Numerical solution of truss and beam made of hyperelastic material in different dimensional configurations

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***Abstract*— Mostly applied materials in constructions and structures are metals, the most famous of which are steel, aluminum, etc. Their structural analysis is mostly based on linear elastic analysis, where is the key to keep the behaviour of material in elastic area and thus prevent plastic deformations. Materials like natural rubber, polymers have their role as a construction materials in practice and their behaviour can be expressed as a hyperelastic whose main characteristics are large, nonlinear deformations and displacements. Because of that, these materials require different tools and knowledge to predict their behaviour under load. This paper will describe some basic models of hyperelastic materials with their corresponding equations. Finally, simple examples of truss and beam will be solved with selected solver to give an insight how to model and solve static structural problems of hyperelastic materials.**

***Index Terms*—hyperelasticity, nonlinenarity, finite element method, Newton-Raphson method, FLagSHyP**

I. Introduction

Materials with large deformations and displacements can not be resolved with linear analysis (Hooke's law). In that case, hyperelasticity theory offers the necesarry means to describe behaviour of that materials, especially in establishing $\sigma - \varepsilon$ diagrams from which the principal constants of material are obtained. [3] Mostly used hyperelastic models are so called *phenomenological* models, which are: Mooney-Rivlin's, polynominal's, Ogden's, etc., and *mechanistic* models of which is the neo-Hook's model most famous and used. [6] Each model has its own characteristic properties and depending on loads and deformations, appropriate model is used to describe certain material. In this paper, neo-Hook's model will be used to describe solved examples of truss and beam, where the truss and beam will be defined in each dimensional configurations (one-dimensional, two-dimensional and three-dimensional,). Overall, this paper will present some basic hyperelastic material models with their governing equations, altogether with numerical model of hyperelastic truss and beam and their solution which is obtained with certain software.

II. Hyperelasticity Theory

A. Basic theoretical terms

When the work done by internal forces (stresses) is only path dependant, then it can be said that the material is *elastic*. In special case of *hyperelastic* material, internal work is only time dependant ie. depends on the initial state at time t_0 and final configuration t. Stored strain energy function or energy potential Ψ is then equal to:

$$\Psi(\boldsymbol{F}(\boldsymbol{X}), \boldsymbol{X}) = \int_{t_0}^{t} \boldsymbol{P}(\boldsymbol{F}(\boldsymbol{X}), \boldsymbol{X}) : \dot{\boldsymbol{F}}\, dt \tag{1}$$

where $\boldsymbol{P}$ is first Piola-Kirchhoff stress tensor and his corresponding work conjugate deformation tensor $\dot{\boldsymbol{F}}$, which is rate of deformation gradient. For convenience, elastic potential Ψ is usually expressed in terms of right Cauchy-Green deformation tensor $\boldsymbol{C}$ or material deformation tensor $\boldsymbol{E}$ whose relation is $2\boldsymbol{E} = \boldsymbol{C}$ and second Piola-Kirchhoff stress tensor $\boldsymbol{S}$. Work conjugate deformation tensor of second Piola-Kirchhoff's stress tensor $\boldsymbol{S}$ is the time derivative of right Cauchy-Greens's tensor $\dot{\boldsymbol{C}}$ or material deformation tensor $\dot{\boldsymbol{E}}$. Second Piola-Kirchhoff stress tensor can now be expressed as:

$$\boldsymbol{S}(\boldsymbol{C}(\boldsymbol{X}), \boldsymbol{X}) = 2\frac{\partial \Psi}{\partial \boldsymbol{C}} \tag{2}$$

Fourth-order elasticity tensor $\boldsymbol{\mathcal{C}}$ then can be obtained as:

$$\boldsymbol{\mathcal{C}} = \frac{4\partial^2 \Psi}{\partial \boldsymbol{C} \partial \boldsymbol{C}} \tag{3}$$

Mostly used stress tensor, Cauchy stress tensor $\boldsymbol{\sigma}$, can be expressed in terms of second Piola-Kirchhoff's:

$$\boldsymbol{\sigma} = J^{-1} \boldsymbol{F} \boldsymbol{S} \boldsymbol{F}^{\boldsymbol{T}} \tag{4}$$

where $\boldsymbol{F}$ is deformation gradient tensor and J is Jacobian, which represents determinant of $\boldsymbol{F}$. [1]

B. Hyperelastic material models

1) St. Vennant-Kirchoff model: One of the simplest examples of hyperelastic model is St. Venant-Kirchhoff's model, whose elastic potential is given by equation:

$$\Psi(\boldsymbol{E}) = \frac{1}{2}\lambda(tr\boldsymbol{E})^2 + \mu\boldsymbol{E} : \boldsymbol{E} \tag{5}$$

where λ and μ are Lame's first and second constants, respectively.
This type of material is only relevant for rubber like materials with small range of strain.

2) Mooney-Rivlin model: One of the first introduced models. Elastic potential Ψ is expressed in terms of invariants of tensor $\boldsymbol{C}$:

$$\Psi(\boldsymbol{C}) = \sum_{r,s\geq 0} \mu_{rs}(I_{\boldsymbol{C}} - 3)^r(II^*_{\boldsymbol{C}} - 3)^s \tag{6}$$

where $II^*_{\boldsymbol{C}}$ is second invariant defined as:

$$II^*_{\boldsymbol{C}} = \frac{1}{2}(I^2_{\boldsymbol{C}} - II_{\boldsymbol{C}}); \quad II_{\boldsymbol{C}} = \boldsymbol{C} : \boldsymbol{C} \tag{7}$$

Number of sums from eq. 6 defines the parameter order of Mooney-Rivlin's model. In some other literature, constants μ_{rs} are expressed by linear dependant constants C_{pq}.
Elastic response of incompressible rubber-like materials are often modeled based on the Mooney–Rivlin model. The two parameter Mooney–Rivlin model is usually valid for strains less than 100 %, and higher parameter order Mooney-Rivlin can be used for higher values of strain. [7]

3) Compressible neo-Hook model: This hyperelastic material model will be used to model truss and beam later in paper, with version of neo-Hook defined in principal directions. Elastic potential is given by equation:

$$\Psi(\boldsymbol{C}) = \frac{\mu}{2}(I_{\boldsymbol{C}} - 3) - \mu\, lnJ + \frac{\lambda}{2}(lnJ)^2 \tag{8}$$

Cauchy stress tensor is defined in terms of left Cauchy-Green's deformation tensor $\boldsymbol{b}$:

$$\boldsymbol{\sigma} = \frac{\mu}{J}(\boldsymbol{b} - \boldsymbol{I}) + \frac{\lambda}{J}(lnJ)\boldsymbol{I} \tag{9}$$

Euler's elastic tensor $\mathcal{c}$ can now be defined:

$$\mathcal{c}_{ijkl} = \lambda'\delta_{ij}\delta_{kl} + \lambda'(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{10}$$

where effective constants λ' and μ' are expressed in terms of J, the determinant of deformation gradient tensor:

$$\lambda' = \frac{\lambda}{J}; \quad \mu' = \frac{\mu - \lambda lnJ}{J} \tag{11}$$

This model is used for modelling materials with similar characteristics like materials used in linear analysis, but with higher values of strain.
For three-dimensional neo-Hook in principal directions, which will be used for modelling three-dimenional truss and beam, Cauchy stress tensor in principal directions is given by equation

$$\boldsymbol{\sigma}_{\alpha\alpha} = \frac{2\mu}{J}ln\,\lambda_\alpha + \frac{\lambda}{J}ln\,J \tag{12}$$

where λ_α defines stretch ratio in principal directions. As for the two-dimensional truss and beam, expression for Cauchy's stress in principal directions is the same with exception to second Lame's constant, which is defined as $\overline{\lambda}$

$$\gamma = \frac{2\mu}{\lambda + 2\mu} \qquad \overline{\lambda} = \gamma\lambda \tag{13}$$

and Jacobian $J = dv/dV$, which is in this case expressed with $j = da/dA$ by relation

$$J = j^\gamma \tag{14}$$

Cauchy stress tensor in principal directions for two-dimensional plane stress is

$$\boldsymbol{\sigma}_{\alpha\alpha} = \frac{2\mu}{J}ln\,\lambda_\alpha + \frac{\overline{\lambda}}{J}ln\,j \tag{15}$$

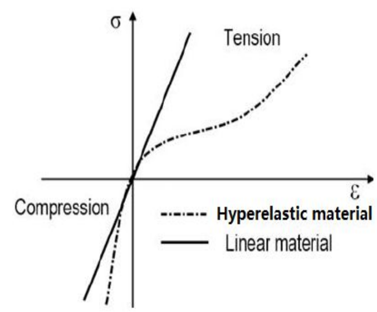

Fig. 1: Main difference between linear material and hyperelastic material behaviour under uniaxial stress test [2]

Fig. 1 shows the results of uniaxial tensile and compressive test of hyperelastic material and linear material and main differences between their behaviour under such loads.

III. FINITE ELEMENT FORMULATION

Finite element formulation is based on virtual work (power) equation, which is given in spatial configuration [1], [5]

$$\delta W(\phi, \delta\boldsymbol{v}) = \int_v \boldsymbol{\sigma} : \delta\boldsymbol{d}\,dv - \int_v \boldsymbol{f}\cdot\delta\boldsymbol{v}\,dv + \int_{\partial v} \boldsymbol{t}\cdot\delta\boldsymbol{v}\,da \tag{16}$$

Virtual work equation can be divided in two components, internal and external virtual work. Discretization of virtual work equation yields equations for internal and external equivalent nodal forces, where expression for internal equivalent nodal forces $\boldsymbol{T}^e_a$ is

$$\boldsymbol{T}^e_a = \int_{v^e} \boldsymbol{\sigma}\nabla N_a\,dv \tag{17}$$

and for the external $\boldsymbol{F}^e_a$ nodal forces

$$\boldsymbol{F}^e_a = \int_{v^e} N_a\boldsymbol{f}\,dv + \int_{\partial v^e} N_a\boldsymbol{t}\,da \tag{18}$$

Eq. 16 represents set of nonlinear equations, where is the current nodal position unknown. Goal is to solve the equations by applying Newton-Raphson iterative procedure, so linearization in the direction of displacement $\boldsymbol{u}$ is needed. Applying linearization and dividing virtual work equation in two parts, internal and external gives

$$D\delta W(\phi, \delta \boldsymbol{v})[\boldsymbol{u}] = D\delta W_{int}(\phi, \delta \boldsymbol{v})[\boldsymbol{u}] - D\delta W_{ext}(\phi, \delta \boldsymbol{v})[\boldsymbol{u}] \tag{19}$$

Linearization of internal part of virtual work equation gives two components of equation, constitutive and initial stress component

$$\begin{aligned} D\delta W_{int}(\phi, \delta \boldsymbol{v})[\boldsymbol{u}] &= D\delta W_{c}(\phi, \delta \boldsymbol{v})[\boldsymbol{u}] + D\delta W_{\sigma}(\phi, \delta \boldsymbol{v})[\boldsymbol{u}] \\ &= \int_{v} \delta \boldsymbol{d} : \boldsymbol{c} : \varepsilon \, dv + \int_{v} \boldsymbol{\sigma} : [(\nabla \boldsymbol{u})^{T} \nabla \delta \boldsymbol{v}] \, dv \end{aligned} \tag{20}$$

Discretization of this equation gives two tangent stiffness matrix, constitutive stiffness matrix $\boldsymbol{K}_c$ which is mostly used in linear analysis and initial stress matrix $\boldsymbol{K}_\sigma$ which represents nonlinear part of total tangent stiffness matrix $\boldsymbol{K}$. In element e, for components of stiffness matrix linking nodes a and b, expressions for constitutive and initial stress matrix are given as, respectively

$$[\boldsymbol{K}^{e}_{c,ab}]_{ij} = \int_{v^e} \sum_{k,l=1}^{3} \frac{\partial N_a}{\partial x_k} \mathcal{c}_{ikjl} \frac{\partial N_b}{\partial x_l} \, dv; \; i,j = 1,2,3. \tag{21}$$

$$[\boldsymbol{K}^{e}_{\sigma,ab}]_{ij} = \int_{v^e} \sum_{k,l=1}^{3} \frac{\partial N_a}{\partial x_k} \sigma_{kl} \frac{\partial N_b}{\partial x_l} \delta_{ij} dv; \; i,j = 1,2,3. \tag{22}$$

Total stiffnes matrix per element e, linking nodes a and b, $\boldsymbol{K}^{e}_{ab}$ is equal to sum of constitutive and initial stress stiffnes matrix

$$\boldsymbol{K}^{e}_{ab} = \boldsymbol{K}^{e}_{c,ab} + \boldsymbol{K}^{e}_{\sigma,ab} \tag{23}$$

Total tangent stiffness matrix $\boldsymbol{K}$ is defined by assembling nodal components of each $\boldsymbol{K}^{e}_{ab}$.

IV. TRUSS AND BEAM CONFIGURATION

Truss and beam are defined in more dimensional configurations. Material model used to describe examples is neo-Hook in principal directions [1]. *FLagShyP* softvare [4] is used to solve problems, which is a Matlab script that contains neccessary functions used to describe geometry, material and to solve static problem by applying Newton-Raphson procedure on discretizated equation $\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{F}$.

TABLE I: Truss and beam dimensions

Length (m)	1
Width (m)	0.1
Height (m)	0.1

Tab. 1 shows the dimensions of truss and beam model, which are the same for each example.

A. Truss configuration

Truss is defined by material from *FLagShyP* software library which enables truss to be defined in one-dimensional, two-dimensional and three-dimensional configuration, because it contains one-dimensional truss elements with regular two-dimensional and three-dimensional elements.

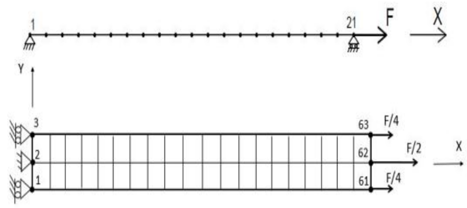

Fig. 2: Spatial discretization and boundary coniditons of one-dimensional and two-dimensional truss

Fig. 2 shows the discretization and boundary conditions of truss in one-dimensional and two-dimensional configuration. Discretization of one-dimensional truss contains 20 truss elements and 21 nodes, while two-dimensional truss contains 40 quadrilateral elements and 63 nodes.

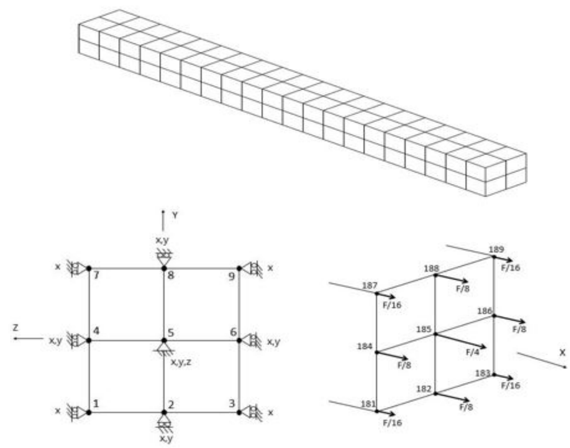

Fig. 3: Spatial discretization and boundary coniditons of three-dimensional truss

Fig. 3 shows the three-dimensional configuration of truss with discretization and boundary conditions. Discretization consists of 80 elements and 189 nodes, where coordinate axis on lower figure denotes prescribed axis with no displacement for each node. Also lower right figure denotes the loads in nodes on front face of the truss.

B. Beam configuration

Beam is defined in two-dimensional and three-dimensional configuration, because *FLagShyP* library doesn't contatin one-dimensional beam elements. Discretization was retained as in truss configuration with exception to boundary conditions.

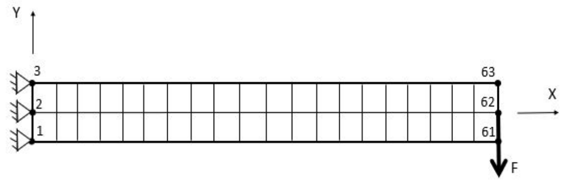

Fig. 4: Spatial discretization and boundary conditions of two-dimensional beam

Fig. 4 shows discretizatioan (which is the same as in two-dimensional truss example) and boundary conditions of two-dimensional beam.

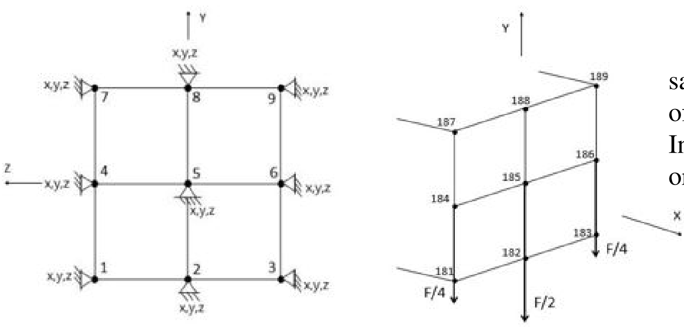

Fig. 5: Boundary conditions of three-dimensional beam

Fig. 5 shows the boundary conditions for three-dimenional beam, where left figure shows the prescribed zero displacement in nodes for back face of the beam and right figure shows the loads in nodes of the front face of the beam. Discretization is identical to the discretization of three-dimensional truss.
Dimensional configurations for each example are defined to be equivalent to each other, seperately.

V. TRUSS AND BEAM SOLUTION

Examples are solved using *FLagShyP* software library in MatLab. Loads are being incremented and results are shown in $F - \lambda$ graph, where λ defines stretch ratio of current and initial length. Referent value for deformation is displacement of a node in the middle of beam and truss face where the force is applied, in the direction of applied force.

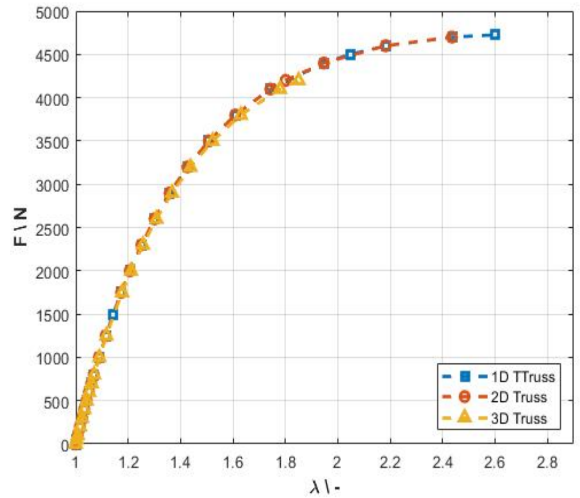

Fig. 6: Results of trusses under axial load

Fig. 6 shows the results of tensile loaded trusses with the same dependance between all three dimensional configurations of truss. Load-deformation dependance is obviously nonlinear. Interesting thing to notice is that the truss of lower dimensional order have higher maximum values of load and deformation.

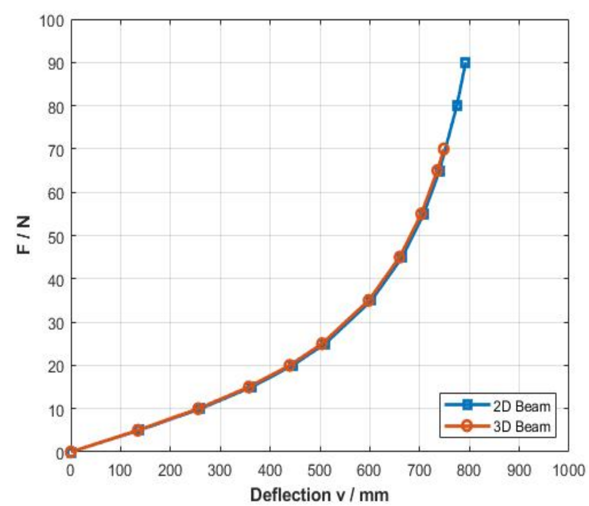

Fig. 7: Results of beam under bending load

Results in Fig. 7 show the deflections of two-dimensional and three-dimensional beam under bending force where results again show nonlinear dependance between load and deformation. Initially, for the first increments of load the $F-\lambda$ relation is approximately linear. Then for the values of $F > 20$ N, relation starts to develop highly nonlineary. The same pattern as in truss example appears, where two-dimensional beam has higher value of maximum load and deformation, in regarding to three-dimensional beam.
These maximum values of deformation and load can represent fracture of beam or truss in realistic conditions.

The higher dimensional order truss or beam has more complex finite elements and geometry, therefore it cointains more information and data about model which includes more equations to be solved in solver. Because of these facts, *FlagSHyP* solver "bursts" under relatively lower loads. These facts could explain why higher dimensional order examples achive lower maximum values of load and deformation.

VI. CONCLUSION

In this paper, basic concepts about hyperelastic material and model were explained with nonlinear finite element formulation. Examples of truss and beam were solved in one-dimensional, two-dimensional and three-dimensional configuration without one-dimensional beam. Each configuration was defined with the same material and dimensions making the examples equivalent to each other, regarding the truss or beam example. Load-deformation results showed expected nonlinear relation because of the nature of hyperelastic material. Individually, example of truss or beam didn't show same results for every dimensional order because of complexity of higher dimension examples. These examples were too robust for *FLagSHyP* solver and therefore solver couldn't reach same maximum values for each dimensional problem.

REFERENCES

[1] J. Bonet, A.J. Gil, R.D. Wood: „Nonlinear Solid Mechanics for Finite Element Analysis: Statics", Cambridge University Press, University of Cambridge, UK, 2016.

[2] WELSIM, Finite Element Analysis Software: „Hyperelastic material model Arruda-Boyce for Nonlinear Finite Element Analysis", URL:https://welsim.com/blogs/post/Pfsrf2CXMcMs3xC9i (accesed on: 01.09.2021).

[3] Wikipedia: „Hyperelastic material", URL:https://en.wikipedia.org/wiki/Hyperelastic_material (accesed on: 01.09.2021)..

[4] FLagSHyP software: „Nonlinear Solid Mechanics for Finite Element Analysys: Statics - FLaGSHyP Software",URL: http://www.flagshyp.com/ (accesed on: 01.09.2021).

[5] Ž. Lozina: „Finite Element Method", Class materials, Faculty of Electrical Engineering, Mechanical Engineering and Naval Architecture, University of Split.

[6] FunctionBay, Tech Support: „What is the difference between a hyperelastic material and an elastic material in RecurDyn?", URL:https://support.functionbay.com/en/faq/single/87/difference-hyperelastic-material-elastic-material-recurdyn (accesed on: 01.09.2021).

[7] Handwiki: „Physics:Mooney–Rivlin solid", URL:https://handwiki.org/wiki/Physics:Mooney%E2%80%93Rivlin_solid (accesed on: 01.09.2021).