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ON THE LUBRICATION OF A ROTATING SHAFT WITH INCOMPRESSIBLE MICROPOLAR FLUID

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Abstract

In this work, we investigate the lubrication process of a slipper bearing. The slipper bearing consists of two coaxial cylinders in relative motion, where an incompressible micropolar fluid (lubricant) is injected in a thin gap between them. We compute the asymptotic approximation of the solution to the governing micropolar system as a power series in terms of the small parameter ε representing the thickness of the shaft. The proposed approximation is given in the explicit form, allowing us to clearly observe the effects of the micropolar nature of the fluid.

Keywords: lubrication process, rotating shaft, micropolar fluid, asymptotic approximation.

1 Introduction

The model of micropolar fluids was introduced in 60's by Eringen in his famous paper [1] and provides a generalization of the classical Navier–Stokes model in the sense that it takes into account the microstructure of the fluid. This is accomplished by introducing the microrotation field (angular velocity field of rotation) describing effects such as rotation and shrinking and, accordingly, a new vector equation coming from the conservation of angular momentum. In this way, we obtain a coupled system of PDEs successfully describing various non–Newtonian fluids, including liquid crystals, animal blood, muddy fluids, certain polymeric fluids and even water in small scales. Thus, there exists a vast number of recent results concerning the engineering applications, primarily in biomedicine (see e.g. [2], [3], [4]) as well as rigorous results (see e.g. [5], [6], [7], [8]) providing various effective models for micropolar fluid flows. A comprehensive survey of the mathematical theory underlying the micropolar fluid model can be found in [9].

The study of lubrication problems can be traced back to the celebrating work of Reynolds [10] in 1886, where the author studied the thin film flow without giving a relation between his model and the Navier–Stokes equations. A formal relation between the Navier–Stokes equations in a thin domain and the Reynolds equation has been provided by Capriz [11], Elrod [12] and Wannier [13] via asymptotic analysis. Rigorous justification of the Reynolds equation for a flow between two plain surfaces has been provided by Bayada and Chambat [14] and Cimatti [15], whereas the study of the asymptotic behavior of the viscous flow in an infinite thin layer between two fixed, plain surfaces is due to Nazarov [16]. Motivated by the engineering applications, the investigation of the lubrication process of a rotating shaft with classical, Newtonian fluid has been carried out by Duvnjak and Marušić–Paloka in [17] and [18].

The aim of the present work is to generalize the results from [17], [18] to a case when lubricant is assumed to be an incompressible micropolar fluid. In view of that, we study the lubrication process of a slipper bearing which consists of two circular surfaces in relative motion. One belongs to the shaft which is rotating with some constant angular velocity ω, while the other is a lubricated surface of support. The shaft is of radius R and height l. Between the shaft and the support there is a thin domain $\mathcal{C}_\varepsilon$ of thickness $\varepsilon \ll l$ completely filled with an incompressible micropolar fluid.

Starting from the system of micropolar equations posed in the thin domain $\mathcal{C}_\varepsilon$, we first rewrite the problem in cylindrical coordinates. We then construct an asymptotic approximation of the solution as a power series in terms of the small parameter $\varepsilon > 0$ representing the film thickness. In this way, a higher–order approximation is proposed, taking into account the microstructure of the fluid. Since the approximation is given in the explicit form, we believe the presented result will contribute to the engineering practice, namely to optimal design of the lubrication devices consisting of slipper bearings, appearing in industrial machinery with a large horse power having high loads and speeds including steam turbines, pumps, compressors and motors (see [19]).

2 Problem Settings

We describe the geometry of the thin film using cylindrical coordinates (r, φ, z). We denote by $\Xi\colon \mathbb{R}^3 \to \mathbb{R}^3$ the change of variables $\Xi(x_1, x_2, x_3) = (r, \varphi, z)$ and assume that the film thickness is $\varepsilon h(\varphi)$, where $\varepsilon > 0$ is a small positive parameter. The flow domain is given by

$$\mathcal{C}_\varepsilon = \{\Sigma^{-1}(r, \varphi, z) \in \mathbb{R}^3 \colon \varphi \in \langle 0, 2\pi \rangle,\ z \in \langle 0, l \rangle,\ r \in \langle R, R + \varepsilon h(\varphi) \rangle\},$$

where the function $h\colon \langle 0, 2\pi \rangle \to \langle 0, \infty \rangle$ is of class C^2, 2π–periodic and bounded in the sense $0 < \beta_1 \le h(\varphi) \le \beta_2$, for $\varphi \in \langle 0, 2\pi \rangle$ (see Figure 1).

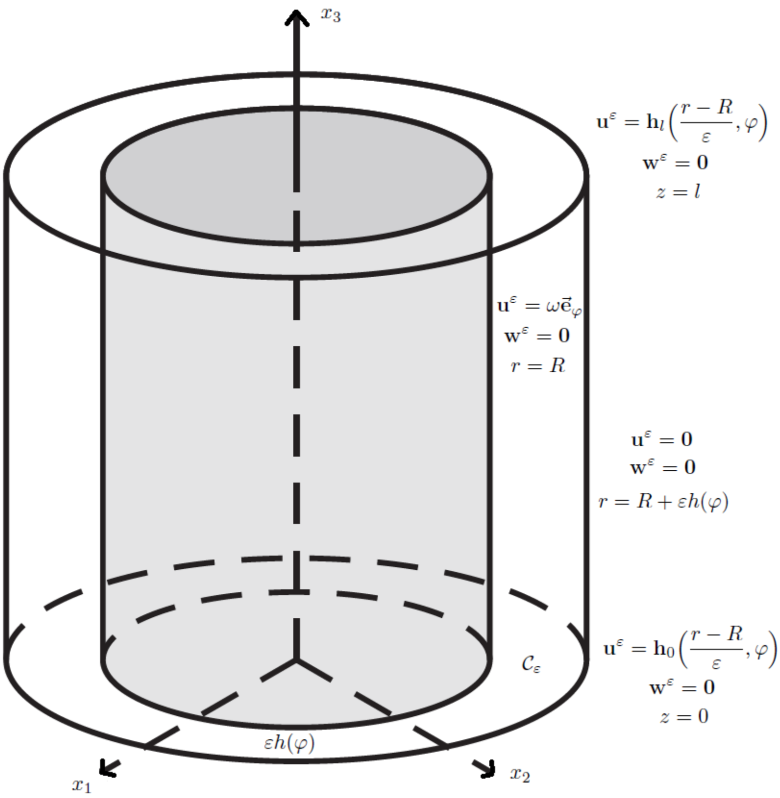

Figure 1: The considered domain $\mathcal{C}_\varepsilon$.

Assuming a low Reynolds number regime, the flow in $\mathcal{C}_\varepsilon$ is assumed to be governed by the following system of micropolar equations [9]:

$$\begin{aligned} -\mu\Delta\mathbf{u}^\varepsilon + \nabla p^\varepsilon &= a\operatorname{rot}\mathbf{w}^\varepsilon + \mathbf{f}^\varepsilon, \\ \operatorname{div}\mathbf{u}^\varepsilon &= 0 \quad \text{in } \mathcal{C}_\varepsilon, \\ -\alpha\Delta\mathbf{w}^\varepsilon - \beta\nabla\operatorname{div}\mathbf{w}^\varepsilon + 2a\mathbf{w}^\varepsilon &= a\operatorname{rot}\mathbf{u}^\varepsilon + \mathbf{g}^\varepsilon. \end{aligned} \tag{1}$$

In view of the application we want to model, we prescribe the following boundary conditions:

$$\begin{aligned} &\mathbf{u}^\varepsilon = \mathbf{0} \quad \text{for } r = R + \varepsilon h(\varphi), \\ &\mathbf{u}^\varepsilon = \omega\vec{\mathbf{e}}_\varphi \quad \text{for } r = R, \\ &\mathbf{u}^\varepsilon = \mathbf{h}_0\Big(\frac{r-R}{\varepsilon}, \varphi\Big) \quad \text{for } z = 0, \\ &\mathbf{u}^\varepsilon = \mathbf{h}_l\Big(\frac{r-R}{\varepsilon}, \varphi\Big) \quad \text{for } z = l, \\ &\mathbf{w}^\varepsilon = \mathbf{0} \quad \text{for } \partial C_\varepsilon, \end{aligned} \tag{2}$$

where $\mathbf{u}^\varepsilon$, p^ε and $\mathbf{w}^\varepsilon$ represent the velocity, pressure and microrotation, respectively. The given constants are defined by $\mu = \nu + \nu_r$, $\alpha = c_a + c_d$, $\beta = c_0 - c_a + c_d$ and $a = 2\nu_r$, where ν is the Newtonian viscosity, ν_r is the microroration viscosity, while c_0, c_a and c_d are the coefficients of angular viscosities. The external sources of linear and angular momentum are denoted by the functions $\mathbf{f}^\varepsilon$ and $\mathbf{g}^\varepsilon$.

Furthermore, we assume that the functions $\mathbf{h}_\alpha \in C^2(\mathcal{S})$, $\alpha = 0, l$, where

$$\mathcal{S} = \{(\rho, \varphi)\colon\ \rho \in \langle 0, h(\varphi)\rangle,\ \varphi \in \langle 0, 2\pi\rangle\},$$

are 2π–periodic with respect to φ and satisfy the following relations:

$$\begin{gathered}
\mathbf{h}_\alpha(h(\varphi), \varphi) = \mathbf{0}, \quad \mathbf{h}_\alpha(0, \varphi) = \omega \vec{\mathbf{e}}_\varphi,\ \alpha = 0, l, \\
\int_0^{2\pi}\int_0^{h(\varphi)} \rho \vec{\mathbf{e}}_z \cdot \mathbf{h}_0(\rho, \varphi) d\rho d\varphi = \int_0^{2\pi}\int_0^{h(\varphi)} \rho \vec{\mathbf{e}}_z \cdot \mathbf{h}_l(\rho, \varphi) d\rho d\varphi, \\
\int_0^{2\pi}\int_0^{h(\varphi)} \vec{\mathbf{e}}_z \cdot \mathbf{h}_0(\rho, \varphi) d\rho d\varphi = \int_0^{2\pi}\int_0^{h(\varphi)} \vec{\mathbf{e}}_z \cdot \mathbf{h}_l(\rho, \varphi) d\rho d\varphi.
\end{gathered} \tag{3}$$

The well–posedness of the above problem can be established in a standard manner (see e.g. [9]). Our goal here is to investigate the asymptotic behavior of the problem (1)–(3), as $\varepsilon \to 0$.

3 Asymptotic Analysis

Taking into account the particular geometry of our domain, it is reasonable to rewrite write the original problem (1)–(3) in cylindrical coordinates. The micropolar equations in cylindrical coordinates takes the following form (see e.g. [20]):

$$\begin{gathered}
-\mu\Big(\Delta u_r^\varepsilon - \frac{u_r^\varepsilon}{r^2} - \frac{2}{r^2}\frac{\partial u_\varphi^\varepsilon}{\partial \varphi}\Big) + \frac{\partial p^\varepsilon}{\partial r} = a\Big(\frac{1}{r}\frac{\partial w_z^\varepsilon}{\partial \varphi} - \frac{\partial w_\varphi^\varepsilon}{\partial z}\Big) + f_r^\varepsilon, \\
-\mu\Big(\Delta u_\varphi^\varepsilon - \frac{u_\varphi^\varepsilon}{r^2} + \frac{2}{r^2}\frac{\partial u_r^\varepsilon}{\partial \varphi}\Big) + \frac{1}{r}\frac{\partial p^\varepsilon}{\partial \varphi} = a\Big(\frac{\partial w_r^\varepsilon}{\partial z} - \frac{\partial w_z^\varepsilon}{\partial r}\Big) + f_\varphi^\varepsilon, \\
-\mu\Delta u_z^\varepsilon + \frac{\partial p^\varepsilon}{\partial z} = a\Big(\frac{\partial w_\varphi^\varepsilon}{\partial r} + \frac{w_\varphi^\varepsilon}{r} - \frac{1}{r}\frac{\partial w_r^\varepsilon}{\partial \varphi}\Big) + f_z^\varepsilon, \\
\frac{\partial u_r^\varepsilon}{\partial r} + \frac{1}{r}u_r^\varepsilon + \frac{1}{r}\frac{\partial u_\varphi^\varepsilon}{\partial \varphi} + \frac{\partial u_z^\varepsilon}{\partial z} = 0,
\end{gathered} \tag{4}$$

$$\begin{aligned}
-\alpha\Big(\Delta w_r^\varepsilon - \frac{w_r^\varepsilon}{r^2} - \frac{2}{r^2}\frac{\partial w_\varphi^\varepsilon}{\partial \varphi}\Big) - \beta\Big(\frac{\partial^2 w_r^\varepsilon}{\partial r^2} - \frac{w_r^\varepsilon}{r^2} + \frac{1}{r}\frac{\partial w_r^\varepsilon}{\partial r} - \frac{1}{r^2}\frac{\partial w_\varphi^\varepsilon}{\partial \varphi} + \frac{1}{r}\frac{\partial^2 w_\varphi^\varepsilon}{\partial \varphi \partial r} + \frac{\partial^2 w_z^\varepsilon}{\partial z \partial r}\Big) + 2a w_r^\varepsilon \\
= a\Big(\frac{1}{r}\frac{\partial u_z^\varepsilon}{\partial \varphi} - \frac{\partial u_\varphi^\varepsilon}{\partial z}\Big) + g_r^\varepsilon, \\
-\alpha\Big(\Delta w_\varphi^\varepsilon - \frac{w_\varphi^\varepsilon}{r^2} + \frac{2}{r^2}\frac{\partial w_r^\varepsilon}{\partial \varphi}\Big) - \frac{\beta}{r}\Big(\frac{\partial^2 w_r^\varepsilon}{\partial \varphi \partial r} + \frac{1}{r}\frac{\partial w_r^\varepsilon}{\partial \varphi} + \frac{1}{r}\frac{\partial^2 w_\varphi^\varepsilon}{\partial \varphi^2} + \frac{\partial^2 w_z^\varepsilon}{\partial \varphi \partial z}\Big) + 2a w_\varphi^\varepsilon \\
= a\Big(\frac{\partial u_r^\varepsilon}{\partial z} - \frac{\partial u_z^\varepsilon}{\partial r}\Big) + g_\varphi^\varepsilon, \\
-\alpha\Delta w_z^\varepsilon - \beta\Big(\frac{\partial^2 w_r^\varepsilon}{\partial z \partial r} + \frac{1}{r}\frac{\partial w_r^\varepsilon}{\partial z} + \frac{1}{r}\frac{\partial^2 w_\varphi^\varepsilon}{\partial z \partial \varphi} + \frac{\partial^2 w_z^\varepsilon}{\partial z^2}\Big) + 2a w_z^\varepsilon = \\
= a\Big(\frac{\partial u_\varphi^\varepsilon}{\partial r} + \frac{u_\varphi^\varepsilon}{r} - \frac{1}{r}\frac{\partial u_r^\varepsilon}{\partial \varphi}\Big) + g_z^\varepsilon,
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\mathbf{u}^\varepsilon &= u_r^\varepsilon \vec{\mathbf{e}}_r + u_\varphi^\varepsilon \vec{\mathbf{e}}_\varphi + u_z^\varepsilon \vec{\mathbf{e}}_z, \\
\mathbf{w}^\varepsilon &= w_r^\varepsilon \vec{\mathbf{e}}_r + w_\varphi^\varepsilon \vec{\mathbf{e}}_\varphi + w_z^\varepsilon \vec{\mathbf{e}}_z,
\end{aligned}$$

and

$$\Delta s^\varepsilon = \frac{\partial^2 s^\varepsilon}{\partial r^2} + \frac{1}{r}\frac{\partial s^\varepsilon}{\partial r} + \frac{1}{r^2}\frac{\partial^2 s^\varepsilon}{\partial \varphi^2} + \frac{\partial^2 s^\varepsilon}{\partial z^2},$$

for a scalar function s^ε.

3.1 Asymptotic Expansion

We seek the solution of the problem (1)–(3) in the following form:

$$\begin{aligned} \mathbf{u}^\varepsilon &\sim \mathbf{u}^0(\rho,\varphi,z) + \varepsilon\mathbf{u}^1(\rho,\varphi,z) + \dots, \\ \mathbf{w}^\varepsilon &\sim \mathbf{w}^0(\rho,\varphi,z) + \varepsilon\mathbf{w}^1(\rho,\varphi,z) + \dots, \\ p^\varepsilon &\sim \frac{1}{\varepsilon^2}p^0(\rho,\varphi,z) + \frac{1}{\varepsilon}p^1(\rho,\varphi,z) + \dots, \end{aligned} \tag{6}$$

where $\rho = \frac{r-R}{\varepsilon}$, while the external force functions are given by:

$$\begin{aligned} \mathbf{f}^\varepsilon(\varphi,z) &\sim \frac{1}{\varepsilon^2}\mathbf{f}^0(\varphi,z) + \frac{1}{\varepsilon}\mathbf{f}^1(\varphi,z) + \dots, \\ \mathbf{g}^\varepsilon(\varphi,z) &\sim \frac{1}{\varepsilon^2}\mathbf{g}^0(\varphi,z) + \frac{1}{\varepsilon}\mathbf{g}^1(\varphi,z) + \dots. \end{aligned}$$

Let us note that f_r^ε and g_r^ε can be neglected due to the small thickness of the domain in the radial direction. For the same reason, we can assume that the external force functions $\mathbf{f}^\varepsilon$ and $\mathbf{g}^\varepsilon$ are independent of ρ.

Plugging the asymptotic expansion (6) into the system of equations (4)–(5) and collecting terms by the same powers of ε, we obtain a recursive sequence of problems that can be solved explicitly.

3.2 Zero–Order Approximation

The zero–order approximation for the velocity and pressure $(\mathbf{u}^0, p^0)$ is the solution of the following system:

$$\begin{aligned} \frac{1}{\varepsilon^2}:&\quad -\mu\frac{\partial^2 u_r^0}{\partial\rho^2} + \frac{\partial p^1}{\partial\rho} = 0, \\ \frac{1}{\varepsilon^2}:&\quad -\mu\frac{\partial^2 u_\varphi^0}{\partial\rho^2} + \frac{1}{R}\frac{\partial p^0}{\partial\varphi} = f_\varphi^0, \\ \frac{1}{\varepsilon^2}:&\quad -\mu\frac{\partial^2 u_z^0}{\partial\rho^2} + \frac{\partial p^0}{\partial z} = f_z^0, \end{aligned} \tag{7}$$

with the boundary conditions

$$\begin{aligned} \varepsilon^0:&\ \ u_r^0(0,\varphi,z) = u_z^0(0,\varphi,z) = 0, \quad u_\varphi^0(0,\varphi,z) = \omega, \\ \varepsilon^0:&\ \ u_r^0(h,\varphi,z) = u_\varphi^0(h,\varphi,z) = u_z^0(h,\varphi,z) = 0. \end{aligned} \tag{8}$$

From the incompressibility equation (4.4) we obtain:

$$\begin{aligned} \frac{1}{\varepsilon}:&\ \ \frac{\partial u_r^0}{\partial\rho} = 0, \\ \varepsilon^0:&\ \ R\frac{\partial u_r^1}{\partial\rho} + u_r^0 + \frac{\partial u_\varphi^0}{\partial\varphi} + R\frac{\partial u_z^0}{\partial z} = 0. \end{aligned} \tag{9}$$

We conclude $u_r^0 = 0$, $p^0 = p^0(\varphi,z)$, $p^1 = p^1(\varphi,z)$. The solution of (7)–(8) is given in the following form:

$$\begin{aligned} u_\varphi^0 &= \frac{1}{2\mu}(\rho-h)\rho\Big(\frac{1}{R}\frac{\partial p^0}{\partial\varphi} - f_\varphi^0\Big) + \omega\Big(1-\frac{\rho}{h}\Big), \\ u_z^0 &= \frac{1}{2\mu}(\rho-h)\rho\Big(\frac{\partial p^0}{\partial z} - f_z^0\Big). \end{aligned} \tag{10}$$

From (9.2), we obtain the equation for the zero–order pressure approximation p^0:

$$R\frac{\partial u_r^1}{\partial\rho}+\frac{R}{2\mu}(\rho-h)\rho\Big(\frac{\partial^2 p^0}{\partial z^2}-\frac{\partial f_z^0}{\partial z}\Big)+\frac{1}{2\mu}\frac{\partial}{\partial\varphi}\Big((\rho-h)\rho\Big(\frac{1}{R}\frac{\partial p^0}{\partial\varphi}-f_\varphi^0\Big)\Big)=-\frac{h'}{h^2}\rho\omega. \tag{11}$$

Now, integrating the equation (11) with respect to ρ over $\langle 0,h(\varphi)\rangle$ and taking into account the boundary conditions $u_r^1(0,\varphi,z)=u_r^1(h,\varphi,z)=0$, we obtain the Reynolds equation for p^0:

$$\frac{Rh^3}{12}\Big(\frac{\partial^2 p^0}{\partial z^2}-\frac{\partial f_z^0}{\partial z}\Big)+\frac{1}{12}\frac{\partial}{\partial\varphi}\Big(h^3\Big(\frac{1}{R}\frac{\partial p^0}{\partial\varphi}-f_\varphi^0\Big)\Big)=\frac{h'\mu\omega}{2}. \tag{12}$$

We endow it with the boundary conditions

$$\begin{aligned}\frac{\partial p^0}{\partial z}&=\lambda_z(\varphi)\quad\text{for } z=0,l,\\ p^0 &\text{ is } 2\pi \text{ periodic in } \varphi,\end{aligned} \tag{13}$$

where $\lambda_z(\varphi)$ can be chosen arbitrarily. The problem (12)–(13) is well–posed (see e.g. [17], [18]).

The zero–order approximation for the microrotation $\mathbf{w}^0$ is given by:

$$\begin{aligned}&\frac{1}{\varepsilon^2}:\ -\alpha\frac{\partial^2 w_r^0}{\partial\rho^2}-\beta\frac{\partial^2 w_r^0}{\partial\rho^2}=0,\\ &\frac{1}{\varepsilon^2}:\ -\alpha\frac{\partial^2 w_\varphi^0}{\partial\rho^2}=g_\varphi^0,\\ &\frac{1}{\varepsilon^2}:\ -\alpha\frac{\partial^2 w_z^0}{\partial\rho^2}=g_z^0,\\ &\varepsilon^0:\ w_r^0(0,\varphi,z)=w_\varphi^0(0,\varphi,z)=w_z^0(0,\varphi,z)=0,\\ &\varepsilon^0:\ w_r^0(h,\varphi,z)=w_\varphi^0(h,\varphi,z)=w_z^0(h,\varphi,z)=0.\end{aligned} \tag{14}$$

The problem (14) can be solved by putting:

$$\begin{aligned}w_r^0&=0,\\ w_\varphi^0&=\frac{1}{2\alpha}(h-\rho)\rho g_\varphi^0,\\ w_z^0&=\frac{1}{2\alpha}(h-\rho)\rho g_z^0.\end{aligned} \tag{15}$$

As expected, the zero–order approximation for the velocity given by (10) does not feel the effects of the micropolar nature of the fluid. For this reason, we need to continue the computation and construct the correctors.

3.3 First–Order Corrector

The first–order corrector for the velocity and pressure $(\mathbf{u}^1,p^1)$ is the solution of the following system:

$$\begin{aligned}&\frac{1}{\varepsilon}:\ -\mu\frac{\partial^2 u_r^1}{\partial\rho^2}+\frac{\partial p^2}{\partial\rho}=0,\\ &\frac{1}{\varepsilon}:\ -\mu\frac{\partial^2 u_\varphi^1}{\partial\rho^2}-\frac{\mu}{R}\frac{\partial u_\varphi^0}{\partial\rho}+\frac{1}{R}\frac{\partial p^1}{\partial\varphi}-\frac{\rho}{R^2}\frac{\partial p^0}{\partial\varphi}=-a\frac{\partial w_z^0}{\partial\rho}+f_\varphi^1,\\ &\frac{1}{\varepsilon}:\ -\mu\frac{\partial^2 u_z^1}{\partial\rho^2}-\frac{\mu}{R}\frac{\partial u_z^0}{\partial\rho}+\frac{\partial p^1}{\partial z}=a\frac{\partial w_\varphi^0}{\partial\rho}+f_z^1,\\ &\varepsilon:\ u_r^1(0,\varphi,z)=u_\varphi^1(0,\varphi,z)=u_z^1(0,\varphi,z)=0,\\ &\varepsilon:\ u_r^1(h,\varphi,z)=u_\varphi^1(h,\varphi,z)=u_z^1(h,\varphi,z)=0.\end{aligned} \tag{16}$$

From the incompressibility equation (4.4) we deduce:

$$\begin{aligned}
&\varepsilon^0\colon\ R\frac{\partial u_r^1}{\partial\rho}+\frac{\partial u_\varphi^0}{\partial\varphi}+R\frac{\partial u_z^0}{\partial z}=0,\\
&\varepsilon\colon\ u_r^1+\rho\frac{\partial u_r^1}{\partial\rho}+\frac{\partial u_\varphi^1}{\partial\varphi}+R\frac{\partial u_z^1}{\partial z}+\rho\frac{\partial u_z^0}{\partial z}=0.
\end{aligned}\tag{17}$$

The solution of (16) is now given in the following form:

$$\begin{aligned}
u_r^1=&-\frac{\omega}{2R}\frac{h'}{h^2}\rho^2+\frac{h'}{4R\mu}\rho^2\Big(\frac{1}{R}\frac{\partial p^0}{\partial\varphi}-f_\varphi^0\Big)+\frac{1}{12\mu}\rho^2(3h-2\rho)\Big(\frac{1}{R^2}\frac{\partial^2p^0}{\partial\varphi^2}-\frac{1}{R}\frac{\partial f_\varphi^0}{\partial\varphi}+\frac{\partial^2p^0}{\partial z^2}-\frac{\partial f_z^0}{\partial z}\Big),\\
u_\varphi^1=&\frac{1}{2\mu R}\rho(\rho-h)\frac{\partial p^1}{\partial\varphi}+\frac{1}{12R^2\mu}\rho(h-\rho)(h+4\rho)\frac{\partial p^0}{\partial\varphi}+\frac{\omega}{2Rh}\rho(\rho-h)\\
&+\frac{1}{12\mu R}\rho(\rho-h)(2\rho-h)f_\varphi^0-\frac{1}{2\mu}\rho(\rho-h)f_\varphi^1+\frac{a}{12\alpha\mu}\rho(\rho-h)(h-2\rho)g_z^0,\\
u_z^1=&\frac{1}{12R\mu}\rho(\rho-h)(h-2\rho)\Big(\frac{\partial p^0}{\partial z}-f_z^0\Big)+\frac{1}{2\mu}\rho(\rho-h)\frac{\partial p^1}{\partial z}\\
&-\frac{a}{12\alpha\mu}\rho(\rho-h)(h-2\rho)g_\varphi^0-\frac{1}{2\mu}\rho(\rho-h)f_z^1.
\end{aligned}\tag{18}$$

From the equation (17.2), we obtain the Reynolds equation for the first–order pressure corrector p^1:

$$\begin{aligned}
\frac{Rh^3}{12\mu}\frac{\partial^2p^1}{\partial z^2}+\frac{1}{12\mu R}\frac{\partial}{\partial\varphi}\Big(h^3\frac{\partial p^1}{\partial\varphi}\Big)=&\frac{3h^4}{24R^2\mu}\frac{\partial^2p^0}{\partial\varphi^2}+\frac{5h'h^3}{12R^2\mu}\frac{\partial p^0}{\partial\varphi}-\frac{3h'h^2}{12R\mu}f_\varphi^0\\
&-\frac{h^4}{12\mu R}\frac{\partial f_\varphi^0}{\partial\varphi}+\frac{h^4}{24\mu}\Big(\frac{\partial^2p^0}{\partial z^2}-\frac{\partial f_z^0}{\partial z}\Big)-\frac{2\omega h'h}{3R}\\
&+\frac{h'h^2}{4\mu}f_\varphi^1+\frac{h^3}{12\mu}\frac{\partial f_\varphi^1}{\partial\varphi}+\frac{Rh^3}{12\mu}\frac{\partial f_z^1}{\partial z}.
\end{aligned}\tag{19}$$

We complete it with the boundary conditions:

$$\begin{aligned}
&\frac{\partial p^1}{\partial z}=\tau_z(\varphi)\quad\text{for } z=0,l,\\
&p^1 \text{ is } 2\pi \text{ periodic in } \varphi,
\end{aligned}\tag{20}$$

where $\tau_z(\varphi)$ can be chosen arbitrarily. The problem (19)–(20) is well–posed (see e.g. [17], [18]).

The first–order corrector for the microrotation $\mathbf{w}^1$ is the solution of the following problem:

$$\begin{aligned}
&\frac{1}{\varepsilon}\colon\ -\alpha\frac{\partial^2w_r^1}{\partial\rho^2}-\frac{\alpha}{R}\frac{\partial w_r^0}{\partial\rho}-\beta\frac{\partial^2w_r^1}{\partial\rho^2}-\frac{\beta}{R}\frac{\partial w_r^0}{\partial\rho}-\frac{\beta}{R}\frac{\partial^2w_\varphi^0}{\partial\varphi\partial\rho}-\beta\frac{\partial^2w_z^0}{\partial z\partial\rho}=0,\\
&\frac{1}{\varepsilon}\colon\ -\alpha\frac{\partial^2w_\varphi^1}{\partial\rho^2}-\frac{\alpha}{R}\frac{\partial w_\varphi^0}{\partial\rho}-\frac{\beta}{R}\frac{\partial^2w_r^0}{\partial\varphi\partial\rho}=-a\frac{\partial u_z^0}{\partial\rho}+g_\varphi^1,\\
&\frac{1}{\varepsilon}\colon\ -\alpha\frac{\partial^2w_z^1}{\partial\rho^2}-\frac{\alpha}{R}\frac{\partial w_z^0}{\partial\rho}-\beta\frac{\partial^2w_r^0}{\partial z\partial\rho}=a\frac{\partial u_\varphi^0}{\partial\rho}+g_z^1,\\
&\varepsilon\colon\ w_r^1(0,\varphi,z)=w_\varphi^1(0,\varphi,z)=w_z^1(0,\varphi,z)=0,\\
&\varepsilon\colon\ w_r^1(h,\varphi,z)=w_\varphi^1(h,\varphi,z)=w_z^1(h,\varphi,z)=0,
\end{aligned}\tag{21}$$

leading to

$$
\begin{aligned}
w_r^1 &= -\frac{\beta}{12\alpha(\alpha+\beta)}\rho(\rho-h)(h-2\rho)\Big(\frac{1}{R}\frac{\partial g_\varphi^0}{\partial\varphi}+\frac{\partial g_z^0}{\partial z}\Big),\\
w_\varphi^1 &= \frac{a}{12\mu\alpha}\rho(\rho-h)(2\rho-h)\Big(\frac{\partial p^0}{\partial z}-f_z^0\Big)-\frac{1}{2\alpha}\rho(\rho-h)g_\varphi^1-\frac{1}{12\alpha R}\rho(\rho-h)(h-2\rho)g_\varphi^0,\\
w_z^1 &= -\frac{1}{12\alpha R}\rho(\rho-h)(h-2\rho)g_z^0-\frac{a}{12\mu\alpha}\rho(\rho-h)(2\rho-h)\Big(\frac{1}{R}\frac{\partial p^0}{\partial\varphi}-f_\varphi^0\Big)\\
&\quad+\frac{a\omega}{2\alpha h}\rho(\rho-h)-\frac{1}{2\alpha}\rho(\rho-h)g_z^1.
\end{aligned}
\tag{22}
$$

3.4 Asymptotic Solution

The derived asymptotic solution of problem (1)–(3) reads:

$$
\begin{aligned}
\mathbf{u}^\varepsilon_{app}(\rho,\varphi,z) &= \varepsilon u_r^1(\rho,\varphi,z)\vec{\mathbf{e}}_r+(u_\varphi^0(\rho,\varphi,z)+\varepsilon u_\varphi^1(\rho,\varphi,z))\vec{\mathbf{e}}_\varphi+(u_z^0(\rho,\varphi,z)+\varepsilon u_z^1(\rho,\varphi,z))\vec{\mathbf{e}}_z,\\
p^\varepsilon_{app}(\varphi,z) &= \frac{1}{\varepsilon^2}p^0(\varphi,z)+\frac{1}{\varepsilon}p^1(\varphi,z),\\
\mathbf{w}^\varepsilon_{app}(\rho,\varphi,z) &= \varepsilon w_r^1(\rho,\varphi,z)\vec{\mathbf{e}}_r+(w_\varphi^0(\rho,\varphi,z)+\varepsilon w_\varphi^1(\rho,\varphi,z))\vec{\mathbf{e}}_\varphi+(w_z^0(\rho,\varphi,z)+\varepsilon w_z^1(\rho,\varphi,z))\vec{\mathbf{e}}_z,
\end{aligned}
\tag{23}
$$

where $u_\varphi^0, u_z^0, w_\varphi^0, w_z^0, u_r^1, u_\varphi^1, u_z^1, w_r^1, w_\varphi^1, w_z^1$ are given (in the explicit form) by (10), (15), (18) and (22), while p^0 and p^1 are the unique solutions (up to a constant) of the equations (12)–(13) and (19)–(20), respectively.

4 Conclusion

In this work, we have studied the lubrication process of a slipper bearing consisting of a circular shaft rotating on lubricated support with some constant angular velocity ω. The thin gap between the shaft and the support is filled with an incompressible micropolar fluid. We have written the governing micropolar equations (1) in cylindrical coordinates (see (4)–(5)) and applied a two-scale expansion method (see (6)). We have derived the explicit expressions for the velocity and microrotation zero–order approximation and the corresponding first–order correctors (see (10), (15), (18) and (22)). By doing that, we directly verified that the asymptotic expansion given by (23) feels the effects of the micropolarity of the fluid (see (18) and (22)).

It should be emphasized that the asymptotic approximation given by (23) was computed to satisfy the governing equations (1) with the boundary conditions (2.1)–(2.2) and (2.5) for $r = R$ and $r = R + \varepsilon h(\varphi)$. The boundary conditions at $z = 0$ and $z = l$ were not taken into account and, as a result, the computed asymptotic solution does not necessarily satisfy these conditions. For this reason, the boundary layer phenomena takes place. This can be corrected in a standard manner by introducing the appropriate boundary layer correctors in the vicinity of the lower and upper part of the shaft (see e.g. [17], [18]). It is important to note that these correctors would have exponential decay towards zero and would not affect the approximation outside the boundary layers. However, in this way, we would improve the convergence rate and provide a way of justification for our derived model. A boundary layer analysis and the rigorous mathematical justification of the proposed model via error estimates will be the subject of our future investigation.

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