Image reconstruction by the weighted median

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Abstract

This paper considers an image denoising by the weighted medina, where the impulse noise, or "salt and pepper" noise is taken into the consideration. Reconstruction of a noisy image is conducted by the weighted median where the central weight of the weighted window is taken into the consideration. The MAE (Mean Absolute Error) and MSE (Mean Squared Error) measurements are considered for determination of the central weight, where the experimental results are conducted for various noise ratio.

Keywords: weighted medina, image denoising, optimization.

1. INTRODUCTION

In this research the weighted median of data is considered in order to reconstruct noisy image. For that purpose a certain component of the weighted vector is taken into the consideration [1], which corresponds to the corresponding central weight of the weighted window.

In the image processing area denoising of an image present a process in which a noisy image is reconstructed via different measurements, where commonly used measurements are MAE and MSE. For that purpose a different types of methods, i.e. filters, are constantly used and developed [2], where filters based on the weighted median is commonly used [3].

For that purpose the experimental results are conducted on the central weight in order to optimize reconstruction of an image via MAE and MSE, where the well known impulse noise, or so called "salt and pepper" noise is observed.

2. THE WEIGHTED MEDIAN

The weighted median of data has great application in different branches of statistics, applied mathematics, and image processing [1], [3], [4]. So, in the next theorem is directly presented the determination of the weighted median $\operatorname{med}(\omega, z)$ for the given input vector $z = (z_1, \dots, z_n) \in \mathbb{R}^n$, and its corresponding vector of weights $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}_+^n$.

Theorem (The Weighted Median). *Let denote the input vector as* $z = (z_1, \dots, z_n) \in \mathbb{R}^n$, *and its corresponding weighted vector* $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}_+^n$, *and let denote set*

$$T = \left\{ t : 2\sum_{i=1}^{t-1} \omega_{(i)} \le \sum_{i=1}^{n} \omega_{(i)} \right\}, \quad t \in \{1, \dots, n\}, \qquad (1)$$

where

$$z_{(1)} \le \cdots \le z_{(n)}, \qquad (2)$$

presents sorted elements of input vector. Then it holds:

i.) $2\sum_{i=1}^{m-1} \omega_{(i)} < \sum_{i=1}^{n} \omega_{(i)} \Rightarrow \operatorname{med}(\omega, z) = z_{(m)};$

ii.) $2\sum_{i=1}^{m-1} \omega_{(i)} = \sum_{i=1}^{n} \omega_{(i)} \Rightarrow \operatorname{med}(\omega, z) = (1-\alpha) z_{(m-1)} + \alpha z_{(m)};$

where $m = \max T$, *and* $\alpha \in [0,1]$.

3. IMAGE DENOISING

In this section the image denoising is presented, where filter is used to reconstruct noisy image which is based on the weighted median.

3.1. Image presentation

In image processing area image of $N \times M$ dimensions can be presented in matrix form as

$$Y = \begin{bmatrix} y_{1,1} & \cdots & y_{1,M} \\ \vdots & \ddots & \vdots \\ y_{N,1} & \cdots & y_{N,M} \end{bmatrix}, \qquad (3)$$

where each matrix element $y_{i,j}$ presents image pixel which denotes colour intensity of image. Colour intensity can be presented in different colours types and scales. For that purpose the grayscale level of image colour is observed, which present an image in monochromatic shades of grey in integer scale, i.e. $y_{i,j} \in [0, 255]$. Next figure present the image of an astronaut in grayscale level of a dimensions 100×100. In figure the part of the image is extracted were pixel values are denoted to an each pixel in range of 0 to 255 (0 denotes black, while 255 denotes white colour).

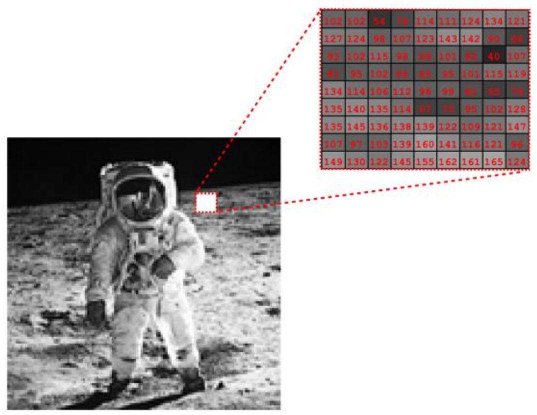


Fig. 1. Image presentation

3.2. Noise model

In image processing a noises are generated by different mathematical models. One of the most known “salt-and-pepper” noise is modeled as

$$x_{i,j} = \begin{cases} \xi_{i,j}, & \text{with probability } \rho; \\ y_{i,j}, & \text{with probability } 1-\rho; \end{cases} \qquad (4)$$

where $\rho \in [0,1]$ present noise ratio, and $\xi_{i,j}$ presents random variable with probability density function which is define as

$$P(t) = \begin{cases} P_p, & t = p; \\ P_s, & t = s; \\ 0, & \text{otherwise}; \end{cases} \qquad (5)$$

where $P_p, P_s \geq 0$, $P_p + P_s = 1$ (we observe when $P_p, P_s = 0.5$), presents occurrence probabilities of value p, and s. For an impulse noise, i.e. “salt-and-pepper” noise, it is usual to observe a values which reach maximum and minimum value of observed scale, i.e. $p = 0$ and $s = 255$. In that way a noisy image is generated which is denoted by

$$X = \begin{bmatrix} x_{1,1} & \cdots & x_{1,M} \\ \vdots & \ddots & \vdots \\ x_{N,1} & \cdots & x_{N,M} \end{bmatrix}. \qquad (6)$$

3.3. Filtering scheme

Reconstruction of a noisy image X is constructed to process each pixel $x_{i,j}$, and thus reconstructed image is created which is denoted by

$$X^* = \begin{bmatrix} x^*_{1,1} & \cdots & x^*_{1,M} \\ \vdots & \ddots & \vdots \\ x^*_{N,1} & \cdots & x^*_{N,M} \end{bmatrix}. \qquad (7)$$

Filter process each pixel $x_{i,j}$ in such a way that it considered all neighbourhood of $x_{i,j}$ into the process. An observed neighbourhood of $x_{i,j}$ is centered around the observed pixel, which is called the filtering window and is denoted as

$$X_{i,j}=\begin{bmatrix} x_{i-C,j-C} & \cdots & x_{i-C,j} & \cdots & x_{i-C,j+C} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ x_{i,j-C} & \cdots & x_{i,j} & \cdots & x_{i,j+C} \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ x_{i+C,j-C} & \cdots & x_{i+C,j} & \cdots & x_{i+C,j+C} \end{bmatrix}. \quad (8)$$

A dimension of the filtering window is $D\times D$, $D=2C+1$, $C\in\mathbb{N}$, and it slides trough all image in order to reconstruct noisy image. In the image processing area filters are often modelled in order to use the weighted median into a process of reconstruction of a noisy image. For that purpose the weighted window W of a dimensions $D\times D$ is also constructed, i.e.

$$W=\begin{bmatrix} w_{1,1} & \cdots & w_{1,C+1} & \cdots & w_{1,D} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ w_{C+1,1} & \cdots & w_{C+1,C+1} & \cdots & w_{C+1,D} \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ w_{D,1} & \cdots & w_{D,C+1} & \cdots & w_{D,D} \end{bmatrix}, \quad (9)$$

which correspond to the filtering window $X_{i,j}$. Each element of the weighted window W corresponds to an element in the filtering window $X_{i,j}$, which is settled on the same position. In order to apply the weighted median to the filtering window $X_{i,j}$ with corresponding weighted window W, the mapping which transforms matrix form into a vector is constructed. So, let us define the mapping

$$u(k)=\sum_{l=1}^{n}\left(\left\lfloor\frac{l-1}{D}\right\rfloor-C\right)\cdot\chi_{\{l\}}(k), \quad (10)$$

$$v(k)=\sum_{l=1}^{n}\left(l-1-C-D\left\lfloor\frac{l-1}{D}\right\rfloor\right)\cdot\chi_{\{l\}}(k), \quad (11)$$

where in this situation follows that $n=D^2$, and

$$\chi_A(x)=\begin{cases}1, & x\in A;\\ 0, & x\notin A;\end{cases} \quad (12)$$

denotes the indicator function, where $A\subseteq\mathbb{R}$. In that way, to an each element z_k, $k\in\{1,\ldots,n\}$, of input vector $z=(z_1,\ldots,z_n)\in\mathbb{R}^n$, and to an each element ω_k, $k\in\{1,\ldots,n\}$, of the weighted vector $\omega=(\omega_1,\ldots,\omega_n)\in\mathbb{R}_+^n$, a corresponding element of the filtering window $X_{i,j}$, and the weighted window W, are mapped, i.e.

$$z_k=x_{i+u(k),j+v(k)},\quad \omega_k=w_{C+1+u(k),C+1+v(k)}. \quad (13)$$

In this situation, a reconstructed pixel $x^*_{i,j}$ is obtained as an output of the weighted median which process the filtering window $X_{i,j}$ with the corresponding weighted window W, what can be written as

$$x^*_{i,j}=\mathrm{med}(\omega,z). \quad (14)$$

3.4. Reconstruction measurement

Quality of reconstructed image are most commonly measured by MAE (Mean Absolute Error) and MSE (Mean Squared Error), which are defined as

$$\mathrm{MAE}=\frac{\sum_{i=1}^{N}\sum_{j=1}^{M}\left|y_{ij}-x^*_{ij}\right|}{N\cdot M}, \quad (15)$$

$$\mathrm{MSE}=\frac{\sum_{i=1}^{N}\sum_{j=1}^{M}\left(y_{ij}-x^*_{ij}\right)^2}{N\cdot M}. \quad (16)$$

MAE measurement are commonly used for insight of detail and edges reconstruction. This property are inherited from robustness on outliers. Otherwise, MSE measurement are used as an information for an impulse noise removal. This is because MSE possesses sensibility to an outliers [4].

4. EXPERIMENTAL RESULTS

In this section an experimental research is conducted in order to optimize reconstruction measurements, i.e. MAE, and MSE. This task yield to a reconstruction of a noisy image which is affected by an impulse noise that is generated by model (4). The optimization problem is set in such a way that a central weight $\omega_{(n+1)/2}=w_{C+1,C+1}$ is observed as a unknown parameter, while other weights are set to one, i.e. $\omega_k = 1$, $k \neq (n+1)/2$. In this way, a regulation to a filtering process is managed, and thus invariability to an unnecessary filtering process of an observed pixel can be provided. This process yields to a preservation of a fine details and edges of an image, and a prevention to an image blurring. In the Fig. 2 and Fig. 3 the experimental results are presented which are conducted on the central weight $\omega_{(n+1)/2}$ respect to the different noise ratios ρ. The research is conducted on discrete set for the central weight, i.e. $\omega_{(n+1)/2} \in \{1,\ldots,n\}$, where in [1] it is shown that on this discrete set MAE, and MSE reaches its global optimum.

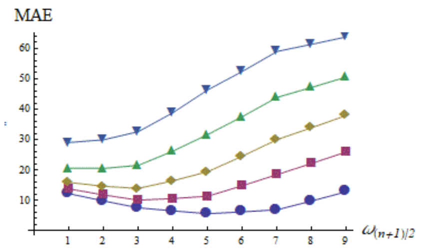


Fig. 2. MAE

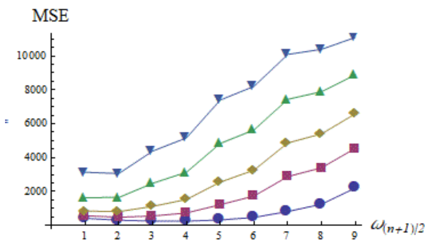
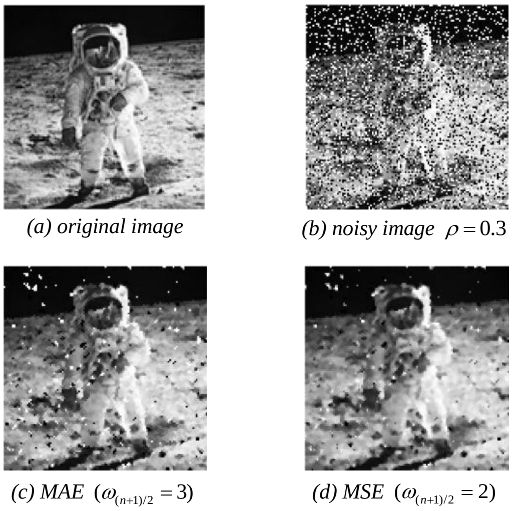


Fig. 3. MSE

In Fig. 4(a) experimental image is presented, while in Fig. 4(b) the noisy image with the presence of the “salt-and-pepper” noise with the noise ratio $\rho = 0.3$ is shown. In Fig. 4(c) and Fig. 4(d) results of filtering with the proposed method is presented, where optimal center weight is implemented. The results which are presented in Fig. 2, shows that for noise ratio $\rho = 0.3$ the optimal center weight, with the respect to MAE, is reached at $\omega_{(n+1)/2} = 3$, while the results which are presented in Fig 3. shows that the optimal center weight with the respect to MSE, is reached at $\omega_{(n+1)/2} = 2$.



(a) original image *(b) noisy image $\rho = 0.3$*

(c) MAE $(\omega_{(n+1)/2} = 3)$ *(d) MSE $(\omega_{(n+1)/2} = 2)$*

Fig. 4. Filtering results

5. CONCLUSION

In this research a reconstruction of a noisy image is conducted by the filter which is based on the weighted median. The problem is set as the optimization problem which considers MAE and MSE, in order to determined central weight of the weighted window.

6. ACKNOWLEDGEMENTS

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