realize 60% of his aspiration respecting priority, the second one 50%, the third 80% and the fourth 110%. We see that $\lambda^* = \lambda_2 = 0.5$ which means that the realization for each DM is at least 50%. And why not more? Because aspiration a_2 is set too high which disables better realization for all DMs. The other aspirations could be even increased up to

$$a_1' = \frac{0.6}{0.5}a_1 = 1.2a_1, \quad a_3' = \frac{0.8}{0.5}a_3 = 1.6a_3, \quad a_4' = \frac{1.1}{0.5}a_4 = 2.2a_4,$$

which will not affect the optimal solution. Thus, if we want any improvements then a_2 has to be decreased and problem (3) solved again. In this way the indicators point to those DMs who set their aspirations too high (for them active constraint $f_i(x^*) = \lambda^* p_i a_i$ holds) and so they caused unsatisfactory realization for all of them. To redefine aspirations we propose the following sustainable rules (Matejaš et al. 2019).

Rule 1 *Any aspiration has to be attainable.*

It means that aspiration a_i should belong to interval $[m_i, M_i]$ where $m_i = \min_{x \in S} f_i(x), M_i = \max_{x \in S} f_i(x)$. Although the method allows any aspiration this is a natural rule, it is in vain to look for something that is out of reach (e.g. that is beyond the maximum). This rule serves to check the aspirations for all DMs at the beginning of each iteration.

Rule 2 *Any DM may require to increase his possible realization at most as much as he is ready to give up from his aspiration.*

This rule means that new aspiration a_i' should satisfy

$$p_i a_i' - f_i(x^*) \le p_i a_i - p_i a_i' \Rightarrow a_i' \le \frac{1+\lambda_i}{2}a_i = \left(1 + \frac{\lambda_i - 1}{2}\right)a_i.$$

We see that the high aspiration ($\lambda_i < 1$) should be decreased while the modest one ($\lambda_i > 1$) could be even increased.

Rule 3 *Any DM has the right to retain at most as much of his possible realization as he is ready to give up.*

Using this rule DM could set the lower bound for his realization, $f_i(x) \ge p_i l_i$, where l_i satisfies

$$p_i l_i \le f_i(x^*) - p_i l_i \Rightarrow p_i l_i \le \frac{f_i(x^*)}{2} \Rightarrow l_i \le \frac{\lambda_i}{2} \cdot a_i.$$

Rule 4 *Any DM has the right to set his possible realization as his aspiration.*

This rule yields the lower bound for the next aspiration a_i',

$$p_i a_i' \ge f_i(x^*) \Rightarrow a_i' \ge \lambda_i a_i.$$

DMs set their initial aspirations according to Rule 1. After that any DM, at any stage of the process when the aspirations need to be changed, may apply any of Rules 2–4. If the solution is not satisfactory for all DMs and they decide to perform a new iteration, then at least one active DM (one for whom the constraint in the optimal point is active, $f_i(x^*) = \lambda^* p_i a_i$) *has to apply* Rule 2 or Rule 4 (to decrease his aspiration). In such case, for any possible improvement, the application of these rules is mandatory, otherwise it is optional. Namely, the other passive DMs (whose constraints are passive, $f_i(x^*) > \lambda^* p_i a_i$) may do nothing or apply any of Rules 2–4 if they want so.

Now, let us apply the proposed method to Example 1. We transform initial MOPP (1) into single objective programming problem (3),

$$\max\lambda, \quad \text{where} \;\; \lambda, x_1, x_2, \ldots, x_k \geq 0, \quad x_1 + x_2 + \cdots + x_k \leq 100, \\ x_i \geq \lambda p_i a_i, \;\; i = 1, 2, \ldots, k. \tag{4}$$

which has a unique efficient solution

$$\lambda^* = \frac{100}{p_1 a_1 + p_2 a_2 + \cdots + p_k a_k}, \quad x_i^* = \lambda^* p_i a_i, \quad i = 1, 2, \ldots, k.$$

We see that the indicators, $\lambda_i = x_i^*/p_i a_i = \lambda^*$, are all optimal which means that decrease of any aspiration improves the optimal solution (that is also obvious from the expression for λ^*). If we vary aspiration a_i, when all other aspirations are fixed, and consider x_i^* as a function of a_i then we have bijection between the sets $a_i \in [0, +\infty]$ and $x_i^* \in [0, 100]$ that justifies our approach. There are no preferred points in the set of efficient points, $x_1 + x_2 + \cdots + x_k = 100$, each one could be a solution depending on the chosen values for aspirations and priorities.

To illustrate how the iterations and applications of the stated rules take place we consider the case $k = 2$ with priorities $p_1 = 1.5$, $p_2 = 1$. We suppose that the first DM in each iteration wants to increase his realization (Rule 2) while the second DM only want to retain what he has (Rule 4). Let $s = 1, 2, 3, \ldots$ be the successive iterations, $a_1^{(s)}, a_2^{(s)}$ aspirations and $\lambda^{(s)}, x_1^{(s)}, x_2^{(s)}$ solution in iteration s. Thus, problem (4) for iteration s reads

$$\max\lambda, \quad \text{where} \;\; \lambda, x_1, x_2 \geq 0, \quad x_1 + x_2 \leq 100, \quad x_1 \geq 1.5\lambda a_1^{(s)}, \quad x_2 \geq \lambda a_2^{(s)},$$

and the solution is $\lambda^{(s)} = 100/(1.5a_1^{(s)} + a_2^{(s)})$, $x_1^{(s)} = 1.5\lambda^{(s)} a_1^{(s)}$, $x_2^{(s)} = \lambda^{(s)} a_2^{(s)}$. The aspirations for the next iteration $s + 1$ is given by

$$a_1^{(s+1)} = \frac{1 + \lambda^{(s)}}{2} a_1^{(s)} \;\; \text{(Rule 2)}, \qquad a_2^{(s+1)} = x_2^{(s)} = \lambda^{(s)} a_2^{(s)} \;\; \text{(Rule 4)}.$$

If we assume maximal initial aspirations, $a_1^{(1)} = a_2^{(1)} = 100$, we obtain the following results for the first ten iterations.

Since the initial aspirations are equal, we see how realizations in the first iteration are in tune with priorities, $60 : 40 = 1.5 : 1$. Note that the second DM could restrict the budget after first iteration by additional constraint $x_2 \geq 1 \cdot (0.4/2) \cdot 100 = 20$

Table 1 Successive iterations of the solution

s	$a_1^{(s)}$	$x_1^{(s)}$	$x_2^{(s)} = a_2^{(s+1)}$	$\lambda^{(s)}$
1	100	60	40	0.4
2	70	72.413793	27.586207	0.689655
3	59.137931	76.278725	23.721275	0.859896
4	54.995207	77.666545	22.333455	0.941495
5	53.386452	78.192782	21.807218	0.976437
6	52.757487	78.396599	21.603401	0.990654
7	52.510943	78.476197	21.523803	0.996315
8	52.414204	78.507385	21.492615	0.998551
9	52.376230	78.519620	21.480380	0.999431
10	52.361322	78.524432	21.475577	0.999776

(Rule 3), which would be his protection in further iterations, but it was superfluous. From Table 1 we see how method respects but controls 'the strong' and protects 'the weak'. Namely, method respects constant request of the first DM to increase his realization (Rule 2) but controls it: we obtain increase convergent sequence $x_1^{(s)} \to x_1^*$ where $78.52 < x_1^* < 78.53$. At the same time method protects the second, modest DM who only want to retain his possible realization (Rule 4): we obtain decrease convergent sequence $x_2^{(s)} \to x_2^*$ where $21.47 < x_2^* < 21.48$. Note also how the application of the stated rules gradually corrects the aspirations to be realistic in the frame of given budget that results with realization convergent sequence, $\lambda^{(s)} \to 1 = 100\%$.

In this way, from the practical standpoint, the method points to those DMs with too high, unreal aspirations and prevents their megalomania. At the same time it prevents the bankruptcy of DMs whose aspiration are, perhaps for some objective reasons, too low. The method provides moves that gradually lead, through successive iterations, to the best possible equilibrium state, satisfactory for everyone.

3 Application to bilevel programming problems

It is widely recognized that the existing methods for solving the bilevel (and especially multilevel) programming problem are usually quite complex and the solutions are often inefficient (see Hansen et al. 1992; Deng 1998; Sinha et al. 2018). We show here how to obtain an efficient solution, using the above presented method. The method can begin with a solution obtained by any existing method or with an arbitrarily chosen feasible point proposed in any way as a reference point. The priority of one level over another is defined and quantified in this method. The solution procedure respects the chosen point, the given priority and maximal possible aspirations, and it always yields a unique, efficient solution. Besides, any feasible efficient point can be an optimal solution for suitably chosen level priority. This is illustrated by several examples for linear and nonlinear case. For the linear bilevel problem the method consists of one or two standard linear problems. The method can be easily extended to multilevel problems with any number of levels.

We state the bilevel programming problem (BLPP). Let $S \subseteq R^n$ be the given budget where the optimization process occurs. Here again, we assume that S is a bounded closed set that is usually defined as the intersection of several constraints. Let $(x, y) \in S$ be partitioned between levels in such a way that the first-level DM (leader) controls $x \in R^{n_1}$ and the second-level DM (follower) controls $y \in R^{n_2}$, where $n_1 + n_2 = n$. If $F : S \to R$, $f : S \to R$ are the objective functions for the leader and follower, respectively; then, the BLPP can be stated as follows:

$$\begin{aligned} &\max_{x,y} F(x, y) \qquad \text{(level 1)} \\ &\text{where} \quad y \text{ solves} \\ &\max_{y} f(x, y) \qquad \text{(level 2)} \\ &\text{subject to } (x, y) \in S. \end{aligned} \tag{5}$$

If the objective functions are linear and the budget is defined as the intersection of linear constraints (polyhedron), we have a linear bilevel programming (LBLPP) problem.

The BLPP and especially the LBLPP has been thoroughly studied in the literature, and various methods have been proposed for solving it (Hansen et al. 1992; Deng 1998; Sinha et al. 2016, 2018). Using the standard basic definition of the solution to the BLPP (5) can be expressed as follows:

$$\max\{F(x, y(x)) : \ (x, y(x)) \in \mathrm{IR}(f)\}, \quad \mathrm{IR} = \text{inducible region},$$

where $y(x) \in \arg\max_z\{f(x, z) : \ (x, z) \in S\}$ is the follower's rational reaction to the leader's choice of x. Since the inducible region is usually not a convex set, the obtained solution is often inefficient. This is particularly the case when the objectives are contradictory to a certain extent.

To apply our method, we begin with the solution (x^*, y^*) that is obtained by using any existing appropriate method for solving the BLPP. For the LBLPP, there are over 20 such methods in the literature. Thus, we do not discuss the methodology for obtaining (x^*, y^*). Generally, in our analysis, any point from S can be proposed, chosen, or suggested in any way to be a solution to the BLPP. Beginning with (x^*, y^*), our method yields

- an efficient solution if (x^*, y^*) is inefficient,
- (x^*, y^*) if it is efficient.

For this purpose, we compute

$$\begin{aligned} &F^* = F(x^*, y^*), \quad f^* = f(x^*, y^*), \\ &F_M = \max\{F(x, y) : \ (x, y) \in S\}, \quad f_M = \max\{f(x, y) : \ (x, y) \in S\}, \end{aligned} \tag{6}$$

and we define

$$\Delta_F = F_M - F^*, \quad \Delta_f = f_M - f^*. \tag{7}$$

The expression Δ_F (Δ_f) represents the additional possible amount that the leader (follower) could receive if the follower (leader) is absent from the considered problem. So, these amounts can be (maximal) aspirations for DMs. If (x^*, y^*) is not an efficient point, then both DMs, leader and follower, can obtain a part of Δ_F and Δ_f, respectively. This part is entirely dependent on the priority of each DM. Therefore, it is necessary to quantify this property in the solution procedure. In our opinion, such quantification should be a part of the definition of a general BLPP. Each level should be assigned with a numerical indicator that will distinguish between leader and follower. We provide the following definition.

Definition 1 Let $\alpha \geq 0$ and $\beta \geq 0$, where $\alpha + \beta > 0$, be the level priorities for the first and the second level in the BLPP (5). For the given reference point $(x^*, y^*) \in S$, the optimal solution to problem (5), with respect to (x^*, y^*) and according to priorities α, β, is given by the solution to the problem

$$\begin{aligned} &\max \ \lambda \\ \text{subject to } &F(x, y) \geq F^* + \lambda\alpha\, \Delta_F \\ &f(x, y) \geq f^* + \lambda\beta\, \Delta_f \\ &\lambda \geq 0, \quad (x, y) \in S, \end{aligned} \tag{8}$$

where $F^*, \ f^*, \ \Delta_F, \ \Delta_f$ are defined by relations (6) and (7).

Thus, we are looking for the maximal part λ of the additional amounts that the leader and follower could obtain, keeping in mind their priorities α and β. We state here the main result:

Theorem 1 *There exists an optimal point for problem* (8) *which is an efficient point of the BLPP* (5).

Proof The statement of the theorem can be split into the following two parts:

(i) if the optimal point is unique, then it is an efficient point of (5),
(ii) if the optimal point is not unique, then there exists one which is an efficient point of (5). □

We prove (i).

If $\tilde{\lambda}$ is the optimal solution of (8) in the unique optimal point $(\tilde{x}, \tilde{y})$, then we claim that

$$F(\tilde{x}, \tilde{y}) = F^* + \tilde{\lambda}\alpha\, \Delta_F, \quad f(\tilde{x}, \tilde{y}) = f^* + \tilde{\lambda}\beta\, \Delta_f. \tag{9}$$

Namely, at least one of these equalities must hold because $\tilde{\lambda}$ is the optimal solution. If one does not hold, say if $F(\tilde{x}, \tilde{y}) > F^* + \tilde{\lambda}\alpha\, \Delta_F$, then there exists another point $(x', y') \in S$ at the intersection of the sheets $F(x, y) = F^* + \tilde{\lambda}\alpha\, \Delta_F, \ f(x, y) = f^* + \tilde{\lambda}\beta\, \Delta_f$, which is also an optimal point of (8) and which contradicts the assumption of uniqueness. Now, in the same way, from (9), it easily follows that if there exists another point $(x'', y'') \in S$ such that $F(x'', y'') > F(\tilde{x}, \tilde{y})$ or $f(x'', y'') > f(\tilde{x}, \tilde{y})$, then it is also an optimal point of (8) that is contradictory. So, $(\tilde{x}, \tilde{y})$ is the efficient point of (5).

We prove (ii).

Let $\tilde{\lambda}$ be the optimal solution and Ω the set of optimal points.

If (9) holds for each $(\tilde{x}, \tilde{y}) \in \Omega$, then Ω is the set of efficient points of (5) for the same reasons as in (i).

If there exists $(\tilde{x}, \tilde{y}) \in \Omega$ for which one of the equalities in (9) does not hold, then this level can increase its objective without affecting the other one. If the first equality in (9) does not hold and the second one does, then we solve

$$\begin{aligned} &\max \ \lambda_1 \\ \text{subject to } &F(x, y) \geq F^* + (\tilde{\lambda} + \lambda_1)\,\alpha\,\Delta_F \\ &f(x, y) = f^* + \tilde{\lambda}\beta\,\Delta_f \\ &\lambda_1 \geq 0, \quad (x, y) \in S. \end{aligned} \tag{10}$$

If $\tilde{\lambda}_1$ is the optimal solution of (10), then $(\tilde{\lambda} + \tilde{\lambda}_1)\alpha\,\Delta_F$ is the maximal additional amount for the leader and $\tilde{\lambda}\beta\,\Delta_f$ for the follower. Now, for the same reasons as before, each optimal point of problem (10) is an efficient point of (5). Similarly, if the first equality in (9) holds and the second one does not, we solve

$$\begin{aligned} &\max \ \lambda_2 \\ \text{subject to } &F(x, y) = F^* + \tilde{\lambda}\alpha\,\Delta_F \\ &f(x, y) \geq f^* + (\tilde{\lambda} + \lambda_2)\beta\,\Delta_f \\ &\lambda_2 \geq 0, \quad (x, y) \in S. \end{aligned} \tag{11}$$

If $\tilde{\lambda}_2$ is the optimal solution of (11), then $\tilde{\lambda}\alpha\,\Delta_F$ is the maximal additional amount for the leader and $(\tilde{\lambda} + \tilde{\lambda}_2)\beta\,\Delta_f$ for the follower, and each optimal point of problem (11) is an efficient point of (5), that completes the proof.

The proof of Theorem 1 is also the constructive one. So, if we want to find an efficient solution to BLPP (5) by employing our method, which is given in Definition 1, the solution procedure is as follows:

- Beginning with the reference point $(x^*, y^*) \in S$, we solve (8).
- If (9) holds for the optimal point(s) of (8) then it is an (they are) efficient point(s) of (5).
- If the first (second) equality in (9) does not hold for each optimal point of (8), then we solve (10) (11) and the obtained optimal point(s) is an (are) efficient one(s) of (5).

Thus, the optimal solution of (8) yields the efficient solution of (5). Is it true for any efficient solution of (5)? The answer is affirmative.

Theorem 2 *Let* $(x^*, y^*) \in S$ *be the given reference point for BLPP* (5) *and let* $F^*, f^*, \Delta_F, \Delta_f$ *be defined by the relations* (6) and (7). *If* $(\tilde{x}, \tilde{y}) \in S$ *is an efficient point of* (5) *for which* $F(\tilde{x}, \tilde{y}) \geq F^*$, $f(\tilde{x}, \tilde{y}) \geq f^*$ *holds, then there exists* $\alpha, \beta \geq 0$ *such that* $(\tilde{x}, \tilde{y})$ *is optimal point of problem* (8).

Proof If $\Delta_F = 0$ or $\Delta_f = 0$ then the assertion is obvious for arbitrary $\alpha, \beta > 0$, otherwise we set

$$\alpha = \frac{F(\tilde{x}, \tilde{y}) - F^*}{\Delta_F}, \qquad \beta = \frac{f(\tilde{x}, \tilde{y}) - f^*}{\Delta_f}.$$

Now, problem (8) reads

$$\begin{aligned} &\max\ \lambda \\ \text{subject to } &F(x, y) \geq (1 - \lambda)F^* + \lambda F(\tilde{x}, \tilde{y}) \\ &f(x, y) \geq (1 - \lambda)f^* + \lambda f(\tilde{x}, \tilde{y}) \\ &\lambda \geq 0, \quad (x, y) \in S, \end{aligned}$$

and its solution is $\tilde{\lambda} = 1$ in the point $(\tilde{x}, \tilde{y})$. Namely, that $\tilde{\lambda} = 1 + \varepsilon > 1$ is the solution in the point (x', y') then $F(x', y') \geq F(\tilde{x}, \tilde{y}) + \varepsilon\big[F(\tilde{x}, \tilde{y}) - F^*\big]$, $f(x', y') \geq f(\tilde{x}, \tilde{y}) + \varepsilon\big[f(\tilde{x}, \tilde{y}) - f^*\big]$ will hold. These two inequalities contradict the assumption of efficiency if $F(\tilde{x}, \tilde{y}) > F^*$ or $f(\tilde{x}, \tilde{y}) > f^*$. Otherwise, if $F(\tilde{x}, \tilde{y}) = F^*$ and $f(\tilde{x}, \tilde{y}) = f^*$ then $F(x', y') = F(\tilde{x}, \tilde{y})$, $f(x', y') = f(\tilde{x}, \tilde{y})$ which means that $(\tilde{x}, \tilde{y})$ is an optimal point but it is not the unique one. □

Theorems 1 and 2 also show two-way connection between the set of efficient points of (8) and the set of values of priorities. This connection is visible and explained in detail in the examples below.

Finally, we provide some remarks:

Remark 1 If (x^*, y^*) is already an efficient point, then the solution of (8) will be $\tilde{\lambda} = 0$ and (x^*, y^*) will be the optimal point of (5) (or one of them).

Remark 2 The method satisfies the well-known properties of a compromise solution (see Yu 1985; Wen and Hsu 1991): feasibility, individual rationality, no dictatorship, uniqueness, symmetry, and Pareto optimality. It is also independent of scaling, particularly, if $\tilde{\lambda}$ is the optimal solution of (8) for the given α, β then $\tilde{\lambda}/k$ is the solution for $k\alpha, k\beta$ in the same optimal point(s).

Remark 3 If the method is applied to the LBLPP, then (8), (10) and (11) are standard linear programming problems; otherwise, these are nonlinear problems where computational problems can occur. In this case, application (or development) of suitable numerical methods is required.

Remark 4 The method can be easily extended to multilevel problems with any number of levels. If we have k levels with objectives $F_i : S \to R,\ \ i = 1, 2, \ldots, k,\ \ S \subseteq R^n$, then we assign them priorities $\alpha_i \geq 0,\ \ i = 1, 2, \ldots, k$. For the given reference point $x^* \in S$, we solve

$$\begin{aligned} &\max\ \lambda \\ \text{subject to } &F_i(x) \geq F(x^*) + \lambda\alpha_i\, \Delta_{F_i}, \quad i = 1, 2, \ldots, k \\ &\lambda \geq 0, \quad x \in S. \end{aligned}$$

where $\Delta_{F_i} = \max\{F_i(x) : \ x \in S\} - F_i(x^*), \quad i = 1, 2, \ldots, k.$

We apply and explain the method with the help of some examples. The following example represents the case of the linear BLPP with a unique optimal point.

Example 2 We consider the following two-dimensional LBLPP (5),

$$\begin{aligned}
&\max_{x,y} \quad 2y - x && \text{(level 1)}\\
&\text{where } y \text{ solves}\\
&\max_{y} \quad 34 - 3y - x && \text{(level 2)}\\
&\text{subject to} \quad y \le 3x - 2, \quad y \le 18 - 2x, \quad y \ge x.
\end{aligned}$$

The solution process is illustrated in Fig. 1. The budget S is the triangle MNP. Using the standard basic definition of the BLPP solution, we determine that the inducible region is the line MN, and the optimal point is N, $(x^*, y^*) = (6, 6)$, while the optimal solution (6) is $F^* = F(6, 6) = 6$, $f^* = f(6, 6) = 10$. We see that the obtained solution is inefficient in the triangle NQT because both levels can increase their objectives. Using (6) and (7), we have $F_M = F(P) = F(4, 10) = 16$, $f_M = f(M) = f(1, 1) = 30$, $\Delta_F = 16 - 6 = 10$, $\Delta_f = 30 - 10 = 20$. Now, we state problem (8),

$$\begin{aligned}
&\max \ \lambda\\
\text{subject to } &2y - x \ge 6 + \lambda\alpha \cdot 10\\
&34 - 3y - x \ge 10 + \lambda\beta \cdot 20\\
&\lambda \ge 0, \quad y \le 3x - 2\,.
\end{aligned} \tag{12}$$

The optimal solution is

$$\begin{aligned}
\tilde{\lambda} &= \frac{1}{2(\alpha+\beta)}\,, \quad \tilde{x} = 2 + \frac{\mu}{\mu+1} = 3 - \frac{1}{\mu+1}\,,\\
\tilde{y} &= 4 + \frac{3\mu}{\mu+1} = 7 - \frac{3}{\mu+1}\,, \quad \mu = \frac{\alpha}{\beta}\,,
\end{aligned}$$

which yields the efficient point of LBLPP, where the solution is

$$\begin{aligned}
F(\tilde{x}, \tilde{y}) &= 6 + \frac{\alpha}{2(\alpha+\beta)} \cdot 10 = 6 + \frac{5\mu}{\mu+1}\,,\\
f(\tilde{x}, \tilde{y}) &= 10 + \frac{\beta}{2(\alpha+\beta)} \cdot 20 = 10 + \frac{10}{\mu+1}\,.
\end{aligned}$$

For the given α, β, we obtain a unique efficient point that depends only on the ratio μ. With varying μ, each efficient point Z on the line QT can be obtained: $T(2, 4)$ for $\mu = 0$ ($\alpha = 0$, $\beta > 0$), $Q(3, 7)$ for $\mu = \infty$ ($\alpha > 0$, $\beta = 0$), and $Z(\tilde{x}, \tilde{y})$ for $0 < \mu < \infty$ ($\alpha > 0$, $\beta > 0$). Thus, each μ, $0 \le \mu \le \infty$ yields the unique efficient point and the corresponding solution, and vice versa; for each feasible efficient point,

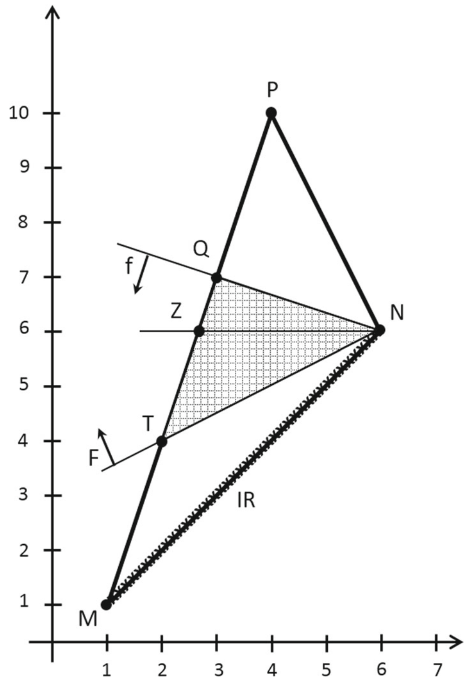


Fig. 1 Example 2—the linear case with unique optimal point

there exists the unique $\mu \geq 0$ that yields the solution in that point. Therefore, we have a bijection between the set of efficient points and the interval $[0, \infty]$. In this way, no point is preferred over the other. Each one is a potential candidate for the solution, depending on the priority of the levels. This fact fully justifies the reason for introducing the indicators α, β.

Geometrically speaking (see Fig. 1), the intersection of $2y - x = 6 + \lambda\alpha \cdot 10$ and $34 - 3y - x = 10 + \lambda\beta \cdot 20$ is

$$\mu = 0 \;\Rightarrow\; y = \frac{1}{2}x + 3 \text{ (line } NT\text{): } f \text{ increases, } F \text{ is constant,}$$

$$\mu = \infty \;\Rightarrow\; y = -\frac{1}{3}x + 8 \text{ (line } NQ\text{): } F \text{ increases, } f \text{ is constant,}$$

$$0 < \mu < \infty \;\Rightarrow\; y = \frac{2-\mu}{4+3\mu}x + 12\frac{2\mu+1}{4+3\mu} \text{ (line } NZ\text{): both } F \text{ and } f \text{ increases.}$$

We can see that the parameter μ determines direction, within the feasible region NQT, in which the DM's objectives will be increased.

We can often find a post-optimality analysis for the LBLPP in the literature (see Wen and Hsu 1991). Some rational solution concepts for finding an efficient solution are suggested. We shall compare our approach with that of Wen and Hsu where three categories of compromise solutions are proposed.

The threat-point dependent solution is defined by solving the problem

$$\max_{(x,y)\,\in\, E^{*}\cap\, S} \quad \pi(x, y) = \left[F(x, y) - F^{*}\right]\cdot\left[f(x, y) - f^{*}\right],$$

where $E^{*} = \{(x, y) \in R^{n} :\ F(x, y) \geq F^{*},\ f(x, y) \geq f^{*}\}$. In our example, the problem is

$$\begin{gathered}\max\ [2y - x - 6]\cdot[24 - 3y - x]\\ \text{subject to } 2y - x \geq 6,\quad 34 - 3y - x \geq 10,\quad y \leq 3x - 2\,,\end{gathered}$$

which yields the efficient point $Z(2.5,\ 5.5)$ and $F(Z) = 8.5,\ \ f(Z) = 15$. This solution can be obtained by our method from (12), as $\mu = 1$ ($\alpha = \beta$).

The ideal-point dependent solution is defined by minimizing l_p norm distance to the ideal point $(x^{(0)}, y^{(0)})$, which is defined by

$$\begin{aligned}F(x^{(0)}, y^{(0)}) &= F^{(0)} = \max\left\{F(x, y) :\ (x, y) \in E^{*} \cap S\right\},\\ f(x^{(0)}, y^{(0)}) &= f^{(0)} = \max\left\{f(x, y) :\ (x, y) \in E^{*} \cap S\right\}.\end{aligned}$$

The problem that has to be solved is

$$\min_{(x,y)\,\in\, E^{*}\cap\, S} \quad \pi_p(x, y) = \left[\left|F(x, y) - F^{(0)}\right|^{p} + \left|f(x, y) - f^{(0)}\right|^{p}\right]^{1/p}.$$

In our example, we have $(x^{(0)}, y^{(0)}) = (-1,\ 5)$ and $F^{(0)} = F(3, 7) = 11,\ \ f^{(0)} = f(2, 4) = 20$. The problem is

$$\begin{gathered}\min\ \left[|2y - x - 11|^{p} + |14 - 3y - x|^{p}\right]^{1/p}\\ \text{subject to } 2y - x \geq 6,\quad 34 - 3y - x \geq 10,\quad y \leq 3x - 2\,.\end{gathered}$$

For each p, the solution can be obtained from our linear problem (12) with the appropriate μ. For instance, for the usual l_p norms, l_1, l_2, l_∞, we obtain

$$\begin{aligned}&p = 1: && Z(2,\ 4),\quad F(Z) = 6,\quad f(Z) = 20\quad (\mu = 0),\\ &p = 2: && Z(2.2,\ 4.6),\quad F(Z) = 7,\quad f(Z) = 18\quad (\mu = 0.25),\\ &p = \infty: && Z(2\tfrac{1}{3},\ 5),\quad F(Z) = 7\tfrac{2}{3},\quad f(Z) = 16\tfrac{2}{3}\quad (\mu = 0.5).\end{aligned}$$

The ideal-threat-point dependent solution is defined by solving the problem

$$\min_{(x,y)\,\in\, E^{*}\cap\, L} \quad \pi(x, y) = \left[F(x, y) - F^{*}\right]^{2} + \left[f(x, y) - f^{*}\right]^{2},$$

where L is the line that connects the ideal point $(x^{(0)}, y^{(0)})$ with the threat point (x^{*}, y^{*}). In our example, the line L is $x - 7y + 36 = 0$ and the problem is

$$\min \ (2y - x - 11)^2 + (14 - 3y - x)^2$$
$$\text{subject to } 2y - x \geq 6, \quad 34 - 3y - x \geq 10, \quad y \leq 3x - 2, \quad x - 7y + 36 = 0\,,$$

which yields $Z(2.5,\ 5.5)$ and $F(Z) = 8.5, \quad f(Z) = 15$ (in our method for $\mu = 1$).

We see that all these solutions can be obtained not by solving non-linear problems but by solving the simple linear problem (12). Moreover, any other efficient solution can be obtained, depending on the relationship between the levels that are determined by level priorities. This is the reason we propose that priorities should be initially defined in multilevel programming problems.

Now, we present the case of the LBLPP where the optimal point is not unique.

Example 3 We consider the following two-dimensional LBLPP problem (5),

$$\begin{aligned}
&\max_{x,y} \quad 2y - x \qquad\qquad\qquad \text{(level 1)}\\
&\text{where } y \text{ solves}\\
&\max_{y} \quad 18 - 2y - x \qquad\qquad\quad \text{(level 2)}\\
&\text{subject to } \ 0 \leq x \leq 4, \quad y \geq x, \quad 2y - x \leq 8\,.
\end{aligned}$$

The solution process is illustrated in Fig. 2. The budget S is the trapeze $MNPT$, the inducible region is the line MN, and the optimal solution is $F^* = F(N) = F(4,4) = 4$, $f^* = f(N) = f(4,4) = 6$. The solution is inefficient, since it can be improved (i.e., increased) in the trapeze $NQTV$. Using (6) and (7), we have $F_M = F(PT) = 8$, $f_M = f(M) = f(0,0) = 18, \Delta_F = 8 - 4 = 4, \quad \Delta_f = 18 - 6 = 12$. Thus, problem (8) is

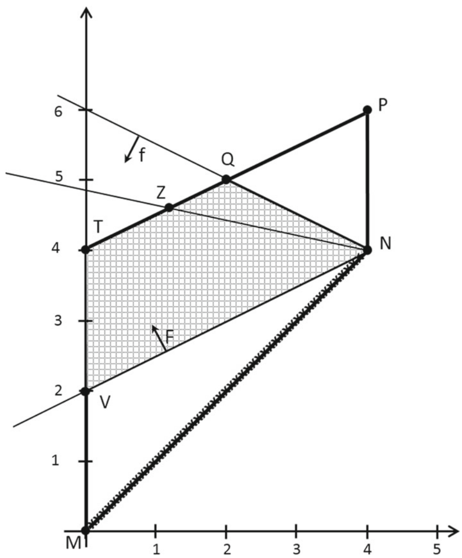
$$\begin{aligned}
&\max \ \lambda\\
&\text{subject to } 2y - x \geq 4 + \lambda\alpha \cdot 4\\
&\qquad 18 - 2y - x \geq 6 + \lambda\beta \cdot 12\\
&\qquad \lambda \geq 0, \quad x \geq 0, \quad 2y - x \leq 8\,.
\end{aligned} \tag{13}$$

Now (see Fig. 2), the intersection of $2y - x = 4 + \lambda\alpha \cdot 4$ and $18 - 2y - x = 6 + \lambda\beta \cdot 12$ is

$$\mu = 0 \Rightarrow y = \frac{1}{2}x + 2 \text{ (line } NV\text{): } f \text{ increases}, \ F \text{ is constant},$$
$$\mu = \infty \Rightarrow y = -\frac{1}{2}x + 6 \text{ (line } NQ\text{): } F \text{ increases}, \ f \text{ is constant},$$
$$0 < \mu < \infty \Rightarrow y = \frac{3-\mu}{6+2\mu}x + 6\frac{\mu+1}{\mu+3} \text{ (line } NZ\text{): both } F \text{ and } f \text{ increases},$$

where again $\mu = \alpha/\beta$. The optimal solution of (13) is:

Fig. 2 Example 3—the linear case where the optimal point is not unique



If $0 \leq \mu \leq 3$, then

$$\tilde{\lambda} = \frac{2}{\alpha + 3\beta}, \quad \tilde{x} = 0, \quad \tilde{y} = 2 + \frac{4\mu}{\mu + 3} = 4 - 2\frac{3 - \mu}{\mu + 3},$$

if $3 < \mu \leq \infty$, then

$$\tilde{\lambda} = \frac{1}{\alpha}, \quad \tilde{x} = 2t\left(1 - \frac{3}{\mu}\right), \quad \tilde{y} = 4 + t\left(1 - \frac{3}{\mu}\right), \quad 0 \leq t \leq 1.$$

Thus, for $0 \leq \mu \leq 3$, we obtain the unique optimal point of (13) at line VT, which is the efficient point of the considered BLPP, where the solution is

$$F(\tilde{x}, \tilde{y}) = 4 + \frac{2\alpha}{\alpha + 3\beta} \cdot 4 = 4 + \frac{8\mu}{\mu + 3}, \quad f(\tilde{x}, \tilde{y}) = 6 + \frac{2\beta}{\alpha + 3\beta} \cdot 12 = 6 + \frac{24}{\mu + 3}.$$

Note that for $\mu = 0$, the solution is $F(V) = F(0, 2) = 4,\ f(V) = f(0, 2) = 14$; for $\mu = 3$, it is $F(T) = F(0, 4) = 8,\ f(T) = f(0, 4) = 10$.

For $3 < \mu \leq \infty$, we obtain the set of optimal points, $\Omega =$ line TZ, where $T(0,\ 4)$ $(t = 0)$ and $Z(2 - 6/\mu,\ 5 - 3/\mu)$ $(t = 1)$. Note that if $\mu \to 3$ then $Z \to T$ and if $\mu \to \infty$ then $Z \to Q$ (see Fig. 2). In point Z, relation (9) holds,

$$2\tilde{y} - \tilde{x} = 8 = 4 + \tilde{\lambda}\alpha \cdot 4, \quad 18 - 2\tilde{y} - \tilde{x} = 6 + \frac{12}{\mu} = 6 + \tilde{\lambda}\beta \cdot 12.$$

In any other point on the line TZ $(0 \leq t < 1)$, the second equality in (9) does not hold,

$$2\tilde{y} - \tilde{x} = 8 = 4 + \tilde{\lambda}\alpha \cdot 4, \quad 18 - 2\tilde{y} - \tilde{x} = 6 + \frac{12}{\mu} + (1-t)\left(4 - \frac{12}{\mu}\right)$$
$$> 6 + \frac{12}{\mu} = 6 + \tilde{\lambda}\beta \cdot 12,$$

which means that the follower (second level) can still increase his objective along the line TZ, while the leader (first level) keeps the constant amount 8. We state problem (11):

$$\begin{aligned} &\max \ \lambda_2 \\ &\text{subject to } 2y - x = 8 \\ &\qquad 18 - 2y - x \geq 6 + \left(\frac{1}{\alpha} + \lambda_2\right)\beta \cdot 12 \\ &\qquad \lambda_2 \geq 0, \quad x \geq 0\,. \end{aligned} \tag{14}$$

The solution of (14) is $\tilde{\lambda}_2 = \frac{1}{3\beta} - \frac{1}{\alpha}$ in the unique optimal point $T(0,\ 4)$. Note that $\tilde{\lambda}_2 > 0$ because of $\mu = \alpha/\beta > 3$. The point T is an efficient point of considered BLPP and the solution is

$$F(0,4) = F^* + \tilde{\lambda}\alpha\,\Delta_F = 4 + \frac{1}{\alpha}\cdot\alpha\cdot 4 = 8\,,$$
$$f(0,4) = f^* + (\tilde{\lambda} + \tilde{\lambda}_2)\beta\,\Delta_f = 6 + \left[\frac{1}{\alpha} + \left(\frac{1}{3\beta} - \frac{1}{\alpha}\right)\right]\cdot\beta\cdot 12 = 10\,.$$

We see that, for $3 < \mu \leq \infty$, the solution is in the common efficient point T, while for $0 \leq \mu \leq 3$, each μ yields another efficient point. Note again that each μ is associated with the unique efficient point and; for each feasible efficient point, there exists μ, which yields the solution at that point. Here, we have a surjection between the set of feasible efficient points and the interval $[0,\ \infty]$.

Finally, in the following example, we provide a nonlinear BLPP.

Example 4 We consider the following two-dimensional BLPP problem (5),

$$\begin{aligned} &\max_{x,y} \quad y - \sqrt{x} \qquad\qquad \text{(level 1)} \\ &\text{where } y \text{ solves} \\ &\max_{y} \quad x^2 - y \qquad\qquad \text{(level 2)} \\ &\text{subject to } \ (x-2)^2 + (y-1)^2 \leq 1\ . \end{aligned}$$

The solution process is illustrated in Fig. 3. The budget S is the circle of radius 1 with the centre at the point (2, 1). The inducible region is the circular arc MTN and the optimal point is M, $(x^*, y^*) = (1, 1)$, while the optimal solution is $F^* = F(1,1) = 0$, $f^* = f(1,1) = 0$. The obtained solution is inefficient in the region $MPZQ$, as

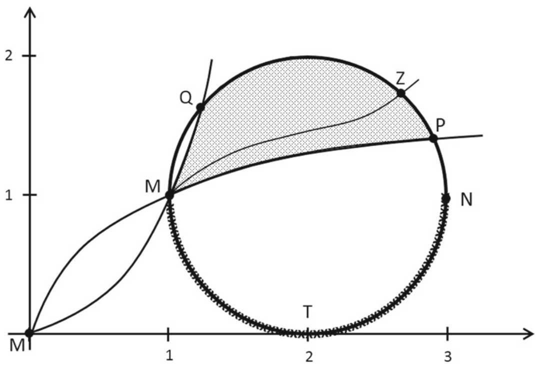


Fig. 3 Example 4—the non-linear case

both levels can increase their objectives. In further calculations, we require numerical methods for solving non-linear systems of equations. We provide all the results with 6 correct decimal digits. Relations (6) and (7) are solved in the following way:

$$\left.\begin{matrix}(x-2)^2(4x+1)=1\\(x-2)^2+(y-1)^2=1\end{matrix}\right\} \Rightarrow \left\{\begin{matrix}x=1.635909\\y=1.931363\end{matrix}\right\} \Rightarrow F_M = y-\sqrt{x} = 0.652337 = \Delta_F,$$

$$\left.\begin{matrix}(x-2)^2(4x^2+1)=4x^2\\(x-2)^2+(y-1)^2=1\end{matrix}\right\} \Rightarrow \left\{\begin{matrix}x=2.986271\\y=0.834866\end{matrix}\right\} \Rightarrow f_M = x^2-y = 8.082949 = \Delta_f.$$

We state problem (8),

$$\begin{aligned} &\max\ \lambda \\ \text{subject to } &y-\sqrt{x} \geq \lambda\alpha\cdot\Delta_F \\ &x^2-y \geq \lambda\beta\cdot\Delta_f \\ &\lambda \geq 0, \quad (x-2)^2+(y-1)^2 \leq 1. \end{aligned}$$

The optimal point is given by the system

$$y = \frac{\mu\Delta_F}{\mu\Delta_F+\Delta_f}x^2 + \frac{\Delta_f}{\mu\Delta_F+\Delta_f}\sqrt{x}, \quad (x-2)^2+(y-1)^2=1,$$

where $\mu = \alpha/\beta$. We see that the first equation is a convex combination of $y = x^2$ and $y = \sqrt{x}$ which covers all possible optimal points on the arc QZP for $0 \leq \mu \leq \infty$. For instance, the choice $\mu = 0, 1, \infty$ yields the optimal points P, Z, Q (the efficient points of the considered BLPP), respectively:

$$\begin{aligned} &\mu = 0 \Rightarrow P(2.752172,\ 1.658967), \quad F(P)=0, \quad f(P)=5.915482, \\ &\mu = 1 \Rightarrow Z(2.446917,\ 1.894575), \quad F(Z)=0.330313, \quad f(Z)=4.092828, \\ &\mu = \infty \Rightarrow Q(1.314596,\ 1.728163), \quad F(Q)=0.581605, \quad f(Q)=0. \end{aligned}$$

We can again see how the priorities α, β determine the solution. For $\mu = 0$ ($\alpha = 0$), the first level keeps the constant amount $F(P) = F(M) = F^* = 0$, while the second one increases it to the highest possible. Similarly, for $\mu = \infty$ ($\beta = 0$), the second level keeps the constant amount $f(Q) = f(M) = f^* = 0$, while the first one increases it to the highest possible. If $0 < \mu < \infty$, both levels increase their objectives. For $\mu = 1$ ($\alpha = \beta$), each level obtains 50.635% of the possible additional amounts Δ_F, Δ_f.

4 Conclusions

If we apply different existing methods to solve a multi-objective programming problem we usually obtain different results. So, which method and which solution from the set of all possible efficient solutions should be taken? In the paper we present criteria and numerical method which try to answer this question. Priorities and aspirations of the decision makers are two categories naturally imposed by the practical situations which can be stated in the form of multi-objective programming problem. It is shown that choice of the solution entirely depends on these categories. The method presented in the paper, respects the priorities and aspirations of the decision makers and it allows iteration (improvement) of the obtained solution, according to the sustainable rules which respect (protect) decision makers with higher (lower) priorities. Decision makers can understand and trust the obtained results because of simple computational procedure.

The method is also applied to bilevel programming problems where, in spite of the significant amount of published research, unsatisfactory results remain from both a theoretical and practical perspective: the existing solution procedures are generally complex and difficult to understand and the obtained optimal solution is often inefficient, which makes this kind of optimality questionable. In other words, if the solution could be improved for both levels, then such methods or the prescribed solution procedures seem to be incomplete.

To remedy such disadvantages or to complete the procedure, we present a simple computational method that includes some kind of cooperation between the decision makers on the both levels and always yields an efficient solution to the bilevel programming problem. It begins with a reference point from the given budget (constraint set), which may be chosen arbitrarily or proposed as a solution by any existing method. The priority (weight of importance, influence or preference) of each level in relation to the others is measured by nonnegative real numbers. By respecting the reference point and the priorities, the method yields a unique efficient solution to the problem. Besides, for any feasible efficient point there exists level priorities such that the optimal solution is in this point. For linear problems, the computational task consists of solving one, or at the most two, standard linear programming problems. For non-linear problems, computational difficulties may occur that require application (or development) of suitable numerical methods. The method can be easily extended to multilevel problems with any number of levels.

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