Associative realizations of the extended Snyder model

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The star product usually associated with the Snyder model of noncommutative geometry is nonassociative, and this property prevents the construction of a proper Hopf algebra. It is however possible to introduce a well-defined Hopf algebra by including the Lorentz generators and their conjugate momenta into the algebra. In this paper, we study the realizations of this extended Snyder spacetime, and obtain the coproduct and twist and the associative star product in a Weyl-ordered realization, to first order in the noncommutativity parameter. We then extend our results to the most general realizations of the extended Snyder spacetime, always up to first order.

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I. INTRODUCTION

Since the advent of quantum field theory there have been proposals to add a new length scale to the theory in order to solve the problems connected to ultraviolet divergences. Later, the necessity of introducing a fundamental length scale also arose in several attempts to build a theory of quantum gravity. In these cases, the scale could be identified in a natural way with the Planck length $L_p = \sqrt{\frac{\hbar G}{c^3}} \sim 1.6 \times 10^{-35}$ m [1].

A naive application of the idea of a minimal length (such as, for example, a lattice field theory) would however break Lorentz invariance. A way to reconcile the discreteness of spacetime with Lorentz invariance was originally proposed by Snyder [2] a long time ago. This was the first example of a noncommutative geometry: the length scale should enter the theory through the commutators of spacetime coordinates; see Refs. [3,4]. In particular, the position operators obey the commutation relations

$$[x_\mu, x_\nu] = i\beta J_{\mu\nu}, \tag{1}$$

where $J_{\mu\nu}$ are the generators of the Lorentz transformations and β is a parameter of dimension length squared that sets the scale of noncommutativity.[1]

In more recent times, the coproduct and star product structures induced by the position operators of the Snyder model were calculated [5,6] using ideas coming from the development of noncommutative geometry [7]. However, in the Snyder model the algebra of the position operators does not close, as is evident from Eq. (1), and hence the bialgebra resulting from the implementation of the coproduct is not strictly speaking a Hopf algebra, as in other noncommutative geometries. In particular, the coproduct is not coassociative and the star product is not associative [5].

However, a closed Lie algebra can be obtained if one adds to the position generators the generators of the Lorentz algebra [6]. In this way, one can define a proper Hopf algebra with a coassociative coproduct.[2] The price to pay is the addition to the formalism of tensorial degrees of freedom and their conjugate momenta. To distinguish from the standard noncommutative realization of the Snyder model [5], we call the algebra where the Lorentz generators are added as extended coordinates the extended Snyder algebra, and the theory based on it the extended Snyder model. However, the physical interpretation of the new degrees of freedom is not evident; they may be viewed, for example, as coordinates parametrizing extra dimensions [6].

In this paper, we construct new realizations of this extended algebra, perturbatively in the parameter β. In order to construct them, we define an extended Heisenberg algebra, which includes the Lorentz generators and their conjugate momenta. Then, we consider a Weyl realization of the algebra in terms of the extended Heisenberg algebra, and then generalize it to the most general one compatible with Lorentz invariance at order β, including the one obtained in Ref. [6], and compute the coproduct and the

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[1]Throughout this paper we adopt natural units $\hbar = c = 1$.

[2]Generally, Lie-deformed quantum Minkowski spaces admit both Hopf algebra and Hopf algebroid structures [8].

star product in the general case. We also calculate the twist in the Weyl realization.

We recall here some of the most relevant recent advances in Snyder theory: in Ref. [9] the Snyder algebra was generalized in such a way to maintain Lorentz invariance; the coproduct was calculated in Ref. [5]; in Ref. [6] the same problem was investigated from a geometrical point of view, using the fact that the momentum space of the Snyder model can be identified with a coset space; and the twist was investigated in Refs. [10,11]. The construction of a field theory was first addressed in Refs. [5,6] and then examined in more detail in Ref. [12]. Different applications to phenomenology were considered in Ref. [13]. Finally, the extension to a curved background was proposed in Ref. [14] and further investigated in Ref. [15]. Also, the nonrelativistic limit of the theory was studied in a large number of papers, but we shall omit a discussion of this topic.

The paper is organized as follows. In Sec. II we introduce the extended Snyder model and discuss its Weyl realization in terms of an extended Heisenberg algebra. In Sec. III we compute the coproduct and the star product in this realization. In Sec. IV we calculate the twist. In Sec. V generic realizations up to order β are introduced and coproducts and star products are obtained. Finally, in Sec. VI the relations of these realizations with that of Ref. [6] and with well-known nonassociative ones are discussed. In Sec. VII we draw our conclusions.

II. EXTENDED SNYDER MODEL AND WEYL REALIZATION

As mentioned in the Introduction, the lack of associativity of the standard realization of the Snyder star product is due to the fact that it is built in terms of the position coordinates only, whose commutators do not close [cf. Eq. (1)]. An associative realization of the Snyder model can however be obtained by adding to the algebra generated by the position coordinates $\hat{x}_i$ the tensorial coordinates $\hat{x}_{ij}$, identified with the Lorentz generators, so that they span the closed algebra (2). In fact, all Lie-algebra-type noncommutative spaces induce associative star products and the coproducts of momenta are coassociative. This implies that the star product that we obtain in the present framework in Eqs. (26)–(27) is associative. If instead only the $\mathcal{D}_i$ were present in the star product, without the $\mathcal{D}_{ij}$, associativity would be lost.

We consider the extended Snyder algebra generated by the N position operators $\hat{x}_i$ and the $N(N-1)/2$ antisymmetric Lorentz generators $\hat{x}_{ij}$, with $i = 0\ldots, N-1$,

$$[\hat{x}_i, \hat{x}_j] = i\lambda\beta\hat{x}_{ij}, \qquad [\hat{x}_{ij}, \hat{x}_k] = i\lambda(\eta_{ik}\hat{x}_j - \eta_{jk}\hat{x}_i),$$
$$[\hat{x}_{ij}, \hat{x}_{kl}] = i\lambda(\eta_{ik}\hat{x}_{jl} - \eta_{il}\hat{x}_{jk} - \eta_{jk}\hat{x}_{il} + \eta_{jl}\hat{x}_{ik}), \tag{2}$$

where λ and β are real parameters. In particular, β can be identified with the Snyder parameter, which is usually assumed to be of size L_p^2, while λ is a dimensionless parameter. The parameter β can take both positive and negative values, leading to quite different physical models. However, from an algebraic point of view both cases can be treated in an essentially unified way. For $\beta = 0$, the commutation relations (2) reduce to those of the standard Lorentz algebra acting on commutative coordinates.

The algebra (2) can be realized in terms of an extended Heisenberg algebra, which also includes the Lorentz generators,

$$[x_i, x_j] = [p_i, p_j] = [x_{ij}, x_{kl}] = [p_{ij}, p_{kl}] = 0,$$
$$[x_i, p_j] = i\eta_{ij}, \qquad [x_{ij}, p_{kl}] = i(\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}),$$
$$[x_i, x_{jk}] = [x_i, p_{jk}] = [x_{ij}, x_k] = [x_{ij}, p_k] = 0, \tag{3}$$

where p_i and p_{ij} are momenta canonically conjugate to x_i and x_{ij} respectively, and $p_{ij} = -p_{ji}$. The momenta can be realized in a standard way as

$$p_i = -i\frac{\partial}{\partial x_i}, \qquad p_{ij} = -i\frac{\partial}{\partial x_{ij}}. \tag{4}$$

Note that, by including the momenta p_i in the algebra (2) with the commutation relations

$$[p_i, p_j] = 0, \qquad [\hat{x}_{ij}, p_k] = i\lambda(\eta_{ik}p_j - \eta_{jk}p_i),$$
$$[\hat{x}_i, p_j] = i(\eta_{ij} + \lambda^2\beta p_i p_j), \tag{5}$$

one recovers the full original Snyder algebra [2].

To proceed with the computations, it is convenient to exploit the isomorphism between the extended Snyder algebra and $so(1,N)$, and write the previous formulas in a more compact from by defining, for positive β, $\hat{x}_i \equiv \sqrt{\beta}\hat{x}_{iN}$, $x_i \equiv \sqrt{\beta}x_{iN}$, $p_i \equiv p_{iN}/\sqrt{\beta}$, with $\eta_{NN} = 1$, and $\mu = 0,\ldots N$.[3] The extended Heisenberg algebra (3) then becomes

$$[x_{\mu\nu}, x_{\rho\sigma}] = [p_{\mu\nu}, p_{\rho\sigma}] = 0,$$
$$[x_{\mu\nu}, p_{\rho\sigma}] = i(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}), \tag{6}$$

while the extended Snyder algebra (2) takes the form

$$[\hat{x}_{\mu\nu}, \hat{x}_{\rho\sigma}] = i\lambda C_{\mu\nu,\rho\sigma,\alpha\beta}\hat{x}_{\alpha\beta}, \tag{7}$$

where $C_{\mu\nu,\rho\sigma,\alpha\beta}$ are the structure constants of the $so(1,N)$ algebra,

[3]When $\beta < 0$ the algebra is isomorphic to $so(2, N-1)$. The coordinates are defined in the same way, except that the absolute value of β must be taken under the square root and $\eta_{NN} = -1$. All results are identical, with the appropriate choice of the sign of β.

$$C_{\mu\nu,\rho\sigma,\alpha\beta} = \frac{1}{2}[-\eta_{\nu\rho}(\eta_{\mu\alpha}\eta_{\sigma\beta} - \eta_{\sigma\alpha}\eta_{\mu\beta}) + \eta_{\mu\sigma}(\eta_{\rho\alpha}\eta_{\nu\beta} - \eta_{\nu\alpha}\eta_{\rho\beta}) + \eta_{\mu\rho}(\eta_{\nu\alpha}\eta_{\sigma\beta} - \eta_{\sigma\alpha}\eta_{\mu\beta}) - \eta_{\nu\sigma}(\eta_{\rho\alpha}\eta_{\mu\beta} - \eta_{\mu\alpha}\eta_{\rho\beta})], \tag{8}$$

that satisfy the symmetry properties $C_{\mu\nu,\rho\sigma,\alpha\beta} = -C_{\nu\mu,\rho\sigma,\alpha\beta} = -C_{\mu\nu,\sigma\rho,\alpha\beta} = -C_{\mu\nu,\rho\sigma,\beta\alpha} = -C_{\rho\sigma,\mu\nu,\alpha\beta} = -C_{\mu\nu,\alpha\beta,\rho\sigma}$.

In general, if the coordinates $\hat{x}_\mu$ generate a Lie algebra $[\hat{x}_\mu, \hat{x}_\nu] = iC_{\mu\nu\lambda}\hat{x}_\lambda$ with structure constants $C_{\mu\nu\lambda}$, then the universal realization of $\hat{x}_\mu$ corresponding to Weyl-symmetric ordering is given by [16]

$$\hat{x}_\mu = x_\alpha \phi_{\alpha\mu}(p) = x_\alpha \left(\frac{\mathcal{C}}{1 - e^{-\mathcal{C}}}\right)_{\mu\alpha}, \tag{9}$$

where $\mathcal{C}_{\mu\nu} = C_{\alpha\mu\nu} p_\alpha$. This realization enjoys the property

$$e^{ik_\mu \hat{x}_\mu} \triangleright 1 = e^{ik_\mu x_\mu}, \qquad k_\mu \in \mathbf{R}, \tag{10}$$

where the action $\triangleright$ is given by

$$x_\mu \triangleright f(x_\alpha) = x_\mu f(x_\alpha), \quad p_\mu \triangleright f(x_a) = -i\frac{\partial f(x_a)}{\partial x_\mu}, \tag{11}$$

or, in our case,

$$x_{\mu\nu} \triangleright f(x_{\alpha\beta}) = x_{\mu\nu} f(x_{\alpha\beta}),$$
$$p_{\mu\nu} \triangleright f(x_{\alpha\beta}) = -i\frac{\partial f(x_{\alpha\beta})}{\partial x_{\mu\nu}} = [p_{\mu\nu}, f(x_{\alpha\beta})]. \tag{12}$$

Hence, the corresponding Weyl realization of $\hat{x}_{\mu\nu}$ in terms of the extended Heisenberg algebra (6) reads [16]

$$\hat{x}_{\mu\nu} = x_{\alpha\beta}\left(\frac{\lambda\mathcal{C}}{1 - e^{-\lambda\mathcal{C}}}\right)_{\mu\nu,\alpha\beta}$$
$$= x_{\mu\nu} + \frac{\lambda}{2} x_{\alpha\beta}\mathcal{C}_{\mu\nu,\alpha\beta} + \frac{\lambda^2}{12} x_{\alpha\beta}(\mathcal{C}^2)_{\mu\nu,\alpha\beta} + \mathcal{O}(\lambda^4), \tag{13}$$

where

$$\mathcal{C}_{\mu\nu,\alpha\beta} = \frac{1}{2} C_{\rho\sigma,\mu\nu,\alpha\beta} p_{\rho\sigma}$$
$$= \frac{1}{2}(\eta_{\mu\alpha}p_{\nu\beta} - \eta_{\mu\beta}p_{\nu\alpha} + \eta_{\nu\beta}p_{\mu\alpha} - \eta_{\nu\alpha}p_{\mu\beta}),$$
$$(\mathcal{C}^2)_{\mu\nu,\alpha\beta} = \frac{1}{2}\sum_{k=0}^{2}\binom{2}{k}((p^k)_{\mu\alpha}(p^{2-k})_{\nu\beta} - (p^{2-k})_{\mu\beta}(p^k)_{\nu\alpha}), \tag{14}$$

and $p_{\mu\nu}$ is written in matrix notation.

Inserting $\mathcal{C}$ into Eq. (13), we find up to order λ^2

$$\hat{x}_{\mu\nu} = x_{\mu\nu} + \frac{\lambda}{2}(x_{\mu\alpha}p_{\nu\alpha} - x_{\nu\alpha}p_{\mu\alpha})$$
$$- \frac{\lambda^2}{12}(x_{\mu\alpha}p_{\nu\beta}p_{\alpha\beta} - x_{\nu\alpha}p_{\mu\beta}p_{\alpha\beta} - 2x_{\alpha\beta}p_{\mu\alpha}p_{\nu\beta}). \tag{15}$$

One then has

$$[\hat{x}_{\mu\nu}, p_{\rho\sigma}] = i(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho})$$
$$+ \frac{i\lambda}{2}(\eta_{\mu\rho}p_{\nu\sigma} - \eta_{\nu\rho}p_{\mu\sigma} + \eta_{\nu\sigma}p_{\mu\rho} - \eta_{\mu\sigma}p_{\nu\rho})$$
$$- \frac{i\lambda^2}{12}(\eta_{\mu\rho}p_{\nu\alpha}p_{\sigma\alpha} - \eta_{\mu\sigma}p_{\nu\alpha}p_{\rho\alpha} - \eta_{\nu\rho}p_{\mu\alpha}p_{\sigma\alpha}$$
$$+ \eta_{\nu\sigma}p_{\mu\alpha}p_{\rho\alpha} + 2p_{\mu\rho}p_{\nu\sigma} - 2p_{\nu\rho}p_{\mu\sigma}). \tag{16}$$

One can rewrite Eq. (15) in terms of its components as

$$\hat{x}_i = x_i + \frac{\lambda}{2}(x_k p_{ik} - \beta x_{ik}p_k) - \frac{\lambda^2}{12}(x_\kappa p_{kl}p_{il} + \beta(-x_k p_k p_i$$
$$+ x_i p_k^2 - x_{ik}p_l p_{kl} - 2x_{kl}p_k p_{il})),$$
$$\hat{x}_{ij} = x_{ij} + \frac{\lambda}{2}(x_i p_j + x_{ik}p_{jk} - (i \leftrightarrow j))$$
$$- \frac{\lambda^2}{12}(x_{ik}p_{jl}p_{kl} - x_{kl}p_{ik}p_{jl} - x_i p_k p_{jk} + 2x_k p_i p_{jk}$$
$$+ \beta x_{ik}p_k p_j - (i \leftrightarrow j)). \tag{17}$$

In the limit $\lambda\beta = L_p^2$, $\lambda = 0$, the algebra (2) becomes the Doplicher-Fredenhagen-Roberts (DFR) (Moyal) algebra [3] and the realization (15) takes the form

$$\hat{x}_i = x_i - \frac{L_p^2}{2} x_{ik}p_k, \qquad \hat{x}_{ij} = x_{ij}. \tag{18}$$

The corresponding Lorentz generators are

$$M_{ij} = x_i p_j - x_j p_i + x_{ik}p_{jk} - x_{jk}p_{ik}. \tag{19}$$

III. COPRODUCT AND STAR PRODUCT IN WEYL REALIZATION

In order to compute the coproduct of the Hopf algebra, we use the formalism introduced in Ref. [17]. We define a function $\mathcal{P}_{\mu\nu}(tk_{\alpha\beta})$ that satisfies the differential equation

$$\frac{d\mathcal{P}_{\mu\nu}}{dt}=\frac{i}{2}[p_{\mu\nu},k_{\rho\sigma}\hat{x}_{\rho\sigma}]|_{p\to\mathcal{P}(tk)}=k_{\rho\sigma}\Phi_{\mu\nu,\rho\sigma}(\mathcal{P}_{\alpha\beta}), \quad (20)$$

with the initial condition $\mathcal{P}_{\mu\nu}(0)=q_{\mu\nu}$. The function $\Phi_{\mu\nu,\rho\sigma}(p_{\alpha\beta})$ is defined from Eq. (15) as $\hat{x}_{\mu\nu}=x_{\rho\sigma}\Phi_{\rho\sigma,\mu\nu}$. In our case, Eq. (20) takes the form

$$\frac{d\mathcal{P}_{\mu\nu}}{dt}=k_{\mu\nu}-\frac{\lambda}{2}(k_{\mu\alpha}\mathcal{P}_{\nu\alpha}-k_{\nu\alpha}\mathcal{P}_{\mu\alpha}) -\frac{\lambda^2}{12}(k_{\mu\alpha}\mathcal{P}_{\alpha\beta}\mathcal{P}_{\nu\beta}-k_{\nu\alpha}\mathcal{P}_{\alpha\beta}\mathcal{P}_{\mu\beta}-2k_{\alpha\beta}\mathcal{P}_{\mu\alpha}\mathcal{P}_{\nu\beta}) \quad (21)$$

and with the given initial condition has the solution

$$\mathcal{P}_{\mu\nu}=q_{\mu\nu}+tk_{\mu\nu}-\frac{\lambda t}{2}(k_{\mu\alpha}q_{\nu\alpha}-k_{\nu\alpha}q_{\mu\alpha}) -\frac{\lambda^2}{12}((k_{\mu\alpha}q_{\alpha\beta}q_{\nu\beta}-k_{\nu\alpha}q_{\alpha\beta}q_{\mu\beta}-2k_{\alpha\beta}q_{\mu\alpha}q_{\nu\beta})t +(k_{\mu\alpha}k_{\alpha\beta}q_{\nu\beta}-k_{\nu\alpha}k_{\alpha\beta}q_{\mu\beta}-2k_{\mu\alpha}k_{\nu\beta}q_{\alpha\beta})t^2). \quad (22)$$

We can now define $P_{\mu\nu}(k_{\mu\nu},q_{\mu\nu})\equiv\mathcal{P}_{\mu\nu}(t=1)$, so that

$$P_{\mu\nu}(k_{\mu\nu},q_{\mu\nu})=k_{\mu\nu}+q_{\mu\nu}-\frac{\lambda}{2}(k_{\mu\alpha}q_{\nu\alpha}-k_{\nu\alpha}q_{\mu\alpha}) -\frac{\lambda^2}{12}(k_{\mu\alpha}q_{\alpha\beta}q_{\nu\beta}-k_{\nu\alpha}q_{\alpha\beta}q_{\mu\beta}-2k_{\alpha\beta}q_{\mu\alpha}q_{\nu\beta} +k_{\mu\alpha}k_{\alpha\beta}q_{\nu\beta}-k_{\nu\alpha}k_{\alpha\beta}q_{\mu\beta}-2k_{\mu\alpha}k_{\nu\beta}q_{\alpha\beta}). \quad (23)$$

Defining $\mathcal{K}_{\mu\nu}(k_{\mu\nu})\equiv P_{\mu\nu}(q_{\mu\nu}=0)$, one has $\mathcal{K}_{\mu\nu}=k_{\mu\nu}$, and therefore also its inverse function $\mathcal{K}^{-1}_{\mu\nu}(k_{\mu\nu})=k_{\mu\nu}$.

It can be shown that the generalized momentum addition law is given by [17]

$$k_{\mu\nu}\oplus q_{\mu\nu}\equiv\mathcal{D}_{\mu\nu}(k_{\alpha\beta},q_{\alpha\beta})=P_{\mu\nu}(\mathcal{K}^{-1}_{\alpha\beta},q_{\alpha\beta}), \quad (24)$$

and hence in our case $\mathcal{D}_{\mu\nu}(k_{\alpha\beta},q_{\alpha\beta})=P_{\mu\nu}(k_{\alpha\beta},q_{\alpha\beta})$. This yields the coproduct

$$\Delta p_{\mu\nu}=\mathcal{D}_{\mu\nu}(p_{\mu\nu}\otimes 1,1\otimes p_{\mu\nu}) =\Delta_0 p_{\mu\nu}-\frac{\lambda}{2}(p_{\mu\alpha}\otimes p_{\nu\alpha}-p_{\nu\alpha}\otimes p_{\mu\alpha}) -\frac{\lambda^2}{12}(p_{\mu\alpha}\otimes p_{\alpha\beta}p_{\nu\beta}-p_{\nu\alpha}\otimes p_{\alpha\beta}p_{\mu\beta} -2p_{\alpha\beta}\otimes p_{\mu\alpha}p_{\nu\beta}+p_{\mu\alpha}p_{\alpha\beta}\otimes p_{\nu\beta} -p_{\nu\alpha}p_{\alpha\beta}\otimes p_{\mu\beta}-2p_{\mu\alpha}p_{\nu\beta}\otimes p_{\alpha\beta}), \quad (25)$$

with $\Delta_0 p_{\mu\nu}=p_{\mu\nu}\otimes 1+1\otimes p_{\mu\nu}$. It is straightforward to explicitly check the coassociativity of this coproduct. It is also easy to see that the antipode is trivial, $S(p_{\mu\nu})=-p_{\mu\nu}$.

Recalling our definitions $\hat{x}_i=\sqrt{\beta}\hat{x}_{iN}$ and $p_i=p_{iN}/\sqrt{\beta}$, we can write the functions $\mathcal{D}_{\alpha\beta}$ in terms of their components, namely,

$$\mathcal{D}_i(k,q)=k_i+q_i-\frac{\lambda}{2}[k_jq_{ij}-k_{ij}q_j]+\frac{\lambda^2}{12}[\beta(k_ik_jq_j-k_j^2q_i)-k_jk_{jk}q_{ik}+2k_{ik}k_jq_{jk} +k_{ij}k_{jk}q_k+\beta(k_jq_jq_i-k_iq_j^2)+k_{ij}q_{jk}q_k-2k_{jk}q_kq_{ij}-k_jq_{jk}q_{ik}], \quad (26)$$

$$\mathcal{D}_{ij}(k,q)=k_{ij}+q_{ij}-\frac{\lambda}{2}[k_{ik}q_{jk}+\beta k_iq_j-(i\leftrightarrow j)]+\frac{\lambda^2}{12}[k_{ik}k_{jl}q_{kl}-k_{ik}k_{kl}q_{jl}+\beta(k_ik_kq_{jk}-k_{ik}k_kq_j-2k_ik_{jk}q_k) +k_{kl}q_{ik}q_{jl}-k_{ik}q_{kl}q_{jl}-\beta(k_{ik}q_kq_j-k_iq_kq_{jk}-2k_kq_iq_{jk})-(i\leftrightarrow j)]. \quad (27)$$

The functions $\mathcal{D}(q,k)$ satisfy the symmetry properties

$$\mathcal{D}_i(q,k)|_\lambda=\mathcal{D}_i(k,q)|_{-\lambda},\qquad \mathcal{D}_{ij}(q,k)|_\lambda=\mathcal{D}_{ij}(k,q)|_{-\lambda}. \quad (28)$$

It also holds that

$$e^{\frac{i}{2}k_{\mu\nu}\hat{x}_{\mu\nu}}e^{\frac{i}{2}q_{\rho\sigma}\hat{x}_{\rho\sigma}}=e^{\frac{i}{2}\mathcal{D}_{\mu\nu}(k,q)\hat{x}_{\mu\nu}} \quad (29)$$

and

$$e^{\frac{i}{2}k_{\mu\nu}x_{\mu\nu}}\star e^{\frac{i}{2}q_{\rho\sigma}x_{\rho\sigma}}=e^{\frac{i}{2}k_{\mu\nu}\hat{x}_{\mu\nu}}e^{\frac{i}{2}q_{\rho\sigma}\hat{x}_{\rho\sigma}}\rhd 1=e^{\frac{i}{2}\mathcal{D}_{\mu\nu}(k,q)\hat{x}_{\mu\nu}}\rhd 1=e^{\frac{i}{2}\mathcal{D}_{\mu\nu}(k,q)x_{\mu\nu}}. \quad (30)$$

Moreover, we can write

$$
\begin{aligned}
e^{\frac{i}{2}k_{\mu\nu}\hat{x}_{\mu\nu}} &= e^{ik_i\hat{x}_i+\frac{i}{2}k_{ij}\hat{x}_{ij}}, \\
e^{ik_ix_i+\frac{i}{2}k_{ij}x_{ij}}\star e^{iq_kx_k+\frac{i}{2}q_{kl}x_{kl}} &= e^{i\mathcal{D}_ix_i+\frac{i}{2}\mathcal{D}_{ij}x_{ij}}.
\end{aligned}
\tag{31}
$$

In particular, from Eqs. (26) and (27) one can obtain the star product for plane waves. Notice that the star product of two translations clearly will also have a component in the direction of rotations,

$$
\begin{aligned}
e^{ik_ix_i}\star e^{iq_jx_j} &= e^{i[k_i+q_i-\frac{1}{12}\lambda^2\beta(q_j^2k_i-k_jq_jq_i+k_j^2q_i-k_jq_jk_i)]x_i-\frac{i}{2}\lambda\beta k_iq_jx_{ij}}, \\
e^{\frac{i}{2}k_{ij}x_{ij}}\star e^{\frac{i}{2}q_{kl}x_{kl}} &= e^{\frac{i}{2}[k_{ij}+q_{ij}-\lambda k_{ik}q_{jk}-\frac{1}{6}\lambda^2(k_{ik}q_{kl}q_{jl}-k_{kl}q_{ik}q_{jl}+k_{ik}k_{kl}q_{jl}-k_{ik}k_{jl}q_{kl})]x_{ij}}, \\
e^{ik_kx_k}\star e^{\frac{i}{2}q_{ij}x_{ij}} &= e^{i[k_i-\frac{\lambda}{2}k_jq_{ij}-\frac{1}{12}\lambda^2k_jq_{jk}q_{ik}]x_i+\frac{i}{2}[q_{ij}+\frac{1}{6}\lambda^2\beta k_ik_kq_{jk}]x_{ij}}, \\
e^{\frac{i}{2}k_{ij}x_{ij}}\star e^{\frac{i}{2}q_kx_k} &= e^{i[q_i+\frac{\lambda}{2}k_{ij}q_j+\frac{1}{12}\lambda^2k_{ij}k_{jk}q_k]x_i+\frac{i}{2}[k_{ij}-\frac{1}{6}\lambda^2\beta k_{ik}q_kq_j]x_{ij}}.
\end{aligned}
\tag{32}
$$

This star product is associative. One can also check that the star products of the coordinates x_i and x_{ij} satisfy the extended Snyder algebra. In fact, according to Ref. [6], denoting for brevity k and q the vectors k_i and q_i and $\mathbf{l}$, $\mathbf{r}$ the tensors l_{ij} and r_{ij} and defining $e_{k,\mathbf{l}} = e^{k_ix_i+\mathbf{l}_{jk}x_{jk}}$, the star product of the coordinates can be evaluated as follows:

$$
\begin{aligned}
x_i\star x_j &= \int dkdqd\mathbf{l}d\mathbf{r}\delta(k)\delta(q)\delta(\mathbf{l})\delta(\mathbf{r})\partial_{k_i}\partial_{q_j}(e_{k,\mathbf{l}}\star e_{q,\mathbf{r}}) = \hat{x}_i\triangleright x_j = x_ix_j + i\frac{\lambda\beta}{2}x_{ij}, \\
x_{ij}\star x_{kl} &= \int dkdqd\mathbf{l}d\mathbf{r}\delta(k)\delta(q)\delta(\mathbf{l})\delta(\mathbf{r})\partial_{\mathbf{l}_{ij}}\partial_{\mathbf{r}_{kl}}(e_{k,\mathbf{l}}\star e_{q,\mathbf{r}}) = \hat{x}_{ij}\triangleright x_{kl} = x_{ij}x_{kl} + i\frac{\lambda}{2}(\eta_{ik}x_{jl}-\eta_{jk}x_{il}-\eta_{il}x_{jk}+\eta_{jl}x_{ik}), \\
x_k\star x_{ij} &= \int dkdqd\mathbf{l}d\mathbf{r}\delta(k)\delta(q)\delta(\mathbf{l})\delta(\mathbf{r})\partial_{k_k}\partial_{\mathbf{r}_{ij}}(e_{k,\mathbf{l}}\star e_{q,\mathbf{r}}) = \hat{x}_k\triangleright x_{ij} = x_kx_{ij} - i\frac{\lambda}{2}(\eta_{ik}x_j-\eta_{jk}x_i), \\
x_{ij}\star x_k &= \int dkdqd\mathbf{l}d\mathbf{r}\delta(k)\delta(q)\delta(\mathbf{l})\delta(\mathbf{r})\partial_{\mathbf{l}_{ij}}\partial_{q_k}(e_{k,\mathbf{l}}\star e_{q,\mathbf{r}}) = \hat{x}_{ij}\triangleright x_k = x_{ij}x_k + i\frac{\lambda}{2}(\eta_{ik}x_j-\eta_{jk}x_i).
\end{aligned}
\tag{33}
$$

Therefore,

$$
[x_i, x_j]_\star = i\lambda\beta x_{ij}, \qquad [x_{ij}, x_k]_\star = i\lambda(\eta_{ik}x_j-\eta_{jk}x_i), \qquad [x_{ij}, x_{kl}]_\star = i\lambda(\eta_{ik}x_{jl}-\eta_{jk}x_{il}-\eta_{il}x_{jk}+\eta_{jl}x_{ik}),
\tag{34}
$$

which is isomorphic to the algebra (2).

IV. TWIST FOR THE WEYL REALIZATION

In this section we construct the twist operator at second order in λ, using a perturbative approach. The twist is defined as a bilinear operator such that $\Delta h = \mathcal{F}\Delta_0 h\mathcal{F}^{-1}$ for each $h \in so(1,N)$.

The twist in a Hopf algebroid sense can be computed by means of the formula [10,18]

$$
\mathcal{F}^{-1} \equiv e^F = e^{-\frac{i}{2}p_{\mu\nu}\otimes x_{\mu\nu}}e^{\frac{i}{2}p_{\rho\sigma}\otimes\hat{x}_{\rho\sigma}}.
\tag{35}
$$

By the Baker-Campbell-Hausdorff formula $e^Ae^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$, one gets

$$
F = \frac{i}{2}p_{\mu\nu}\otimes(\hat{x}_{\mu\nu}-x_{\mu\nu}) - \frac{1}{8}p_{\mu\nu}p_{\rho\sigma}\otimes[x_{\mu\nu},\hat{x}_{\rho\sigma}] + \cdots,
\tag{36}
$$

where we can safely ignore further terms because it can be explicitly checked that they give contributions of order λ^3.

Substituting Eq. (15) into Eq. (36), one obtains

$$
F = \frac{i\lambda}{2}p_{\alpha\gamma}\otimes x_{\alpha\beta}p_{\gamma\beta} - \frac{i\lambda^2}{24}(2p_{\alpha\gamma}\otimes x_{\alpha\beta}p_{\beta\delta}p_{\gamma\delta} - 2p_{\gamma\delta}\otimes x_{\alpha\beta}p_{\alpha\gamma}p_{\beta\delta} - p_{\alpha\gamma}p_{\beta\delta}\otimes x_{\alpha\beta}p_{\gamma\delta} + p_{\alpha\gamma}p_{\delta\gamma}\otimes x_{\alpha\beta}p_{\delta\beta}).
\tag{37}
$$

Using the Hadamard formula $e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \cdots$, it is easy to check that

$$\mathcal{F}\Delta_0 p_{\mu\nu}\mathcal{F}^{-1} = \Delta p_{\mu\nu}, \tag{38}$$

with $\Delta p_{\mu\nu}$ given in Eq. (25), as expected.

V. GENERIC REALIZATIONS

We now consider the most general realization of the commutation relations (2) in terms of the elements of the extended Heisenberg algebra (3), up to second order in λ. Of course, this will deform the commutation relations between coordinates and momenta in Eq. (5).

The generic form of the Lorentz-covariant combinations of the generators of the algebra (3), linear in x_i, x_{ij}, up to order λ^2 is given by[4]

$$\begin{aligned}\hat{x}_i &= x_i + \lambda(\beta c_0 x_{ik}p_k + c_1 x_k p_{ik}) + \lambda^2(\beta(c_2 x_i p_k^2 + c_3 x_k p_k p_i + c_4 x_{ik}p_{kl}p_l + c_5 x_{kl}p_k p_{il}) + c_6 x_k p_{kl}p_{il}),\\ \hat{x}_{ij} &= x_{ij} + \lambda(d_0 x_{ik}p_{jk} + d_1 x_i p_j - (i \leftrightarrow j)) + \lambda^2(\beta d_2 x_{ik}p_k p_j + d_3 x_{ik}p_{kl}p_{jl} + d_4 x_{kl}p_{ik}p_{jl}\\ &\quad + d_5 x_i p_k p_{jk} + d_6 x_k p_{ik}p_j - (i \leftrightarrow j)).\end{aligned} \tag{39}$$

In order to satisfy Eq. (2) to first order in λ one must have

$$c_0 = -\frac{1}{2}, \qquad d_0 = \frac{1}{2}, \qquad c_1 + d_1 = 1. \tag{40}$$

Hence, at this order one has one free parameter. In particular, in the Weyl realization (17), $d_1 = c_1 = \frac{1}{2}$.

To second order in λ, one has ten new parameters $c_2, \ldots, c_6$, $d_2, \ldots, d_6$ that must satisfy the six independent relations

$$\begin{aligned}&\frac{c_1}{2} - 2c_2 + c_3 = d_1, \qquad \frac{c_1}{2} + c_4 + c_5 = \frac{1}{2}, \qquad d_3 - 2d_4 = -\frac{1}{4},\\ &c_5 - d_2 = \frac{1}{4}, \qquad \frac{c_1}{2} + c_6 - d_6 = 0, \qquad \frac{c_1}{2} - c_1 d_1 + c_6 + d_5 = 0.\end{aligned} \tag{41}$$

Hence, up to second order one has five free parameters. For example, one may choose as free parameters c_1, c_2, c_4, d_4, and d_5, so that $d_1 = 1 - c_1$ and

$$\begin{aligned}&c_3 = 1 - \frac{3c_1}{2} + 2c_2, \qquad c_5 = \frac{1}{2} - \frac{c_1}{2} - c_4, \qquad c_6 = \frac{c_1}{2} - c_1^2 - d_5,\\ &d_2 = \frac{1}{4} - \frac{c_1}{2} - c_4, \qquad d_3 = -\frac{1}{4} + 2d_4, \qquad d_6 = c_1 - c_1^2 - d_5.\end{aligned} \tag{42}$$

It is easy to verify that the coefficients of the Weyl realization (17) satisfy the above relations with $c_1 = \frac{1}{2}$, $c_2 = -c_4 = -d_4 = -d_5 = -\frac{1}{12}$.

Note that setting $\beta = 0$ in Eq. (39), one obtains realizations of the Poincaré algebra. For example, the Weyl realization for the operators $\hat{x}_i$ and $\hat{x}_{ij}$ of the Poincaré algebra becomes

$$\begin{aligned}\hat{x}_i &= x_i + \frac{\lambda}{2}x_k p_{ik} - \frac{\lambda^2}{12}x_k p_{kl}p_{il},\\ \hat{x}_{ij} &= x_{ij} + \left[\frac{\lambda}{2}(x_i p_j + x_{ik}p_{jk}) - \frac{\lambda^2}{12}(x_{ik}p_{jl}p_{kl} - x_{kl}p_{ik}p_{jl} - x_i p_k p_{jk} + 2x_k p_i p_{jk}) - (i \leftrightarrow j)\right].\end{aligned} \tag{43}$$

Through the same procedure as in the previous section, one can determine the coproduct for the generic realization (39). The differential equations for $\mathcal{P}_i(tk)$ and $\mathcal{P}_{ij}(tk)$ are

[4]In principle, one may add further terms to Eq. (39), namely, the terms $x_i p_{kl} p_{kl}$ and $x_{kl} p_{kl} p_i$ to $\hat{x}_i$, and $x_{ij} p_k p_k$, $x_{ij} p_{kl} p_{kl}$, $x_{kl} p_{kl} p_{ij}$, $x_k p_k p_{ij}$ to $\hat{x}_{ij}$. However, these terms must vanish if one requires that the Snyder algebra be satisfied.

$$\frac{d\mathcal{P}_i}{dt} = i\left[p_i, k_k\hat{x}_k + \frac{1}{2}k_{kl}\hat{x}_{kl}\right]\Big|_{p\to\mathcal{P}(tk)},$$
$$\frac{d\mathcal{P}_{ij}}{dt} = i\left[p_{ij}, k_k\hat{x}_k + \frac{1}{2}k_{kl}\hat{x}_{kl}\right]\Big|_{p\to\mathcal{P}(tk)}, \tag{44}$$

with initial conditions $\mathcal{P}_i(0) = q_i$ and $\mathcal{P}_{ij}(0) = q_{ij}$. After some calculations, one can write down the functions $\mathcal{D}_i(k,q)$ and $\mathcal{D}_{ij}(k,q)$ that appear in the star product of plane waves,

$$\mathcal{D}_i(k,q) = k_i + q_i + \lambda(-c_1k_jq_{ij} + d_1k_{ij}q_j) + \frac{\lambda^2}{2}[\beta(c_0c_1 + c_3)k_j^2q_i + \beta(-c_0c_1 + 2c_2 + c_3)k_ik_jq_j$$
$$+ (c_1^2 - c_1d_0 - c_1d_1 + c_6 + d_6)k_jk_{jk}q_{ik} + (c_1d_0 + c_1d_1 + c_6 - d_5)k_{ik}k_jq_{jk}$$
$$+ (d_1^2 + d_5 - d_6)k_{ij}k_{jk}q_k + 2\beta c_2k_iq_j^2 + 2\beta c_3k_jq_jq_i + 2d_5k_{ij}q_{jk}q_k + 2d_6k_{jk}q_jq_{ik} + 2c_6k_jq_{jk}q_{ik}], \tag{45}$$

and

$$\mathcal{D}_{ij}(k,q) = k_{ij} + q_{ij} + \lambda(-d_0k_{ik}q_{jk} + \beta c_0k_iq_j - (i\leftrightarrow j)) + \frac{\lambda^2}{2}[\beta(-c_0c_1 + c_4 - c_5)k_ik_kq_{jk}$$
$$+ (-d_0^2 + d_3 + 2d_4)k_{ik}k_{kl}q_{jl} + (d_0^2 + d_3)k_{ik}k_{jl}q_{kl} + \beta(c_0d_0 + c_5 + d_2)k_{ik}k_kq_j$$
$$+ \beta(c_0d_0 + c_0d_1 + c_4 - d_2)k_ik_{jk}q_k + 2\beta d_2k_{ik}q_kq_j + 2d_3k_{ik}q_{kl}q_{jl} + 2d_4k_{kl}q_{ik}q_{jl} + 2\beta c_4k_iq_kq_{jk}$$
$$+ 2\beta c_5k_kq_{ik}q_j - (i\leftrightarrow j)]. \tag{46}$$

From these functions one can easily obtain the star product and the coproduct in the general case; see Eqs. (25) and (31). In particular, for $c_0 = -\frac{1}{2}$ and $k_{ij} = q_{ij} = 0$, one has

$$e^{ik_ix_i}\star e^{iq_jx_j} = e^{i[k_i+q_i+\frac{1}{2}\lambda^2\beta(2c_2q_j^2k_i+2c_3k_jq_jq_i+(c_3-\frac{c_1}{2})k_j^2q_i+(2c_2+c_3+\frac{c_1}{2})k_jq_jk_i)]x_i-\frac{i}{2}\lambda\beta k_iq_jx_{ij}}, \tag{47}$$

which for $c_1 = \frac{1}{2}$, $c_2 = -c_3 = -\frac{1}{12}$ reduces to the first relation in Eq. (32).

VI. COMPARISON WITH THE GIRELLI-LEVINE APPROACH

The authors of Ref. [6] studied our model in three-dimensional Euclidean space using geometric methods, with a very different parametrization, adapted to the coset-space nature of the Snyder momentum space. In our notations, their star product for plane waves, at second order in λ, takes the form

$$e^{ik_ix_i}\star e^{iq_jx_j} = \exp\left[i\left(k_i + q_i + \frac{\lambda^2\beta}{2}(k_jq_jk_i + k_j^2q_i + 2k_jq_jq_i)\right)x_i - i\frac{\lambda\beta}{2}k_iq_jx_{ij}\right]. \tag{48}$$

This expression corresponds to the realization (39) with $c_0 = -\frac{1}{2}$, $d_0 = \frac{1}{2}$, and $c_1 = c_2 = 0$. It follows from Eq. (42) that $c_3 = 1$, but the other coefficients are not determined and depend on three free parameters. If one also requires $d_5 = 0$, this may be called a generalized Snyder realization, since it obeys all of the commutation relations of the original Snyder model [2], given by Eqs. (2) and (5). Note that the momenta p_{ij} do not appear in these relations. Of course, additional commutation relations are obeyed by the momenta p_{ij}, but they are not of interest for our considerations.

One may consider more general realizations belonging to the previous class, with $c_0 = -\frac{1}{2}$, $d_0 = \frac{1}{2}$, $c_1 = 0$, and three free parameters. For example, $c_2 = -\frac{1}{2}$ implies $c_3 = 0$ and gives rise to a realization that, for $d_5 = 0$, reproduces at order β the commutation relations of the Maggiore realization introduced in Ref. [9].

More generally, these representations generalize those introduced in Ref. [10], with arbitrary c_2 and $c_3 = 1 + 2c_2$. In particular, one can choose the free parameters such that

$$\hat{x}_i = x_i + \frac{\lambda^2\beta}{2}[(c_3 - 1)x_ip_k^2 + 2c_3x_kp_kp_i] - \frac{\lambda\beta}{2}\hat{m}_{ik}p_k,$$
$$\hat{x}_{ij} = \hat{m}_{ij} + \lambda(x_ip_j - x_jp_i), \tag{49}$$

where the $\hat{m}_{ij}$ generate the Lorentz algebra $so(1, N-1)$ and

$$[\hat{m}_{ij}, x_k] = [\hat{m}_{ij}, p_k] = 0. \tag{50}$$

For example, in the Weyl realization of $\hat{m}_{ij}$, $d_3 = -d_4 = -\frac{1}{12}$, leaving c_3 as a free parameter. In the limit $\beta = 0$, $\hat{x}_i$ reduces to x_i.

VII. CONCLUSIONS

The coalgebra usually associated with the Snyder model is noncoassociative, and this fact prevents the definition of a proper Hopf algebra, whose coproduct is by definition coassociative. The reason is that the algebra of the position operators of the Snyder model does not close. However, this can be remedied by including the Lorentz generators in the defining algebra [6]. In this way, a standard coassociative Hopf algebra can be defined.

In this paper we have studied the realizations of this extended algebra in terms of the deformations of an extended Heisenberg algebra, which contains tensorial elements that in the deformation assume the role of Lorentz generators. We have obtained the coproduct, the star product, and the twist in the case of a Weyl realization. We have also considered the most general realization of the algebra up to second order in the expansion parameter λ (or, equivalently, at first order in the Snyder parameter β) and calculated the corresponding coproduct and star product.

Although this approach may be considered more rigorous than the standard one from a mathematical point of view, the physical interpretation of the new degrees of freedom, related to the Lorentz generators and their momenta, is still an issue. In Ref. [6] the tensorial coordinates x_{ij} were interpreted from a Kaluza-Klein perspective as coordinates of extra dimensions, and hence were not identified with Lorentz generators. It is also important to note that the action of noncommutative tensorial coordinates on Eq. (1) is defined to give commutative tensorial coordinates [see Eqs. (30) and (33)]. The noncommutative tensorial coordinates are related to the parametrization of the dual Lorentz group. This topic is presently being investigated.

In applications, one may for example build a field theory assuming that the fields $\psi(\hat{x}_{\mu\nu})$ depend only on the spacetime coordinates [6], i.e., $\psi(x_{\mu\nu}) = \phi(\hat{x}_i)\delta(\hat{x}_{ij})$. However, in this way one would recover the usual nonassociative star product. Another possibility is that the extra coordinates parametrize a compactified internal space. In this case associativity would be preserved, but nontrivial physical consequences would presumably arise. We leave the investigation of this possibility for future work. In any case, a field theory based on this formalism could avoid the shortcomings due to the nonassociativity of the star product [12], but different problems can arise because of the intertwining between the position and the extra degrees of freedom [6].

To conclude, we also observe that the standard commutative theory, as well as DFR spacetime [3], can be formulated in this extended framework, as we have observed several times in the text. The investigation of these elementary cases could be a good starting point to better understand the physical implications of the present formalism, in particular in relation to quantum field theory.

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