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Distributions of anisotropic order and applications to H-distributions

Abstract

We define distributions of anisotropic order on manifolds, and establish their immediate properties. The central result is the Schwartz kernel theorem for such distributions, allowing the representation of continuous operators from $\mathrm{C}^l_c(X)$ to $(\mathrm{C}^m_c(Y))'$ by kernels, which we prove to be distributions of order l in x, but higher, although still finite, order in y.

Our main motivation for introducing these distributions is to obtain the new result that H-distributions (Antonić and Mitrović, 2011), a recently introduced generalisation of H-measures are, in fact, distributions of order 0 (i.e. Radon measures) in $\mathbf{x} \in \mathbf{R}^d$, and of finite order in $\boldsymbol{\xi} \in \mathrm{S}^{d-1}$. This allows us to obtain some more precise results on H-distributions, hopefully allowing for further applications to partial differential equations.

Keywords: Fourier multiplier, H-measure, H-distribution, distribution of finite order, Schwartz kernel theorem

Mathematics subject classification: 42B15, 35S05, 46E30, 47G30

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1. Introduction

Motivation

The first ideas of notions of generalised functions can be traced back to the eighteen century. Although many mathematicians contributed to the development of the theory, its precise form which is widely accepted nowadays, the notion of the *distribution*, emerged during the Second world war in the studies of Laurent Schwartz, based on the duality theory in functional analysis. In a matter of several years, it proved to be a powerful tool in the study of partial differential equations: let us just mention the Malgrange-Ehrenpreis theorem on the existence of elementary solutions.

The theory of distributions allowed some particular, already known results in mathematical analysis to be written in an abstract general form. For example, it was known that a continuous linear mapping $u \mapsto v$ between two Banach function spaces could in many (but not all) cases be represented in terms of a kernel k, such that (with a suitable notion of the integral) one had

$$v(x) = \int k(x,y)u(y)dy \,.$$

The Schwartz kernel theorem states that such a kernel always exists for operators between spaces of distributions, provided that the kernel is allowed to be a distribution itself, facilitating the extension of the important notion of integral operators to the case of singular kernels. In particular, as a consequence of the kernel theorem, all pseudodifferential operators can be understood as integral operators with a distributional kernel (see, e.g., [37, Chapter 6]).

More precisely, the theorem states that any continuous bilinear form on the Cartesian product of distributional spaces can be represented as a unique distribution in both variables, and vice versa. There are several approaches in proving this theorem, and let us mention just a few: via the decomposition into orthonormal series (see, e.g., [16]), the abstract approach via nuclear spaces due to Alexander Grothendieck (e.g., [39]), and the original Schwartz's proof (cf. [34, 13]).

The kernel theorem played an important role in Tartar's construction of H-measures [38], which was the starting point that motivated our research in this direction. H-measures were introduced by Luc Tartar and independently by Patrick Gérard [17], who called them microlocal defect measures, to study oscillation and concentration effects in partial differential equations. As opposed to Young measures, the H-measures are able to capture propagation effects [1]. In fact, they are (possibly unbounded) positive Radon measures, being positive distributions on $\mathbf{R}^d \times \mathrm{S}^{d-1}$ [35, Section I.4]. On the other hand, any positive Radon measure on (the cospherical bundle) $\mathbf{R}^d \times \mathrm{S}^{d-1}$ can be obtained as an H-measure generated by a weakly converging L^2 sequence of functions defined on $\mathbf{R}^d$ [38, Corollary 2.3]. For some variants of H-measures tailored to parabolic problems see [5], and for further generalisations [14, 3, 31, 32]. However, they are suitable only for problems expressed in the L^2 framework.

In order to overcome that limitation, H-distributions were introduced [7], which are an extension of H-measures to the $\mathrm{L}^p - \mathrm{L}^q$ context. The existence proof in [7] was based on the classical Schwartz kernel theorem, giving that H-distributions were only general distributions in $\mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$, as we lack the positivity here (contrary to the L^2 case). However, in the community there has been no consensus whether to expect that H-distributions are actually Radon measures, as all the available examples so far have been such. In this paper we address this problem and provide (almost) complete answer.

The precise analytic properties of these objects are our main application of the developed theory: these objects (in general) are not Radon measures, but anisotropic distributions, of zero order in $\mathbf{x}$ (usually taken to be in the Euclidean space $\mathbf{R}^d$) and given finite order in $\boldsymbol{\xi}$ taken on the unit sphere S^{d-1}, therefore allowing for applications to partial differential equations with continuous coefficients. Some further variants and existing applications of H-distributions are related to the mixed-norm Lebesgue spaces [4], the velocity averaging [22] and $\mathrm{L}^p - \mathrm{L}^q$ compactness by compensation [25].

Another attempt to extend H-measures in order to encompass L^p weakly converging sequences [33] was based on the extension of generalised Young measures of DiPerna and Majda to *microlocal compactness forms*, which turn out to be closely connected to H-distributions [8].

While we introduced the distributions of anisotropic order with a precise goal of better understanding the H-distributions, namely to precisely formulate how far they are from being Radon measures (i.e. the distributions of order zero), this new notion, together with a more precise form of the Schwartz kernel theorem, can be of interest on its own [24, 26]. More precisely, in [24] a notion of anisotropic distributions was crucial in extending a coordinate-free characterisation of partial differential operators, where every linear operator on C^∞_c which does not increase the support is understood to be a partial differential operator.

Overview of the paper

In order to precisely determine what kind of objects H-distributions are, we need the notion of distributions of anisotropic (finite) order on manifolds without boundary (not necessarily C^∞ smooth), which we introduce in the next section, together with some immediate properties. The third section is devoted to the proof of the Schwartz kernel theorem for distributions of anisotropic order. Let us stress that the spaces we consider are not nuclear, so powerful abstract approach of Grothendieck is not applicable in our case. The kernel theorem is necessary prerequisite for a precise version of the existence of H-distributions presented in the fourth section, together with its basic properties, like the criterion for strong convergence and the connection to defect measures. Our abstract result has an important practical consequence: when applying H-distributions to partial differential equations, we are now able to assume only continuity of coefficients in the equations, not being limited to the C^∞ case.

An important example of a weakly converging sequence is concentration, which is treated in the fifth section, providing an example of an H-distribution which is not a Radon measure on $\Omega \times \mathrm{S}^{d-1}$, thus justifying the introduction of distributions of anisotropic order. This is followed by the investigation how a perturbation of generating sequences can still preserve the H-distribution, by some comments on using different symbols and on applications of compactness by compensation.

Finally, we would like to mention that a preliminary version of some results presented in this paper was a part of Marin Mišur's Ph.D. thesis [23].

Notation

Before we proceed, let us introduce the notation which we shall use in the paper. By $\hat{u}$ and $\mathcal{F}u$ we shall denote the Fourier transform of u: $\hat{u}(\boldsymbol{\xi}) = (\mathcal{F}u)(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} u(\mathbf{x}) d\mathbf{x}$, and by $\check{u}$ and $\bar{\mathcal{F}}u$ it's inverse, while by $\mathcal{A}_\psi$ the Fourier multiplier operator with symbol ψ: $\mathcal{A}_\psi(u) = (\psi\hat{u})^\vee$.

In what follows, we shall use Fourier multiplier operators defined by functions on the unit sphere S^{d-1} in $\mathbf{R}^d$. Whenever we say that a function $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ is a symbol of the Fourier multiplier operator (denoted $\mathcal{A}_\psi$), we shall actually mean that the symbol is $\psi \circ \pi$, a function homogeneous of order zero on $\mathbf{R}^d_* := \mathbf{R}^d \setminus \{0\}$, where $\pi : \mathbf{R}^d_* \longrightarrow \mathrm{S}^{d-1}$ is the projection onto unit sphere along rays: $\pi(\boldsymbol{\xi}) := \boldsymbol{\xi}/|\boldsymbol{\xi}|$. By the Hörmander-Mihlin theorem, if $\kappa \geqslant \lfloor d/2 \rfloor + 1$, such an operator $\mathcal{A}_\psi$ is bounded on $\mathrm{L}^p(\mathbf{R}^d)$ for any $p \in \langle 1, \infty \rangle = \{p \in \mathbf{R} : 1 < p\}$, and we have the bound for the operator norm of $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ [18, Example 5.2.6.] (see also [4, Theorem 7] for the mixed-norm case):

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \leqslant C_d \max\left\{p, \frac{1}{p-1}\right\} \|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})} .$$

It is known that in some particular cases κ can be taken to be less than $\lfloor d/2 \rfloor + 1$ (cf. [12] for the case when $d = 2$).

For $p \in [1, \infty]$, by $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ we denote the space of all distributions u such that the following holds

$$(\forall \varphi \in \mathrm{C}^\infty_c(\mathbf{R}^d)) \quad \varphi u \in \mathrm{L}^p(\mathbf{R}^d) .$$

Actually, $\mathrm{C}_c^\infty(\mathbf{R}^d)$ can be reduced to $\mathcal{G}$, its subset such that $(\forall \mathbf{x} \in \mathbf{R}^d)(\exists \varphi \in \mathcal{G})$ Re $\varphi(\mathbf{x}) > 0$, which can be chosen to be countable. We endow it with the locally convex topology induced by a family of seminorms $|\cdot|_{\varphi,p}$ (for $\varphi \in \mathcal{G}$)

$$|u|_{\varphi,p} := \|\varphi u\|_{\mathrm{L}^p(\mathbf{R}^d)} .$$

It can be shown that neither the definition of $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ nor its topology depend on the choice of family $\mathcal{G}$ with the above property. This definition is equivalent to a definition where one requires that $\mathrm{L}^p_{\mathrm{loc}}$ functions have finite L^p norms over every compact subset of $\mathbf{R}^d$ (indeed, we can take $\mathcal{G}$ to consist of all characteristic functions χ_K of compacts $K \subseteq \mathbf{R}^d$ and notice that smoothness is not actually needed for the definition of $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ space).

We say that a sequence (u_n) is bounded in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ if for every seminorm $|\cdot|_{\varphi,p}$ there exists $C_{\varphi,p} > 0$ such that $|u_n|_{\varphi,p} < C_{\varphi,p}$ uniformly in n. By choosing a countable $\mathcal{G} = \{\vartheta_l \;:\; l \in \mathbf{N}\}$ such that $0 \leqslant \vartheta_l \leqslant 1$ and $\chi_{K_l} \leqslant \vartheta_l \leqslant \chi_{K_{l+1}}$, where $K_l := \mathrm{K}[0, l] \subseteq \mathbf{R}^d$ is a closed ball of radius l centred around the origin, we can define a metric d_p on $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ by

$$d_p(u, v) := \sup_{l \in \mathbf{N}} 2^{-l} \frac{|u - v|_{\vartheta_l,p}}{1 + |u - v|_{\vartheta_l,p}} .$$

With this metric, $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ is a Fréchet space for each $p \in [1, \infty]$, separable for $p \in [1, \infty\rangle$ and reflexive for $p \in \langle 1, \infty\rangle$. For $p \in [1, \infty\rangle$ it is also valid that $\mathrm{C}_c^\infty(\mathbf{R}^d)$ is dense in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$; we also define its dual exponent p' such that $1/p + 1/p' = 1$, and then (for $p < \infty$) $\mathrm{L}^{p'}(\mathbf{R}^d)$ is the dual of $\mathrm{L}^p(\mathbf{R}^d)$, while $\mathrm{L}^{p'}_c(\mathbf{R}^d)$ is a subspace of $\mathrm{L}^{p'}(\mathbf{R}^d)$ consisting of all functions in that space having a compact support, equipped with the topology of strict inductive limit

$$\mathrm{L}^{p'}_c(\mathbf{R}^d) = \bigcup_{l \in \mathbf{N}} \mathrm{L}^{p'}_{K_l}(\mathbf{R}^d) ,$$

and it is the dual of $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$. Of course, we define

$$\mathrm{L}^{p'}_{K_l}(\mathbf{R}^d) := \Bigl\{ f \in \mathrm{L}^{p'}(\mathbf{R}^d) : \operatorname{supp} f \subseteq K_l \Bigr\} \sim \mathrm{L}^{p'}(K_l) ,$$

and we equip it with the $\mathrm{L}^{p'}$ norm topology. Let us just remark that we could have replaced $\mathbf{R}^d$ by any open set $\Omega \subseteq \mathbf{R}^d$ and all of the above definitions and conclusions would remain valid. For omitted proofs and further references, we refer the interested reader to [2].

Our last points regarding the notation are that by $\boxtimes$ we shall denote the tensor product, while by $\langle \cdot, \cdot \rangle$ we shall denote various duality products, always assuming that it is bilinear. In particular, this means that its relation to the L^2 scalar product (which we take to be antilinear in the first variable) is as follows

$$\langle\, f \mid g \,\rangle_{\mathrm{L}^2} = \int \overline{f(\mathbf{x})} g(\mathbf{x}) d\mathbf{x} = \langle \bar{f}, g \rangle .$$

2. Distributions of anisotropic order

Functions differentiable of order l in one variable $\mathbf{x}$, and differentiable of order m in the other variable $\mathbf{y}$, can easily be defined. This notion has been extended to Sobolev functions (see e.g. [29]). However, in the theory of distributions of finite order such distinction, up to our knowledge, has never been made before.

As the objects of our primary interest will turn out to be distributions of order zero in $\mathbf{x}$ variable, and distributions of order $m \in \mathbf{N}$ in another variable $\boldsymbol{\xi}$ (actually, $\boldsymbol{\xi} \in \mathrm{S}^{d-1}$), in this section we shall sketch such a definition for general differential manifolds X and Y, and extend the classical proofs (cf. [9, 13, 21, 35]) to this situation (having also in mind some further possible variants of H-distributions).

We shall first precisely define the spaces of test functions on the flat space and then distributions, as appropriate duals of those spaces. In the next step we shall extend these definitions to product manifolds. In fact, we shall show how the case of manifolds can be reduced to the flat space, as usual. Based on this, we shall show that most of the standard properties of distributions of finite order carry on to this case as well. While the results for distributions of (isotropic) finite order in the flat space have already been presented in Schwartz's book [35], for their extension to manifolds we shall primarily refer to Dieudonné's treatise [13] (some readers might wish to consult [20] as well).

Distributions of anisotropic order on the flat space

Let $\Omega \subseteq \mathbf{R}^d_{\mathbf{x}} \times \mathbf{R}^r_{\mathbf{y}}$ be an open set. For $l, m \in \mathbf{N}_0$, by $\mathrm{C}^{l,m}(\Omega)$ (we are aware that this notation is often used for Hölder spaces; however, we shall not use Hölder spaces in this paper at all, so this simple notation should not cause any confusion) we denote the space of functions f on Ω, such that for any $\boldsymbol{\alpha} \in \mathbf{N}^d_0$ and $\boldsymbol{\beta} \in \mathbf{N}^r_0$, if $|\boldsymbol{\alpha}| \leqslant l$ and $|\boldsymbol{\beta}| \leqslant m$, $\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f = \partial^{\boldsymbol{\alpha}}_{\mathbf{x}} \partial^{\boldsymbol{\beta}}_{\mathbf{y}} f \in \mathrm{C}(\Omega)$. Of course, by the Schwarz theorem, the order in which derivatives are taken is not important. Here we follow the convention $\mathrm{C}^{0,0}(\Omega) = \mathrm{C}(\Omega)$. By $\mathrm{C}^{\infty,m}(\Omega)$ and $\mathrm{C}^{l,\infty}(\Omega)$ we denote the intersection of the decreasing sequence of spaces $(\mathrm{C}^{k,m}(\Omega))_k$ and $(\mathrm{C}^{l,k}(\Omega))_k$, respectively, while $\mathrm{C}^{\infty,\infty}(\Omega) := \mathrm{C}^{\infty}(\Omega)$. Thus, we shall be able to consider $l, m \in \mathbf{N}_0 \cup \{\infty\}$ in the sequel. In order to make our notation simpler, on $\mathbf{N}_0 \cup \{\infty\}$ we define a partial order as an extension of the standard one together with $k \leqslant \infty$, $k \in \mathbf{N}_0 \cup \{\infty\}$.

By choosing a sequence of nested compact sets K_n in Ω, such that $\Omega = \bigcup_{n\in\mathbf{N}} K_n$ and $K_n \subseteq \operatorname{Int} K_{n+1}$ (by $\operatorname{Int} A$ we denote the interior of set A), for $f \in \mathrm{C}^{l,m}(\Omega)$ we define

$$p^{l,m}_{K_n}(f) := \max_{|\boldsymbol{\alpha}|\leqslant l, |\boldsymbol{\beta}|\leqslant m} \|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|_{\mathrm{L}^\infty(K_n)} \,. \tag{1}$$

It is clear that each $p^{l,m}_{K_n}$ is a seminorm on $\mathrm{C}^{l,m}(\Omega)$, while considering the (increasing in $n \in \mathbf{N}$) sequence of these seminorms, $\mathrm{C}^{l,m}(\Omega)$ becomes a locally convex (Hausdorff) metrisable topological vector space. This space has the topology of uniform convergence on compact sets of functions and their derivatives up to order l in $\mathbf{x}$ and m in $\mathbf{y}$.

In the classical (isotropic) case, instead of $p^{l,m}_{K_n}$ we simply take the seminorms

$$p^{l}_{K_n}(f) := \max_{|\boldsymbol{\alpha}|\leqslant l} \|\partial^{\boldsymbol{\alpha}} f\|_{\mathrm{L}^\infty(K_n)} \,, \tag{2}$$

which we shall explicitly use for comparison below.

If $l = \infty$, instead of seminorms in (1), we take the following sequence of seminorms on $\mathrm{C}^{\infty,m}(\Omega)$:

$$p^{n,m}_{K_n}(f) := \max_{|\boldsymbol{\alpha}|\leqslant n, |\boldsymbol{\beta}|\leqslant m} \|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|_{\mathrm{L}^\infty(K_n)} \,,$$

and similarly for $m = \infty$ (of course, $l = m = \infty$ reduces to the standard isotropic case of $\mathrm{C}^\infty(\Omega)$). In fact, we have the following theorem, where in the proof we follow [13, 17.1.2] (covering the isotropic case).

Theorem 1. *For $l, m \in \mathbf{N}_0 \cup \{\infty\}$ the spaces $\mathrm{C}^{l,m}(\Omega)$ are separable Fréchet spaces. More precisely, there exists a sequence of functions in $\mathrm{C}^\infty(\Omega)$, with compact supports contained in Ω, which is dense in each of the spaces $\mathrm{C}^{l,m}(\Omega)$.*

Dem. Let (f_k) be a Cauchy sequence in $\mathrm{C}^{l,m}(\Omega)$. For a fixed $(\mathbf{x}, \mathbf{y}) \in \Omega$, the sequence of numbers $f_k(\mathbf{x}, \mathbf{y})$ forms a Cauchy sequence in $\mathbf{C}$, so it converges to a number, which we denote by $f(\mathbf{x}, \mathbf{y})$. This pointwise limit f is a bounded function on each compact K_n, where (f_k) uniformly converges to f, so f is also a continuous function on each K_n, therefore on Ω as well.

The above argument can be repeated for $\partial^{(\boldsymbol{\alpha},\boldsymbol{\beta})} f_k$, with $|\boldsymbol{\alpha}| \leqslant l$ and $|\boldsymbol{\beta}| \leqslant m$, providing us with continuous functions $f^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, which are uniform limits of $\partial^{(\boldsymbol{\alpha},\boldsymbol{\beta})} f_k$ on each K_n. By the well known result on uniform limits of derivatives, this allows us to identify the derivatives $\partial^{(\boldsymbol{\alpha},\boldsymbol{\beta})} f = f^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, and then to conclude that (f_k) converges to f in $\mathrm{C}^{l,m}(\Omega)$.

In order to construct a dense set, we start with the well known fact that there is a countable set $\{u_k \in \mathrm{C}_c(\Omega) : k \in \mathbf{N}\}$ which is dense in $\mathrm{C}(\Omega)$; furthermore, let $\rho_\kappa(\mathbf{x},\mathbf{y}) := \kappa^{d+r}\rho(\kappa\mathbf{x},\kappa\mathbf{y})$ be the standard regularising sequence, where smooth $\rho \geqslant 0$ is supported in $\mathrm{K}[0,1]$. Now we can define functions $v_{k,\kappa} := \rho_\kappa * u_k$, which are well defined for large enough κ (depending on the support of u_k), and in that case are smooth and compactly supported in Ω. We claim that these functions $v_{k,\kappa}$ form a dense set $\mathcal{G}$ in any $\mathrm{C}^{l,m}(\Omega)$. For this it is enough to prove that for any $f \in \mathrm{C}^{l,m}(\Omega)$, $\varepsilon > 0$ and $n \in \mathbf{N}$ there is a $v_{k,\kappa} \in \mathcal{G}$ such that $p^{l,m}_{K_n}(f - v_{k,\kappa}) < \varepsilon$ (with an obvious change for either l or m being ∞).

Let $F \in \mathrm{C}^{l,m}(\Omega)$ be supported in $\mathrm{Int}\, K_{n+2}$ and equal to f on K_{n+1}; $f_\kappa := F * \rho_\kappa \in \mathrm{C}^\infty(\Omega)$ (for κ large enough; in fact, κ depends on n: we shall assume that $1/\kappa$ is less than the distance between K_n and the complement of K_{n+1}, and the distance between K_{n+1} and the complement of K_{n+2}, in order to limit the convolution effects only to one layer between nested compacts K_n), and we have by properties of mollifiers that for $|\boldsymbol{\alpha}| \leqslant l$ and $|\boldsymbol{\beta}| \leqslant m$ $|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}}(f_\kappa - f)| \leqslant \varepsilon/2$, for large enough κ uniformly on K_n.

On the other hand, by the density of the sequence, we can approximate F by an u_k uniformly on K_{n+1}, thus choosing k. Therefore

$$\begin{aligned}\Big|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}}(f_\kappa - v_{k,\kappa})(\mathbf{x},\mathbf{y})\Big| &\leqslant \left|\int_{K_{n+1}} \partial^{\boldsymbol{\alpha},\boldsymbol{\beta}}\rho_\kappa(\mathbf{x}-\mathbf{x}',\mathbf{y}-\mathbf{y}')\Big(F(\mathbf{x}',\mathbf{y}') - u_k(\mathbf{x}',\mathbf{y}')\Big)d\mathbf{x}'d\mathbf{y}'\right| \\ &\leqslant C \sup_{K_{n+1}} |F - u_k| \,,\end{aligned}$$

where C is the L^1 norm of $\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}}\rho_\kappa$.

Thus, for any $f \in \mathrm{C}^{l,m}(\Omega)$, $\varepsilon > 0$ and $n \in \mathbf{N}$ we can find first $\kappa \in \mathbf{N}$ such that $|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}}(f_\kappa - f)| \leqslant \varepsilon/2$ and then a $k \in \mathbf{N}$ such that $C \sup_{K_{n+1}} |F - u_k| \leqslant \varepsilon/2$, providing us with the required bound.

Q.E.D.

After fixing the notation and constructions in the proof of previous theorem, we shall only state a number of simple results we need, referring to [13, 17.1.3–5] for proofs, which can easily be modified to the anisotropic case.

Corollary 1.

a) For any fixed $g \in \mathrm{C}^{l,m}(\Omega)$, the linear mapping $f \mapsto fg$ is continuous from $\mathrm{C}^{l,m}(\Omega)$ to itself.

b) If φ_1 is a C^l differentiable function from an open set $\Omega_1 \subseteq \mathbf{R}^d$ to $\Omega_1' \subseteq \mathbf{R}^d$, and φ_2 a C^m differentiable function from an open set $\Omega_2 \subseteq \mathbf{R}^r$ to $\Omega_2' \subseteq \mathbf{R}^r$,then the linear mapping $f \mapsto f \circ (\varphi_1 \boxtimes \varphi_2)$, from $\mathrm{C}^{l,m}(\Omega_1' \times \Omega_2')$ to $\mathrm{C}^{l,m}(\Omega_1 \times \Omega_2)$ is continuous. ∎

Having just described the properties of $\mathrm{C}^{l,m}(\Omega)$, our next goal is to describe the spaces of functions with compact support, which duals will be the spaces of anisotropic distributions (on the flat space).

For a compact set $K \subseteq \Omega$ we can consider only those functions which are supported in K, and define a subspace of $\mathrm{C}^{l,m}(\Omega)$

$$\mathrm{C}^{l,m}_K(\Omega) := \Big\{ f \in \mathrm{C}^{l,m}(\Omega) : \ \mathrm{supp}\, f \subseteq K \Big\} \,.$$

This subspace inherits the topology from $\mathrm{C}^{l,m}(\Omega)$, which is, when considered only on the subspace, a norm topology determined by

$$\|f\|_{l,m,K} := p^{l,m}_K(f) \,,$$

and $\mathrm{C}^{l,m}_K(\Omega)$ is a Banach space (it can be identified with a proper subspace of the Banach space $\mathrm{C}^{l,m}(K)$). However, if $l = \infty$ or $m = \infty$ (in order to keep the notation simple, we assume that

$m = \infty$), then we shall not get a Banach space, but a Fréchet space. Note that, similarly as it was in the isotropic case, an increasing sequence of seminorms that makes $\mathrm{C}^{l,\infty}_{K_n}(\Omega)$ a Fréchet space is given by $(p^{l,k}_{K_n}), k \in \mathbf{N}_0$. Throughout the rest of this section, we shall consider $m \in \mathbf{N}_0 \cup \{\infty\}$, unless explicitly stated otherwise.

Of particular importance is the following subspace of $\mathrm{C}^{l,m}(\Omega)$

$$\mathrm{C}^{l,m}_c(\Omega) := \bigcup_{n\in\mathbf{N}} \mathrm{C}^{l,m}_{K_n}(\Omega) ,$$

consisting of all functions with compact support in $\mathrm{C}^{l,m}(\Omega)$, and equipped by a stronger topology than the one inherited from $\mathrm{C}^{l,m}(\Omega)$: by the topology of *strict inductive limit*. More precisely, it can easily be checked that

$$\mathrm{C}^{l,m}_{K_n}(\Omega) \hookrightarrow \mathrm{C}^{l,m}_{K_{n+1}}(\Omega) ,$$

the inclusion being continuous. Also, the topology induced on $\mathrm{C}^{l,m}_{K_n}(\Omega)$ by that of $\mathrm{C}^{l,m}_{K_{n+1}}(\Omega)$ coincides with the original one, and $\mathrm{C}^{l,m}_{K_n}(\Omega)$ (as a Banach space in that topology, or a Fréchet space for $m = \infty$) is a closed subspace of $\mathrm{C}^{l,m}_{K_{n+1}}(\Omega)$. Then we have that the inductive limit topology on $\mathrm{C}^{l,m}_c(\Omega)$ induces on each $\mathrm{C}^{l,m}_{K_n}(\Omega)$ the original topology, while a subset of $\mathrm{C}^{l,m}_c(\Omega)$ is bounded if and only if it is contained in one $\mathrm{C}^{l,m}_{K_n}(\Omega)$, and bounded there [9, Theorem 1.3]. This space is also complete.

After defining the spaces of test functions, we can consider their duals, and finally define the objects in the title of this paper:

Definition. Any continuous linear functional on $\mathrm{C}^{l,m}_c(\Omega)$ we call a *distribution of anisotropic order*, and such functionals form a vector space $\mathcal{D}'_{l,m}(\Omega) = (\mathrm{C}^{l,m}_c(\Omega))'$.
Similarly, any continuous linear functional on $\mathrm{C}^{l,m}(\Omega)$ we call a *distribution of anisotropic order with compact support*, and such functionals form a vector space $\mathcal{E}'_{l,m}(\Omega) = (\mathrm{C}^{l,m}(\Omega))'$.

If we define the support of a distribution from $\mathcal{D}'_{l,m}(\Omega)$ in the usual way, the distributions which have compact support can be extended to continuous functionals on $\mathrm{C}^{l,m}(\Omega)$, as in the standard case, thus justifying the name.

In the latter case we have a Fréchet topology on the base space, while in the former it is the topology of strict inductive limit, which leads to the following characterisation of continuity: a linear functional T on $\mathrm{C}^{l,m}_c(\Omega)$ is continuous (at zero) if and only if its restriction to any space $\mathrm{C}^{l,m}_K(\Omega)$, where $K \subseteq \Omega$ is a compact (denoted also by $K \in \mathcal{K}(\Omega)$ below), is continuous (at zero), meaning in fact that

$$(\forall\, K \in \mathcal{K}(\Omega))(\exists\, C > 0)(\forall\, \varphi \in \mathrm{C}^{l,m}_c(\Omega)) \quad \operatorname{supp}\varphi \subseteq K \quad\Longrightarrow\quad |\langle T, \varphi\rangle| \leqslant C p^{l,m}_K(\varphi) .$$

If either l or m is infinite, we have to modify the above in the obvious way.

We can equip the duals with the weak∗ topology of the dual pair, and this will suffice for many applications. However, in order to study further properties of these spaces, we have to equip them with the *strong topology*, the topology of uniform convergence on bounded sets in $\mathrm{C}^{l,m}_c(\Omega)$. Our spaces have the standard properties (like completeness) as derived in [35], having in mind that, except in the case when $l = m = \infty$, the bounded sets are not necessarily relatively compact in $\mathrm{C}^{l,m}_c(\Omega)$, so the spaces are neither Montel nor reflexive.

Clearly, for $l \geqslant k$ and $m \geqslant n$ we have continuous and dense embedding $\mathrm{C}^{l,m}(\Omega) \hookrightarrow \mathrm{C}^{k,n}(\Omega)$, as well as $\mathrm{C}^{l,m}_c(\Omega) \hookrightarrow \mathrm{C}^{k,n}_c(\Omega)$. In particular, we have that $\mathrm{C}^{\infty}_c(\Omega) \hookrightarrow \mathrm{C}^{l,m}_c(\Omega)$ is a continuous and dense imbedding, as well as that $\mathrm{C}^{l,m}(\Omega) \hookrightarrow \mathrm{C}(\Omega)$, and $\mathrm{C}^{l,m}_c(\Omega) \hookrightarrow \mathrm{C}_c(\Omega)$ ($\mathrm{C}_c(\Omega)$ being the space of continuous functions with compact support, having as a dual the space of Radon measures, i.e. the distributions of order zero).

On the other hand, by using the Lebesgue measure on Ω, we can identify locally summable functions $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ (which subsumes $\mathrm{C}_c(\Omega)$) with some distributions of order zero by the usual formula

$$\varphi \mapsto \int_\Omega f(\mathbf{x})\varphi(\mathbf{x})\,d\mathbf{x}\,. \tag{3}$$

In this way we get continuous and dense embeddings $\mathrm{C}^\infty_c(\Omega) \hookrightarrow \mathrm{C}^{l,m}_c(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$, thus $\mathrm{C}^{l,m}_c(\Omega)$ is a normal space of distributions, hence its dual $\mathcal{D}'_{l,m}(\Omega)$ forms a subspace of $\mathcal{D}'(\Omega)$, when equipped with strong topology.

Distributions of anisotropic order on manifolds

On the flat space we have defined distributions as continuous linear functionals on the space of test functions, and then identified each function as a (regular) distribution by formula (3), where the integration was with respect to the Lebesgue measure on Ω. Unfortunately, this cannot be done on an arbitrary manifold (cf. [20, Sect. 3.1.1] for a nice explanation). Functions cannot be identified as regular distributions, as there is no standard measure or an a priori notion of integration on a general manifold. We have to change the space of test functions, or the space of regular distributions, or both. Following [13] we shall keep the notion of test functions, and the distributions as elements of its dual, but for regular distributions we shall use smooth *densities*.

In this paper by a *differential manifold* we always consider a locally Euclidean (of the fixed dimension, i.e. pure) second countable Hausdorff topological space on which an equivalence class of C^∞ smooth atlases is given. In particular, our definition implies that every differentiable manifold X is paracompact, hemicompact (i.e. admits a compact exhaustion), separable, metrisable, and for any open cover of X there exists a smooth partition of unity subordinate to this cover [13, 16.1–4].

The above constructions of anisotropic function spaces on $\mathbf{R}^d \times \mathbf{R}^r$ can easily be transferred to manifolds. Indeed, let X and Y be differentiable manifolds of dimensions d and r. Then $X\times Y$ is also a smooth manifold of dimension $d+r$ [13, Section 16.6], and if (U,φ) and (V,ψ) are charts of X and Y, respectively, then $(U\times V, \varphi\boxtimes\psi)$ is a chart of $X\times Y$. Let $\Omega \subseteq X\times Y$ be an open subset. For $l,m \in \mathbf{N}_0\cup\{\infty\}$, by $\mathrm{C}^{l,m}(\Omega)$ we denote the space of continuous functions $f:\Omega\to\mathbf{C}$ such that for any chart $(U\times V,\varphi\boxtimes\psi)$ of $X\times Y$ the mapping $f|_{U\times V}\circ(\varphi^{-1}\boxtimes\psi^{-1})$ is contained in $\mathrm{C}^{l,m}(\varphi(U)\times\psi(V))$ [13, 16.3]. Now we introduce a locally convex topology on $\mathrm{C}^{l,m}(\Omega)$ in the same manner as it was done in [13, 17.2] for more general C^∞-sections, here with a simplification in the notation, as we consider only the trivial complex line bundle $(X\times Y)\times\mathbf{C}$ in the codomain of sections.

Let $(U_i\times V_j, \varphi_i\boxtimes\psi_j)$ be at most countable family of charts on $X\times Y$ such that $(U_i\times V_j)$ form a locally finite open cover of Ω. For further reference, we shall denote these sets of indices by $\mathcal{I}$ and $\mathcal{J}$ (so $i\in\mathcal{I}$, and $j\in\mathcal{J}$), and their Cartesian product by $\mathcal{H}$ (thus $(i,j)\in\mathcal{H}$). Furthermore, the *families* of all their finite subsets we shall denote by $\mathcal{I}^{fin}$, $\mathcal{J}^{fin}$ and $\mathcal{H}^{fin}$, respectively.

For any $i\in\mathcal{I}$ and $j\in\mathcal{J}$ let us by $K^{i,j}_n$ denote a sequence of compacts in $\varphi_i(U_i)\times\psi_j(V_j)$ such that $\varphi_i(U_i)\times\psi_j(V_j) = \bigcup_{n\in\mathbf{N}} K^{i,j}_n$ and $K^{i,j}_n \subseteq \operatorname{Int} K^{i,j}_{n+1}$. Then $(p^{l,m}_{K^{i,j}_n})$, given by (1), is an increasing sequence of seminorms on $\mathrm{C}^{l,m}(\varphi_i(U_i)\times\psi_j(V_j))$ (here we limit ourselves to the case when $l,m\neq\infty$). Further on, for $f\in\mathrm{C}^{l,m}(\Omega)$ we define

$$p^{l,m;i,j}_n(f) := p^{l,m}_{K^{i,j}_n}\left(f|_{U_i\times V_j}\circ(\varphi_i^{-1}\boxtimes\psi_j^{-1})\right)\,. \tag{4}$$

If $l=\infty$ or $m=\infty$, one proceeds as in the Euclidean setting. It is clear that each element of a countable family $(p^{l,m;i,j}_n)_{i,j,n}$ is a seminorm, while the family is total (it separates points). Thus, this family of seminorms defines a locally convex Hausdorff topology on $\mathrm{C}^{l,m}(\Omega)$. By the definition, a sequence of functions (f_k) converges to 0 in $\mathrm{C}^{l,m}(\Omega)$ if and only if for each i,j the sequence of restrictions $(f_k|_{U_i\times V_j})$ converges to 0 in $\mathrm{C}^{l,m}(U_i\times V_j)$, which is isomorphic to the Fréchet space $\mathrm{C}^{l,m}(\varphi_i(U_i)\times\psi_j(V_j))$, consisting of functions defined on the flat space.

It can easily be seen that the introduced topology does not depend on the choice of a countable locally finite family of charts $(U_i \times V_j, \varphi_i \boxtimes \psi_j)$ and a sequence of compacts $K_n^{i,j}$. Indeed, the following lemma holds.

Lemma 1. *A sequence of functions (f_k) converges to 0 in $\mathrm{C}^{l,m}(\Omega)$ if and only if for any chart $(U \times V, \varphi \boxtimes \psi)$ intersecting Ω, any compact $K \subseteq \varphi(U) \times \psi(V)$, and multiindices $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ and $\boldsymbol{\beta} \in \mathbf{N}_0^r$, such that $|\boldsymbol{\alpha}| \leqslant l$, $|\boldsymbol{\beta}| \leqslant m$, the sequence $\Big(\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}}\Big(f_k \circ (\varphi^{-1} \boxtimes \psi^{-1})\Big)|_K\Big)$ converges uniformly to 0.*

Dem. By taking φ_i, ψ_j and $K_n^{i,j}$, for φ, ψ and K, respectively, we trivially establish the proof of the reverse implication of the lemma, while the proof of the direct implication is completely analogous to the isotropic case (see [13, 17.2]). Indeed, the local finitness of $(U_i \times V_j, \varphi_i \boxtimes \psi_j)$ implies that only finitely many charts intersect $(\varphi^{-1} \boxtimes \psi^{-1})(K)$, and for each such chart by writing $f_k \circ (\varphi^{-1} \boxtimes \psi^{-1}) = f_k \circ (\varphi_i^{-1} \boxtimes \psi_j^{-1}) \circ (\varphi_i \circ \varphi^{-1} \boxtimes \psi_j \circ \psi^{-1})$ we conclude, after applying the smoothness of transition maps $\varphi_i \circ \varphi^{-1}$ and $\psi_j \circ \psi^{-1}$, and Corollary 1(b).

Q.E.D.

Since the topology is independent of the choice of charts and compacts, for $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 \subseteq \mathbf{R}^d$ and $\Omega_2 \subseteq \mathbf{R}^r$ are open sets, we can take a single chart (Ω, id) to define seminorms, where id here denotes the identity map. Thus, definitions given in this and the preceding section coincide for $\Omega = \Omega_1 \times \Omega_2$. The same could be concluded even when Ω is not a Cartesian product, but then (Ω, id) is not an eligible chart as we need to avoid mixing of first d and last r variables, so one might have to work with possibly countably many charts in (4).

Now we can generalise Theorem 1 to differentiable manifolds, in the same vein as it was done in [13, 17.2.2] for the isotropic case.

Theorem 2. *Let X, Y, Ω, and $p_n^{l,m;i,j}$ be as above. For $l, m \in \mathbf{N}_0 \cup \{\infty\}$ the spaces $\mathrm{C}^{l,m}(\Omega)$ are separable Fréchet spaces. More precisely, there exists a sequence of functions in $\mathrm{C}^\infty(\Omega)$, with compact supports contained in Ω, which is dense in each of the spaces $\mathrm{C}^{l,m}(\Omega)$.*

Dem. The proof follows the same steps as in [13, 17.2.2], but for the clarity of exposition, we shall try to concisely sketch the main ideas.

Let (f_k) be a Cauchy sequence in $\mathrm{C}^{l,m}(\Omega)$. Then for any (i,j) the sequence $(f_k|_{U_i \times V_j})$ is also a Cauchy sequence in $\mathrm{C}^{l,m}(U_i \times V_j)$, which is isomorphic to $\mathrm{C}^{l,m}(\varphi_i(U_i) \times \psi(V_j))$. Thus, applying Theorem 1, for any i, j we get $f^{i,j} \in \mathrm{C}^{l,m}(U_i \times V_j)$, being limits of the above sequence of restrictions. On the intersections of charts, $f^{i,j}$ coincide. Indeed, let $\mathbf{x} \in (U_i \times V_j) \cap (U_{i'} \times V_{j'}) \neq \emptyset$, then

$$|f^{i,j}(\mathbf{x}) - f^{i',j'}(\mathbf{x})| \leqslant p_n^{l,m;i,j}(f^{i,j} - f_k) + p_n^{l,m;i'j'}(f^{i',j'} - f_k)\,,$$

and similarly if $l = \infty$ or $m = \infty$. Since the right hand side is arbitrarily small (for k large enough), we have the claim. Thus, the family of functions $f^{i,j}$ defines $f \in \mathrm{C}^{l,m}(\Omega)$, which is the limit of (f_k).

By Theorem 1 for each i, j there exists a sequence of smooth functions $(f_k^{i,j})$, with $\operatorname{supp} f_k^{i,j} \subseteq U_i \times V_j$, which is dense in $\mathrm{C}^{l,m}(U_i \times V_j)$. In order to have a good behaviour on the intersections of charts, we consider the set of all finite linear combinations of $(\chi_i \boxtimes \theta_j) f_k^{i,j}$, where $\chi_i \boxtimes \theta_j$ is a partition of unity on $X \times Y$ subordinate to $(U_i \times V_j, \varphi_i \boxtimes \psi_j)$. This set is dense in $\mathrm{C}^{l,m}(\Omega)$ by virtue of Corollary 1(a). Indeed, for any compact $K \subseteq \Omega$ and $f \in \mathrm{C}^{l,m}(\Omega)$, $f|_K = (\sum_{(i,j)\in H} (\chi_i \boxtimes \theta_j) f)|_K$, where $H \in \mathcal{H}^{fin}$, so we are left with approximating each (of finitely many) $(\chi_i \boxtimes \theta_j) f$, which is obviously possible in the above set.

Q.E.D.

Having that $\mathrm{C}^{l,m}(\Omega)$ is a Fréchet space, the same construction of the strict inductive limit presented in the previous subsection applies here as well. Thus, for a compact $K \subseteq \Omega$,

$$\mathrm{C}_K^{l,m}(\Omega) := \Big\{ f \in \mathrm{C}^{l,m}(\Omega) :\ \operatorname{supp} f \subseteq K \Big\}$$

is a closed subspace of $\mathrm{C}^{l,m}(\Omega)$, hence a Fréchet space. Since K intersects only finitely many chart domains, the topology on $\mathrm{C}^{l,m}_K(\Omega)$ can be descried by the following increasing sequence of norms

$$\max_{(i,j)\in H} p_n^{l',m';i,j} \,,$$

where $H \in \mathcal{H}^{fin}$ is the finite set of indices (i,j) for which the intersection of K and $U_i \times V_j$ (a chart domain) is non-empty, $n \in \mathbf{N}$ is large enough such that $K \subseteq \bigcup_{(i,j)\in H} (\varphi_i^{-1} \boxtimes \psi_j^{-1})(K_n^{i,j})$, where compacts $K_n^{i,j}$ are defined in (4), and $l', m' \in \mathbf{N}_0$ are less (or equal) than l, m respectively. If $l, m \neq \infty$, then it is sufficient to consider only one norm for $l' = l$ and $m' = m$, implying that $\mathrm{C}^{l,m}_K(\Omega)$ is a Banach space.

For the construction of the topology of strict inductive limit on $\mathrm{C}^{l,m}_c(\Omega)$ (the space of all functions with compact support in $\mathrm{C}^{l,m}(\Omega)$) we take a compact exhaustion, i.e. a sequence of compact sets K_s in Ω, such that $\Omega = \bigcup_{s\in\mathbf{N}} K_s$ and $K_s \subseteq \mathsf{Int}\, K_{s+1}$ for any $s \in \mathbf{N}$. Since

$$\mathrm{C}^{l,m}_c(\Omega) = \bigcup_{s\in\mathbf{N}} \mathrm{C}^{l,m}_{K_s}$$

and, for any $s \in \mathbf{N}$, $\mathrm{C}^{l,m}_{K_s}$ is a closed subspace of $\mathrm{C}^{l,m}_{K_{s+1}}$, the strict inductive limit topology on $\mathrm{C}^{l,m}_c(\Omega)$ determined by the sequence of spaces $\mathrm{C}^{l,m}_{K_s}$ exists, so we equip $\mathrm{C}^{l,m}_c(\Omega)$ with this topology. The space $\mathrm{C}^{l,m}_c(\Omega)$ is then a complete locally convex Hausdorff topological vector space, while $f_k \to f$ in $\mathrm{C}^{l,m}_c(\Omega)$ if and only if there exists $s \in \mathbf{N}$ such that the supports of all f_n are contained in K_s and $f_k \to f$ in $\mathrm{C}^{l,m}_{K_s}(\Omega)$. Moreover, the strict inductive limit topology on $\mathrm{C}^{l,m}_c(\Omega)$ induces on each $\mathrm{C}^{l,m}_{K_s}$ the original Fréchet topology [28, Chapter 12.1].

For dual spaces we use the same notation as in the flat space. Thus, the dual of $\mathrm{C}^{l,m}_c(\Omega)$, i.e. the space of continuous linear functionals on $\mathrm{C}^{l,m}_c(\Omega)$ with respect to the topology of strict inductive limit, is denoted by $\mathcal{D}'_{l,m}(\Omega) = (\mathrm{C}^{l,m}_c(\Omega))'$, while of $\mathrm{C}^{l,m}(\Omega)$ (with respect to the Fréchet topology) by $\mathcal{E}'_{l,m}(\Omega) = (\mathrm{C}^{l,m}(\Omega))'$. Elements of $\mathcal{D}'_{l,m}(\Omega)$ we call *distributions of anisotropic order*. Since $\mathrm{C}^\infty_c(\Omega) \hookrightarrow \mathrm{C}^{l,m}_c(\Omega)$, any distribution of anisotropic order is a distribution, i.e. an element of the space $\mathcal{D}'(\Omega) := (\mathrm{C}^\infty_c(\Omega))'$, and analogously $\mathcal{E}'_{l,m}(\Omega) \subseteq \mathcal{E}'(\Omega) := (\mathrm{C}^\infty(\Omega))'$. In particular, any element of $\mathcal{E}'_{l,m}(\Omega)$ has a compact support (where the support of a distribution is defined in the usual way using the local nature of distributions provided by Theorem 3 below), while it can easily be shown that the reverse implication also holds in the sense that any distribution of anisotropic order with compact support can be extended in the unique way (the uniqueness is due to Theorem 2) to a continuous functional on $\mathrm{C}^{l,m}(\Omega)$, thus elements of $\mathcal{E}'_{l,m}(\Omega)$ are called *distributions of anisotropic order with compact support*.

By definition of the strict inductive limit topology, it is immediate that $u \in \mathcal{D}'_{l,m}(\Omega)$ if and only if for any $s \in \mathbf{N}$ the restriction of u to $\mathrm{C}^{l,m}_{K_s}(\Omega)$ is continuous, and as (K_s) is a compact exhaustion we can generalise this statement by replacing (K_s) by the family of all compact sets in Ω. Therefore, by the above discussed properties of spaces $\mathrm{C}^{l,m}_K(\Omega)$ we have the following corollary (v. [13, 17.3] for the isotropic case, but for general currents).

Corollary 2. *Let X, Y, Ω, and $p_n^{l,m;i,j}$ be as above (for $l, m \in \mathbf{N}_0 \cup \{\infty\}$). For a linear functional u on $\mathrm{C}^{l,m}_c(\Omega)$, the following statements are equivalent*

a) $u \in \mathcal{D}'_{l,m}(\Omega)$;

b) For every sequence (φ_k) converging to zero in $\mathrm{C}^{l,m}_c(\Omega)$ the scalar sequence $(\langle u, \varphi_k\rangle)$ converges to zero;

c) $(\forall\, K \in \mathcal{K}(\Omega))(\exists\, C > 0)(\exists\, n \in \mathbf{N}_0)(\exists\, H \in \mathcal{H}^{fin})(\exists\, l' \in 0..l)(\exists\, m' \in 0..m)(\forall\, \Psi \in \mathrm{C}^{l,m}_K(\Omega))$

$$|\langle u, \Psi\rangle| \leqslant C \max_{(i,j)\in H} p_n^{l',m';i,j}(\Psi) \,, \tag{5}$$

where for brevity by $0..l$ we denote the set $\{l' \in \mathbf{N}_0 : l' \leqslant l\}$ (and similarly we shall do for $1..l$).

∎

If in (5) $l', m' \in \mathbf{N}_0$ are independent of K, we say that u is of (anisotropic) order (in fact, of order at most) (l', m'). We extend this definition allowing for the value ∞ in the case when such l' or m' (or both) does not exist. Thus, $u \in \mathcal{D}'_{l,m}(\Omega)$ is of order at most (l, m). Anisotropic distributions of order (l, m) correspond to distributions of order (at most) $l + m$, hence orders (l, ∞), (∞, m), (∞, ∞) (for $l, m \in \mathbf{N}_0$) provide the infinite order in the setting of (classical isotropic) distributions.

Using this notion of order of anisotropic distributions, one could equivalently define the space $\mathcal{D}'_{l,m}(\Omega)$ as the space of all (classical) distributions which are of order (l, m). Indeed, such objects can then be uniquely extended to continuous linear functionals on $\mathrm{C}_c^{l,m}(\Omega)$ by Theorem 2. However, the possible extension of the results to manifolds which are not (C^∞) smooth have determined our choice above (cf. Remark 1 below).

On each chart domain we can transfer a distribution to the flat space. Indeed, let $u \in \mathcal{D}'_{l,m}(\Omega)$. Then for any i, j the restriction $u_{i,j} := u|_{U_i \times V_j}$ given by

$$\langle u_{i,j}, \Psi \rangle = \langle u, \Psi \rangle \ , \quad \Psi \in \mathrm{C}_c^{l,m}(U_i \times V_j) \, ,$$

is a continuous linear functional on $\mathrm{C}_c^{l,m}(U_i \times V_j)$, i.e. $u_{i,j} \in \mathcal{D}'_{l,m}(U_i \times V_j)$. Thus, the pushforward $\tilde{u}_{i,j} := (\varphi_i \boxtimes \psi_j)_* u_{i,j}$ defined by

$$\langle \tilde{u}_{i,j}, \Psi \rangle = \langle u, \Psi \circ (\varphi_i \boxtimes \psi_j) \rangle \ , \quad \Psi \in \mathrm{C}_c^{l,m}\Big(\varphi_i(U_i) \times \psi_j(V_j)\Big) \, ,$$

belongs trivially to the space $\mathcal{D}'_{l,m}(\varphi_i(U_i) \times \psi_j(V_j))$.

Conversely, if we are given a family of distributions $u_{i,j}$ on chart domains, or equivalently a family of distributions $\tilde{u}_{i,j}$ on the flat space, the question is whether they define a distribution on Ω. The answer to this question is given by the well-known localisation/unification principle [35, Ch. I, Théorème IV (*Principe du recollement des morceaux*)] (for the manifolds see [13, 17.4.2]). The proof relies on the existence of a partition of unity and a consequence of Corollary 1(a) that for any $u \in \mathcal{D}'_{l,m}(\Omega)$ and $\Psi \in \mathrm{C}^{l,m}(\Omega)$ the product $\Psi u : \mathrm{C}_c^{l,m}(\Omega) \to \mathbf{R}$ given by

$$\langle \Psi u, \Phi \rangle = \langle u, \Psi \Phi \rangle \ , \quad \Phi \in \mathrm{C}_c^{l,m}(\Omega) \, ,$$

belongs to the space $\mathcal{D}'_{l,m}(\Omega)$.

As the proof follows the same pattern as in the classical isotropic case, we present only the precise statement of this result.

Theorem 3. *Let X and Y be differential manifolds, and let $(\Omega_\alpha, \alpha \in A)$ be a family of open sets, such that $\bigcup \Omega_\alpha = \Omega \subseteq X \times Y$. Further, let $T_\alpha \in \mathcal{D}'_{l,m}(\Omega_\alpha)$ for each α, in such a way that for $\Omega_\alpha \cap \Omega_\beta \neq \emptyset$, distributions T_α and T_β coincide on this intersection.*

Then there is a unique distribution $T \in \mathcal{D}'_{l,m}(\Omega)$ which, for any $\alpha \in A$, coincides with T_α on Ω_α.

∎

Let us close this subsection with some comments on the identification of functions with distributions. In the case when $X \times Y$ is an oriented manifold, or equivalently, if both X and Y are oriented manifolds, we have a canonical way how to integrate differential n-forms (cf. [13, 16.24]). Furthermore, by fixing a non-vanishing differential n-form v_0 belonging to the orientation of $X \times Y$, for any locally integrable function f on $\Omega \subseteq X \times Y$ we define the mapping on $\mathrm{C}_c^{l,m}(\Omega)$ by

$$\varphi \mapsto \int_\Omega f \varphi v_0 \, ,$$

which is clearly a distribution of anisotropic order. This provides an embedding $\mathrm{L}^1_{\mathrm{loc}}(\Omega) \hookrightarrow \mathcal{D}'_{l,m}(\Omega)$, which was previously commented on the flat space.

However, if $X \times Y$ is not an oriented manifold, we cannot integrate differential n-forms, but only the *densities*, which are sections of the volume bundle. Thus, we cannot identify functions with distributions, but only with locally integrable densities, hence regular distributions are densities (and not functions). A nice exposition of the theory of distributions in this general setting can be found in [20, Chapter 3]. Let us just remark that our notion of (anisotropic) distributions corresponds to *distributional densities* (or more generally to *section distribution densities*) in [20, Definition 3.1.4].

Tensor products

For differential manifolds X and Y, it is straightforward to see that for a linear functional u on $\mathrm{C}_c^{l,m}(X\times Y)$, statement (c) of Corollary 2 implies:

$$
\begin{aligned}
&(\forall K\in\mathcal{K}(X))(\forall L\in\mathcal{K}(Y))(\exists C>0)(\exists n\in\mathbf{N}_0)(\exists I\in\mathcal{I}^{fin})(\exists J\in\mathcal{J}^{fin})\\
&\qquad(\exists l'\in 0..l)(\exists m'\in 0..m)(\forall\varphi\in\mathrm{C}_K^l(X))(\forall\psi\in\mathrm{C}_L^m(Y))\\
&\qquad\qquad|\langle u,\varphi\boxtimes\psi\rangle|\leqslant C\max_{i\in I}p_n^{l';i}(\varphi)\max_{j\in J}p_n^{m';j}(\psi)\ ,
\end{aligned}\tag{6}
$$

where $p_n^{l';i}$ and $p^{m';j}$ are the projections of $p_n^{l',m';i,j}$ (given by (4)) to the first and second variable respectively. More precisely, $p_n^{l';i}(f)=p^{l'}_{\pi_1(K_n^{i,j})}(f|_{U_i}\circ\varphi_i^{-1})$, where $\pi_1:X\times Y\to X$ is the projection to the first argument (for the definition of p_K^l see (2)), and analogously for $p_n^{m';i}$.

The reverse implication would have significantly greater practical use but, as we shall see at the end of this section, it fails to hold in general.

We should first develop some properties of tensor products of test functions.

Lemma 2. *Let X and Y be differential manifolds, $K,K'\subseteq X$ compacts such that $K\subseteq\operatorname{Int}K'$, and similarly $L,L'\subseteq Y$ compacts with $L\subseteq\operatorname{Int}L'$. Then any $f\in\mathrm{C}^{l,m}_{K\times L}(X\times Y)$ can be approximated by a sequence of functions from $\mathrm{C}^\infty_{K'}(X)\boxtimes\mathrm{C}^\infty_{L'}(Y)$ in the topology of $C^{l,m}_{K'\times L'}(X\times Y)$.*

Dem. As $K'\times L'$ is not necessarily subordinate to only one chart, we shall use a partition of unity on $X\times Y$, subordinate to the product atlas $(U_i\times V_j,\varphi_i\boxtimes\psi_j)$. In fact, we shall take a partition of unity (χ_i) subordinate to (U_i) on X, and (θ_j) subordinate to (V_j) on Y. As K' and L' are compacts, they can be covered by only a finite number of U_i-s, say for $i\in I$, and V_j-s, for $j\in J$, respectively. Therefore we have

$$
f=\sum_{i\in I}\sum_{j\in J}f(\chi_i\boxtimes\theta_j)=\sum_{i\in I}\sum_{j\in J}f_{i,j}\ .
$$

By this we have reduced the problem to functions

$$
f_{i,j}=f(\chi_i\boxtimes\theta_j)\in\mathrm{C}^{l,m}_{(K\cap\operatorname{supp}\chi_i)\times(L\cap\operatorname{supp}\theta_j)}(X\times Y)\ .
$$

If we show that each of them can be approximated by a sequence of finite linear combinations of tensor products, then the same will be valid for the finite sum of them.

Let us first notice that we can find compacts G and H such that

$$
\begin{aligned}
K\cap\operatorname{supp}\chi_i&\subseteq\operatorname{Int}G\subseteq G\subseteq\operatorname{Int}K'\cap U_i\\
L\cap\operatorname{supp}\theta_j&\subseteq\operatorname{Int}H\subseteq H\subseteq\operatorname{Int}L'\cap V_j\ .
\end{aligned}
$$

Now we have to reduce the argument to the one on the flat space, by using the charts. We define

$$
\tilde f_{i,j}:=f_{i,j}\circ(\varphi_i^{-1}\boxtimes\psi_j^{-1})\ ,
$$

which is supported in the compact $\varphi_i(K\cap\operatorname{supp}\chi_i)\times\psi_j(L\cap\operatorname{supp}\theta_j)\subseteq\operatorname{Int}\varphi_i(G)\times\operatorname{Int}\psi_j(H)$. At this point we can apply the result for the flat space (which was essentially carried out in [13, 17.10.2]), obtaining the approximation supported in $\varphi_i(G)\times\psi_j(H)$. These approximations can be pulled back to the manifold $X\times Y$, as $G\subseteq U_i$ and $H\subseteq V_j$, and by (4) the approximation remains valid. Extending these functions by zero to the whole manifold we get the required approximation for $f_{i,j}$.

Q.E.D.

The above lemma has an important consequence: a distribution $u \in \mathcal{D}'_{l,m}(X \times Y)$ is uniquely determined by its values on tensor products. It allows us to define the tensor product of two distributions, as the unique distribution given by the following theorem (cf. [13, 17.10.3]).

Theorem 4. *Let X and Y be differential manifolds, $u \in \mathcal{D}'_l(X)$ and $v \in \mathcal{D}'_m(Y)$. Then*

$$\Big(\exists! w \in \mathcal{D}'_{l,m}(X \times Y)\Big)\Big(\forall \varphi \in \mathrm{C}^l_c(X)\Big)\Big(\forall \psi \in \mathrm{C}^m_c(Y)\Big) \quad \langle w, \varphi \boxtimes \psi\rangle = \langle u, \varphi\rangle\langle v, \psi\rangle.$$

Furthermore, for any test function $\Phi \in \mathrm{C}^{l,m}_c(X \times Y)$, function $V_\Phi : \mathbf{x} \mapsto \langle v, \Phi(\mathbf{x}, \cdot)\rangle$ is in $\mathrm{C}^l_c(X)$, while $U_\Phi :\ \mathbf{y} \mapsto \langle u, \Phi(\cdot, \mathbf{y})\rangle$ is in $\mathrm{C}^m_c(Y)$, and we have that

$$\langle w, \Phi\rangle = \langle u, V_\Phi\rangle = \langle v, U_\Phi\rangle\,.$$

Dem. The uniqueness is clear from Lemma 2.

Let us highlight the main steps, paying the attention to the differences from the anisotropic case.

Firstly, for Φ of the form $\varphi \boxtimes \psi$, where $\varphi \in \mathrm{C}^l_c(X)$ and $\psi \in \mathrm{C}^m_c(Y)$, we get $V_{\varphi\boxtimes\psi}(\mathbf{x}) = \varphi(\mathbf{x})\langle v, \psi\rangle$ and $\langle u, V_{\varphi\boxtimes\psi}\rangle = \langle u, \varphi\rangle\langle v, \psi\rangle =: \langle w, \Phi\rangle$. Thus, it is sufficient to prove that the linear mapping $\Phi \mapsto \langle u, V_\Phi\rangle$ is continuous on $\mathrm{C}^{l,m}_c(X \times Y)$, which is equivalent to the continuity on $\mathrm{C}^{l,m}_M(X \times Y)$ for any compact set M in $X \times Y$. By Theorem 3 (or applying the same argument as in the proof of the previous lemma), we can assume that M is contained in a chart of $X \times Y$, hence, applying the pushforward, we are left to consider only the flat case, i.e. X and Y open sets in $\mathbf{R}^d$ and $\mathbf{R}^r$.

From [13, 17.10.1] it follows that for any $\Phi \in \mathrm{C}^{l,m}_M(X \times Y)$, we have $V_\Phi \in \mathrm{C}^l_{\pi_1(M)}(X)$. Let us take a sequence (Φ_j) converging to zero in $\mathrm{C}^{l,m}_M(X\times Y)$. Since v is a distribution, the corresponding (V_j) converges to zero uniformly on $\pi_1(M)$. Noticing that $\partial^{\boldsymbol\alpha}_{\mathbf{x}} V_j(\mathbf{x}) = \langle v, \partial^{\boldsymbol\alpha}_{\mathbf{x}}\Phi_j(\mathbf{x}, \cdot)\rangle$, for $|\boldsymbol\alpha| \leqslant l$, we conclude that derivatives of V_j up to order l converge to zero uniformly on $\pi_1(M)$ as well. From this, we get that $\langle u, V_j\rangle$ converges to zero. Thus, by Corollary 2(b) we get the claim.

Q.E.D.

The distribution $w \in \mathcal{D}'_{l,m}(X \times Y)$ is called the *tensor product* of distributions u and v, and it is denoted by $u \boxtimes v$.

It is not difficult to check (following [13, 17.10.]) that the tensor product of anisotropic distributions satisfies:

a) $\mathrm{supp}\,(u \boxtimes v) = \mathrm{supp}\,u \times \mathrm{supp}\,v$.
b) For each $\varphi \in \mathrm{C}^l(X)$ and $\psi \in \mathrm{C}^m(Y)$ one has $(\varphi \boxtimes \psi)(u \boxtimes v) = (\varphi u) \boxtimes (\psi v)$.

Let us go back to the starting question of this subsection on the validity of the converse implication in (6), i.e. if $u \in \mathcal{D}'(X \times Y)$ satisfies (6) whether it can be uniquely extended to an element of $\mathcal{D}'_{l,m}(X \times Y)$.

Unfortunately, this conjecture fails in general. Let us first see where the standard straightforward approach to the proof would fail, and after that we shall provide a counterexample which was kindly communicated to us by Evgenij Panov. It is sufficient to consider only the simplest case where X and Y are open subsets of Euclidean spaces.

First, one would take an arbitrary compact $M \subseteq X \times Y$. Let L and K be its projections to X and Y, which are compact. Then replacing them by larger compacts K' and L', as it was done in Lemma 2, one would approximate any $\Psi \in \mathrm{C}^{l,m}_{K\times L}(X \times Y)$ by a sequence $(\sum_{k=1}^N \varphi^{(N)}_k \boxtimes \psi^{(N)}_k)$ of functions from $\mathrm{C}^\infty_{K'}(X) \boxtimes \mathrm{C}^\infty_{L'}(Y)$, and one would be tempted to define

$$\langle u, \Psi\rangle = \lim_N \sum_{k=1}^N \langle u, \varphi^{(N)}_k \boxtimes \psi^{(N)}_k\rangle\,.$$

However, the problem with this approach is in obtaining an appropriate bound for $\langle u, \Psi\rangle$. Naturally, one would like to use the already available bound for the tensor product given in (6). Since the number of elements in tensor approximation can be unbounded (although, for each element it is finite), the constant for each seminorm from the definition of anisotropic distributions would be unbounded. Thus, this approach fails to yield the desired bound.

For the counterexample, let us assume $l = m = 0$ and $d = r = 1$. First notice that $\ln|x-y| \in \mathrm{L}^1_{\rm loc}(\mathbf{R}_x \times \mathbf{R}_y)$, so it can be identified with an element of $\mathcal{D}'(\mathbf{R}^2)$. Consider a distribution $u = -\frac{1}{\pi}\partial_y \ln|x-y|$. For $g \in \mathrm{C}^{0,1}(\mathbf{R}\times\mathbf{R})$, we get

$$\langle u, g\rangle = \frac{1}{\pi}\int_{\mathbf{R}^2} \ln|x-y|\partial_y g(x,y)\,dxdy\,.$$

It follows that $u \in \mathcal{D}'_{0,1}(\mathbf{R}\times\mathbf{R})$. Since $\partial_x \ln|x-y| = \partial_y \ln|x-y|$, we have $u \in \mathcal{D}'_{1,0}(\mathbf{R}\times\mathbf{R})$ as well. Thus, $u \in \mathcal{D}'_{0,1}(\mathbf{R}\times\mathbf{R}) \cap \mathcal{D}'_{1,0}(\mathbf{R}\times\mathbf{R})$. Of course, any such functional u can be extended to a linear functional U on $\mathrm{C}_c(\mathbf{R}\times\mathbf{R})$ (for example, take the extension by zero to the algebraic complement N of that subspace $M := \mathcal{D}'_{0,1}(\mathbf{R}\times\mathbf{R}) \cap \mathcal{D}'_{1,0}(\mathbf{R}\times\mathbf{R})$, and use additivity to define $U(m+n) := u(m)$). However, there is no guarantee that such an extension will be continuous.

Take g to be of the form $\varphi(x) \boxtimes \psi(y)$, for $\varphi \in \mathrm{C}_c(\mathbf{R})$ and $\psi \in \mathrm{C}^1_c(\mathbf{R})$. It holds:

$$\langle u, \varphi \boxtimes \psi\rangle = \frac{1}{\pi}\int_{\mathbf{R}} \varphi(x) \int_{\mathbf{R}} \ln|x-y|\psi'(y)\,dydx\,,$$

and the inner integral, after integration by parts, becomes

$$\begin{aligned}\int_{\mathbf{R}} \ln|x-y|\psi'(y)\,dy &= \lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon} \ln|y-x|\psi'(y)\,dy \\ &= \lim_{\varepsilon\to 0}\left((\psi(x-\varepsilon)-\psi(x+\varepsilon))\ln\varepsilon + \int_{|y-x|>\varepsilon} \frac{\psi(y)}{x-y}\,dy\right) \\ &= \text{V.P.}\int_{\mathbf{R}} \frac{\psi(y)}{x-y}\,dy \quad = \pi H\psi(x)\,,\end{aligned}$$

where $H\psi$ denotes the Hilbert transform of function $\psi \in \mathrm{C}^1_c(\mathbf{R})$. Since it is an isometry on $\mathrm{L}^2(\mathbf{R})$ (see [18, Chapter 5.1.1]), we have the following bound

$$|\langle u, \varphi\boxtimes\psi\rangle| = \left|\int_{\mathbf{R}} \varphi(x) H\psi(x)\,dx\right| \leqslant \|\varphi\|_{\mathrm{L}^2(\mathbf{R})}\|\psi\|_{\mathrm{L}^2(\mathbf{R})} \leqslant |K|\|\varphi\|_{\mathrm{L}^\infty(\mathbf{R})}\|\psi\|_{\mathrm{L}^\infty(\mathbf{R})}\,,$$

for smooth functions φ and ψ whose both supports are contained in a compact set $K \subseteq \mathbf{R}$. Thus, all the assumptions of the conjecture are satisfied, but $u \notin \mathcal{D}'_{0,0}(\mathbf{R}\times\mathbf{R})$.

To demonstrate that, assume to the contrary that u is a distribution of order 0 on $\mathbf{R}^2$. Take a test-function g whose support does not intersect the diagonal of $\mathbf{R}^2$. After integration by parts, we get the identity

$$\langle u, g\rangle = \frac{1}{\pi}\int_{\mathbf{R}^2} \frac{g(x,y)}{x-y}\,dxdy\,.$$

Now, take a compact set $K = [0,3]\times[0,3]$, and since u is a distribution of order 0, there exists a constant $C_K > 0$ such that for any test-function g whose support is in K, we get

$$|\langle u, g\rangle| \leqslant C_K \|g\|_{\mathrm{L}^\infty(\mathbf{R}^2)}\,.$$

Take a sequence of non-negative test-functions (g_ε), $0<\varepsilon<1$, whose supports are contained in the triangle with vertices $(0,0), (3,0)$ and $(3,3)$, which are identically equal to 1 in the triangle with vertices $(1+\varepsilon, 1), (2,1)$ and $(2, 2-\varepsilon)$, and $\|g_\varepsilon\|_{\mathrm{L}^\infty(\mathbf{R}^2)} = 1$. Clearly, the supports of all g_ε are contained in K, and they do not intersect the diagonal of $\mathbf{R}^2$. Thus, we can write

$$\begin{aligned}\langle u, g_\varepsilon\rangle &= \frac{1}{\pi}\int_{\mathbf{R}^2} \frac{g_\varepsilon(x,y)}{x-y}\,dxdy \\ &\geqslant \frac{1}{\pi}\int_1^{2-\varepsilon}\int_{y+\varepsilon}^2 \frac{1}{x-y}\,dxdy \\ &= \frac{1}{\pi}\int_1^{2-\varepsilon} \ln(2-y)\,dy - (1-\varepsilon)\frac{\ln\varepsilon}{\pi} \quad \geqslant \frac{-1}{\pi} - (1-\varepsilon)\frac{\ln\varepsilon}{\pi}\,.\end{aligned}$$

On one hand, we have the uniform bound $\langle u, g_\varepsilon\rangle = |\langle u, g_\varepsilon\rangle| \leqslant C_K$, while on the other hand, the above bound implies $\langle u, g_\varepsilon\rangle \to \infty$ as $\varepsilon \to 0$, which is a contradiction.

The lack of the above result is the main reason why we need to consider a variant of the Schwartz kernel theorem for anisotropic distributions.

Remark 1. Many constructions and proofs of this section could be simplified by applying existing results for the (classical) distributions. However, we hope that the readers would benefit more from this construction of anisotropic distributions *from scratch*. An additional reason for this approach is to build a theory that does not depend on the smoothness of the corresponding manifolds. Let us elaborate more on this.

In this section we considered only C^∞ smooth manifolds, denoted by *differential manifolds* (see the beginning of the second subsection for the precise definition). Although any C^r smooth manifold is C^r-diffeomorphic to a C^∞ smooth manifold (differential manifold) (see [19, Chapter 2, Theorem 2.10]), often in applications it is simpler to work with an explicit C^r-structure [14, 22, 25]. Here by C^r smooth manifolds we denote a locally Euclidean (of the fixed dimension, i.e. pure) second countable Hausdorff topological space on which an equivalence class of C^r smooth atlases is given, i.e. the transition maps are (at least) of the class C^r [19, Chapter 1]. Thus, for $r = \infty$ we get differential manifolds that are studied in this paper. It is crucial to notice that if $r < \infty$ we cannot define classical distributions.

An important fact is that all the results developed in this section can be adjusted to such C^r manifolds. Indeed, let X be $C^{\bar{l}}$ smooth manifold of dimension d, and Y $C^{\bar{m}}$ smooth manifold of dimension r, where $\bar{l}, \bar{m} \in \mathbf{N} \cup \{\infty\}$. Then on $X \times Y$ the highest regularity for functions that we can obtain is $(\bar{l}, \bar{m})$, i.e. the space $C^{\bar{l},\bar{m}}(\Omega)$, $\Omega \subseteq X \times Y$ open. Now, it is an easy exercise to check that all the results and all the proofs of this section remain valid under additional assumptions that $l \leqslant \bar{l}$ and $m \leqslant \bar{m}$. In addition, the space $C^\infty(\Omega)$ in the statement of Theorem 2 should be replaced by $C^{\bar{l},\bar{m}}(\Omega)$, and similar changes should be made in Lemma 2.

In the following section we shall return to this topic and state a kernel theorem for C^r manifolds.

∎

3. The Schwartz kernel theorem for distributions of anisotropic order

Starting with a distribution $K \in \mathcal{D}'(X \times Y)$, it is straightforward to define strongly continuous linear operator $A : C_c^\infty(X) \to \mathcal{D}'(Y)$ by formula

$$\langle A\varphi, \psi\rangle = \langle K, \varphi \boxtimes \psi\rangle \ ,$$

where $\varphi \in C_c^\infty(X)$ and $\psi \in C_c^\infty(Y)$ [34, p. 138]. The Schwartz kernel theorem states that the converse is also valid: starting from a linear operator $A : C_c^\infty(X) \to \mathcal{D}'(Y)$, continuous from the strict inductive limit topology on the domain to weak (i.e. weak $*$) topology on its range, one can define the unique distributional kernel K such that the above formula is valid. This theorem was used in [7] to prove the existence of H-distributions, and its form prevented any refinement on whether H-distributions lay in a better (smaller) space than $\mathcal{D}'$.

In this section we shall prove a version of the Schwartz kernel theorem for distributions of anisotropic order. While doing that, we shall follow the proof in [13, 23.9.2] and carefully take note of the order of distributions appearing. For other possible approaches, see the remarks after the proof of the following theorem. However, it should be noted that, up to our knowledge, this is the first version of the Schwartz kernel theorem for distributions of finite order.

Theorem 5. *Let X and Y be differential manifolds, of dimension d and r, and $l, m \in \mathbf{N}_0 \cup \{\infty\}$. Then the following statements hold:*

a) If $K \in \mathcal{D}'_{l,m}(X \times Y)$, then for each $\varphi \in C_c^l(X)$ the linear form K_φ, defined by $\psi \mapsto \langle K, \varphi \boxtimes \psi\rangle$, is a distribution of order not more than m on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $C_c^l(X)$ with its inductive limit topology to $\mathcal{D}'_m(Y)$ with weak $*$ topology, is linear and continuous.

b) *Let* $A : \mathrm{C}^l_c(X) \to \mathcal{D}'_m(Y)$ *be a continuous linear operator, in the pair of topologies as in (a) above. Then there exists a unique distribution of anisotropic order* $K \in \mathcal{D}'_{l,r(m+2)}(X \times Y)$ *such that for any* $\varphi \in \mathrm{C}^l_c(X)$ *and* $\psi \in \mathrm{C}^{r(m+2)}_c(Y)$

$$\langle K, \varphi \boxtimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle \ .$$

Dem. a) Since K is continuous, by Corollary 2(c) (see also (6)) for any $L \in \mathcal{K}(X)$ and $H \in \mathcal{K}(Y)$ there exist $C > 0$, $n \in \mathbf{N}$, finite set of indices I and J, and $l', m' \in \mathbf{N}_0$, $l' \leqslant l$, $m' \leqslant m$, such that for any $\varphi \in \mathrm{C}^l_L(X)$ and $\psi \in \mathrm{C}^m_H(Y)$ it holds

$$|\langle K_\varphi, \psi \rangle| = |\langle K, \varphi \boxtimes \psi \rangle| \leqslant \tilde{C} \max_{i \in I} p^{l';i}_n(\varphi) \max_{j \in J} p^{m';j}_n(\psi) \ . \tag{7}$$

Let $\varphi_0 \in \mathrm{C}^l_c(X)$ be an arbitrary function. K_{φ_0} is linear as the tensor product is bilinear and K is linear, while for $L := \operatorname{supp} \varphi_0$ by the previous estimate we get

$$|\langle K_{\varphi_0}, \psi \rangle| \leqslant C \max_{j \in J} p^{m';j}_n(\psi) \ , \quad \psi \in \mathrm{C}^m_H(Y) \ ,$$

where $C = \tilde{C} \max_{i \in I} p^{l';i}_n(\varphi_0)$. Thus, $K_{\varphi_0} \in \mathcal{D}'_m(Y)$.

The mapping $\varphi \mapsto K_\varphi$ is trivially linear on $\mathrm{C}^l_c(X)$. To show that it is continuous, we need to estimate $|\langle K_\varphi, \psi \rangle|$ by a norm defining the topology on $\mathrm{C}^l_c(X)$ for an arbitrary $\psi \in \mathrm{C}^m_c(Y)$. Let us take an arbitrary $\psi_0 \in \mathrm{C}^m_c(Y)$. For $H := \operatorname{supp} \psi_0$ by (7) we get

$$|\langle K_\varphi, \psi_0 \rangle| \leqslant C \max_{i \in I} p^{l';i}_n(\varphi) \ , \quad \varphi \in \mathrm{C}^l_L(X) \ ,$$

where $C = \tilde{C} \max_{j \in J} p^{m';j}_n(\psi_0)$. Therefore, the mapping $\varphi \mapsto K_\varphi$, from $\mathrm{C}^l_c(X)$ to $\mathcal{D}'_m(Y)$ is linear and continuous.

b) Let us first comment the uniqueness. By formula

$$\langle K, \varphi \boxtimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle \ ,$$

a continuous functional K on $\mathrm{C}^l_c(X) \boxtimes \mathrm{C}^{r(m+2)}_c(Y)$ is defined. As it is defined on a dense subset of $\mathrm{C}^{l,r(m+2)}(X \times Y)$ (see Lemma 2), such K is uniquely determined on the whole $\mathrm{C}^{l,r(m+2)}(X \times Y)$.

The proof of existence will be divided into three steps. In the first step we assume that X and Y are open subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$, and additionally, that the range of A is $\mathrm{C}(Y) \subseteq \mathcal{D}'_m(Y)$ (understood as distributions which can be identified with continuous functions; note that such an embedding exists in the flat space). This will allow us to write explicitly the action of $A\varphi$ on a test function $\psi \in \mathrm{C}^m_c(Y)$, which will finally enable us to define the kernel K. In the second step we briefly comment how the case of general manifolds X and Y can be reduced to Euclidean spaces, while in the last step the structure theorem of distributions is used to reduce the problem to the second step. Let us begin.

Step I. X, Y *open subsets of Euclidean spaces and the range of* A *contained in* $\mathrm{C}(Y)$
Assume that X and Y are open subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$, respectively, and that for any $\varphi \in \mathrm{C}^l_c(X)$, $A\varphi \in \mathrm{C}(Y)$. Its action on a test function $\psi \in \mathrm{C}^m_c(Y)$ is given by

$$\langle A\varphi, \psi \rangle = \int_Y (A\varphi)(\mathbf{y})\psi(\mathbf{y}) d\mathbf{y} \ .$$

The continuity assumption on A implies that $A : \mathrm{C}^l_c(X) \longrightarrow \mathrm{C}(Y)$ is continuous when the range is equipped with the weak $*$ topology inherited from $\mathcal{D}'_m(Y)$.

As the latter is a Hausdorff space, that operator has a closed graph, but this remains true even when we replace the topology on $\mathrm{C}(Y)$ by its standard Fréchet topology [28, Exercise 14.101(a)], which is stronger. Since $\mathrm{C}^l_c(X)$ is barrelled, as a strict inductive limit of barrelled spaces, we can apply the Closed graph theorem [28, Theorem 14.3.4(b)] (the proof of this form is essentially the same as the classical Banach's proof) to conclude that $A : \mathrm{C}^l_c(X) \longrightarrow \mathrm{C}(Y)$ is continuous with usual strong topologies on its domain and range.

For $\mathbf{y} \in Y$ consider a linear functional $F_{\mathbf{y}} : \mathrm{C}^l_c(X) \longrightarrow \mathbf{C}$ defined by

$$F_{\mathbf{y}}(\varphi) = (A\varphi)(\mathbf{y}) \; .$$

Since $A\varphi$ is a continuous function, $F_{\mathbf{y}}$ is well-defined and clearly it is continuous as a composition of continuous mappings, thus a distribution in $\mathcal{D}'_l(X)$.

Let us take a test function $\Psi \in \mathrm{C}^{l,0}_c(X \times Y)$. If we fix its second variable, we can consider it as a function from $\mathrm{C}^l_c(X)$ and apply $F_{\mathbf{y}}$; we are interested in the properties of the following mapping:

$$\mathbf{y} \mapsto F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y})) = \Big(A\Psi(\cdot, \mathbf{y})\Big)(\mathbf{y}) \; .$$

Clearly, it is well defined on Y, with a compact support contained in the projection $\pi_2(\operatorname{supp}\Psi)$. Furthermore, we have the following bounds:

$$\begin{aligned}
\Big|F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y}))\Big| = \Big|\Big(A\Psi(\cdot, \mathbf{y})\Big)(\mathbf{y})\Big| &\leqslant \|A\Psi(\cdot, \mathbf{y})(\cdot)\|_{\mathrm{L}^\infty(\pi_2(\operatorname{supp}\Psi))} \\
&\leqslant C\|\Psi(\cdot, \mathbf{y})\|_{\mathrm{C}^l_{\pi_1(\operatorname{supp}\Psi)}(X)} \leqslant C\|\Psi\|_{\mathrm{C}^{l,0}_{\operatorname{supp}\Psi}(X\times Y)} \; .
\end{aligned} \tag{8}$$

The proof of continuity is a bit more involved; we shall show sequential continuity: take a sequence $\mathbf{y}_n \to \mathbf{y}$ in Y. Denote $H = \pi_1(\operatorname{supp}\Psi)$ and let $L \subseteq Y$ be a compact such that $\mathbf{y}_n, \mathbf{y} \in L$; Ψ is uniformly continuous on compact $H \times L$. This is also valid for $\partial^{\boldsymbol{\alpha}}_{\mathbf{x}}\Psi$, where $|\boldsymbol{\alpha}| \leqslant l$, which results in $\Psi(\cdot, \mathbf{y}_n) \longrightarrow \Psi(\cdot, \mathbf{y})$ in $\mathrm{C}^l_c(X)$. As A is continuous, the convergence is carried to $\mathrm{C}(Y)$, i.e. to the uniform convergence on compacts of the sequence of functions $A\Psi(\cdot, \mathbf{y}_n)$ to $A\Psi(\cdot, \mathbf{y})$. In particular, this gives that $(A\Psi(\cdot, \mathbf{y}_n))(\bar{\mathbf{y}}) - (A\Psi(\cdot, \mathbf{y}))(\bar{\mathbf{y}})$ is arbitrary small independently of $\bar{\mathbf{y}} \in L$, for large enough n.

On the other hand, $A\Psi(\cdot, \mathbf{y})$ is uniformly continuous, thus $(A\Psi(\cdot, \mathbf{y}))(\mathbf{y}_n) - (A\Psi(\cdot, \mathbf{y}))(\mathbf{y})$ is small for large n. In other terms, we have the required convergence

$$F_{\mathbf{y}_n}(\Psi(\cdot, \mathbf{y}_n)) \longrightarrow F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y})) \; .$$

Any continuous function with compact support is summable, so we can define functional K on $\mathrm{C}^{l,0}_c(X \times Y)$:

$$\langle K, \Psi\rangle = \int_Y F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y}))\, d\mathbf{y} \; ,$$

which is obviously linear in Ψ, as $F_{\mathbf{y}}$ is.

For any $H \in \mathcal{K}(X \times Y)$ let us take $\Psi \in \mathrm{C}^{l,0}_H(X \times Y)$. By (8) and the fact that the support of $\mathbf{y} \mapsto F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y}))$ is contained in the projection $\pi_2(H)$ we get

$$|\langle K, \Psi\rangle| \leqslant \int_{\pi_2(H)} |F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y}))|\, d\mathbf{y} \leqslant \mathsf{vol}\Big(\pi_2(H)\Big) C\|\Psi\|_{\mathrm{C}^{l,0}_H(X\times Y)} \; ,$$

thus, by Corollary 2(c), $K \in \mathcal{D}'_{l,0}(X \times Y)$.

Finally, it is easy to check that for $\varphi \in \mathrm{C}^l_c(X)$ and $\psi \in \mathrm{C}_c(Y)$, we have:

$$\langle K, \varphi \boxtimes \psi\rangle = \int_Y F_{\mathbf{y}}(\varphi \boxtimes \psi(\mathbf{y}))d\mathbf{y} = \int_Y F_{\mathbf{y}}(\varphi)\psi(\mathbf{y})d\mathbf{y} = \int_Y (A\varphi)(\mathbf{y})\psi(\mathbf{y})d\mathbf{y} = \langle A\varphi, \psi\rangle \; .$$

Step II. *General X, Y – reduction to Euclidean spaces*
Let $(\tilde{U}_\alpha)$ and $(\tilde{V}_\beta)$ be covers of X and Y consisting of relatively compact open sets and such that each $\tilde{U}_\alpha \times \tilde{V}_\beta$ is contained in a single chart domain $U_i \times V_j$, where $(U_i \times V_j, \varphi_i \boxtimes \psi_j)$ is at

most countable family of charts on $X \times Y$ such that $(U_i \times V_j)$ form a locally finite open cover of $X \times Y$. It is sufficient to show for any α, β the existence of $K_{\alpha\beta} \in \mathcal{D}'_{l,r(m+2)}(\tilde{U}_\alpha \times \tilde{V}_\beta)$ such that for any $\varphi \in \mathrm{C}^l_c(\tilde{U}_\alpha)$ and $\psi \in \mathrm{C}^{r(m+2)}_c(\tilde{V}_\beta)$ it holds $\langle K_{\alpha\beta}, \varphi \boxtimes \psi\rangle = \langle A\varphi, \psi\rangle$. Indeed, by Lemma 2 each $K_{\alpha\beta}$ is unique, thus for any $\alpha, \beta, \gamma, \delta$ distributions $K_{\alpha\beta}$ and $K_{\gamma\delta}$ coincide on open sets $(\tilde{U}_\alpha \cap \tilde{U}\gamma) \times (\tilde{V}_\beta \cap \tilde{V}_\delta)$ of $X \times Y$. Therefore, the existence of $K \in \mathcal{D}'_{l,r(m+2)}(X \times Y)$ follows by Theorem 3.

Moreover, as each set $\tilde{U}_\alpha \times \tilde{V}_\beta$ is contained in a single chart domain, we can transfer for any α, β the objects to the flat space (see the discussion before Theorem 3). Thus, without loss of generality, in the rest of the proof (i.e. in Step III) we assume that $X = \mathbf{R}^d$ and $Y = \mathbf{R}^r$, and we need to prove for arbitrary open and bounded subsets $U \subseteq \mathbf{R}^d$, $V \subseteq \mathbf{R}^r$ the existence of $K \in \mathcal{D}'_{l,r(m+2)}(U \times V)$ such that for any $\varphi \in \mathrm{C}^l_c(U)$ and $\psi \in \mathrm{C}^{r(m+2)}_c(V)$ it holds $\langle K_{\alpha\beta}, \varphi \boxtimes \psi\rangle = \langle A\varphi, \psi\rangle$.

Step III. *General A*
For $X = \mathbf{R}^d$ and $Y = \mathbf{R}^r$, let us choose arbitrary open and bounded subsets $U \subseteq \mathbf{R}^d$ and $V \subseteq \mathbf{R}^r$. By Step II it is sufficient to show the existence of $K \in \mathcal{D}'_{l,r(m+2)}(U \times V)$ such that

$$\langle K, \varphi \boxtimes \psi\rangle = \langle A\varphi, \psi\rangle \ , \quad \varphi \in \mathrm{C}^l_c(U)\, , \ \psi \in \mathrm{C}^{r(m+2)}(V)\, .$$

Let us take a relatively compact open neighbourhood W of the closure of V in Y and pick a smooth cut-off function ρ which is equal to one on the closure of V and whose support is contained in W. By $\tilde{A} : \mathrm{C}^l_c(U) \to \mathcal{D}'_m(W)$ we denote the restriction of A defined by

$$\langle \tilde{A}\varphi, \psi\rangle = \langle A\varphi, \psi\rangle \ , \quad \varphi \in \mathrm{C}^l_c(U)\, , \ \psi \in \mathrm{C}^m_c(W)\, .$$

It is clear that $\tilde{A}$ is well-defined and continuous (in the same sense as A).

Multiplying a distribution of finite order with ρ does not change its order. Thus, for $\varphi \in \mathrm{C}^l_c(U)$, $\rho\tilde{A}\varphi$ is an element of the space $\mathcal{D}'_m(W)$ and has a compact support. The next step is to use the so-called structure theorem for distributions: from the proof of Theorem 5.4.1 of [15], it follows that we can write

$$\rho\tilde{A}\varphi = \left(\partial_1^{m+2} \dots \partial_r^{m+2}\right) \left(E_{m+2} * (\rho\tilde{A}\varphi)\right) \ ,$$

where E_{m+2} is the fundamental solution of the differential operator $\partial_1^{m+2} \dots \partial_r^{m+2}$ (we take partial derivatives with respect to the $\mathbf{y}$ variable), i.e. it satisfies in the sense of distributions the following equation $\left(\partial_1^{m+2} \dots \partial_r^{m+2}\right) E_{m+2} = \delta_0$, where δ_0 is the Dirac measure concentrated at the origin. For the explicit formula of E_{m+2}, see [15, Chapter 5.4]. Furthermore, in the proof of [15, Theorem 5.4.1], it was shown that $E_{m+2} * (\rho\tilde{A}\varphi)$ is a continuous function. Denoting by $\widetilde{E}_{m+2}*$ transpose of the operator $E_{m+2}*$, we write for $\varphi \in \mathrm{C}^l_c(U)$ and $\psi \in \mathrm{C}^m_c(W)$

$$\left\langle E_{m+2} * (\rho\tilde{A}\varphi), \psi\right\rangle = \left\langle \tilde{A}\varphi, \rho\widetilde{E}_{m+2} * \psi\right\rangle \ ,$$

from which we conclude that the mapping $\varphi \mapsto E_{m+2}*(\rho\tilde{A}\varphi)$ is continuous from $\mathrm{C}^l_c(U)$ to $\mathcal{D}'_m(W)$.

Now we can apply Step I for $X = U$, $Y = W$ and $A = E_{m+2} * (\rho\tilde{A}\,\cdot)$. Thus, there exists a distribution $R \in \mathcal{D}'_{l,0}(U \times W)$ such that for all $\varphi \in \mathrm{C}^l_c(U)$ and $\psi \in \mathrm{C}_c(W)$ it holds

$$\left\langle E_{m+2} * (\rho\tilde{A}\varphi), \psi\right\rangle = \langle R, \varphi \boxtimes \psi\rangle \ .$$

Let us define a distribution K on $U \times V$ by $K := (\partial_1^{m+2} \dots \partial_r^{m+2})R$, where the derivatives are taken with respect to the variable $\mathbf{y}$. Since the order of R is $(l, 0)$, K is a distribution of anisotropic order $(l, r(m+2))$ on $U \times V$.

Taking $\varphi \in \mathrm{C}_c^l(U)$ and $\psi \in \mathrm{C}_c^{r(m+2)}(V)$, we have

$$\begin{aligned}\langle K, \varphi \boxtimes \psi\rangle &= (-1)^{r(m+2)} \left\langle R, \varphi \boxtimes \left(\partial_1^{m+2} \dots \partial_r^{m+2}\right) \psi\right\rangle \\ &= (-1)^{r(m+2)} \left\langle E_{m+2} * (\rho\tilde{A}\varphi), \left(\partial_1^{m+2} \dots \partial_r^{m+2}\right) \psi\right\rangle \\ &= \left\langle \left(\partial_1^{m+2} \dots \partial_r^{m+2}\right) \left(E_{m+2} * (\rho\tilde{A}\varphi)\right), \psi\right\rangle \\ &= \left\langle \rho\tilde{A}\varphi, \psi\right\rangle \\ &= \left\langle \tilde{A}\varphi, \rho\psi\right\rangle \\ &= \langle A\varphi, \psi\rangle .\end{aligned}$$

Therefore, K is the desired kernel distribution.

Q.E.D.

Remark 2.

a) Note that in part (b) of the theorem, we did not get that $K \in \mathcal{D}'_{l,m}(X \times Y)$, as one would wish to get while observing the statement in part (a): the order with respect to $\mathbf{x}$ variable remained the same, but the order with respect to the $\mathbf{y}$ variable increased from m to $r(m+2)$. However, this was expected by the example given in the previous section.

b) Under assumptions of Theorem 5(b) we have that the mapping $\psi \mapsto \langle A\cdot, \psi\rangle$ is continuous from $\mathrm{C}_c^m(Y)$ to $\mathcal{D}'_l(X)$ in the analogous pair of topologies as was the case with A. Thus, we can interchange the role of X and Y to get $\tilde{K} \in \mathcal{D}'_{d(l+2),m}(X \times Y)$, where the order with respect to $\mathbf{y}$ remained the same, while the order with respect to $\mathbf{x}$ variable increased from l to $d(l+2)$. Since the uniqueness of the kernel has already been determined, we conclude that the unique kernel K belongs to the space $\mathcal{D}'_{l,r(m+2)}(X \times Y) \cap \mathcal{D}'_{d(l+2),m}(X \times Y)$. It might be interesting to see some additional properties of that intersection.

c) Note that the order up to which we got the increase is determined by the structure theorem for distributions we used in the proof, and it represents an improvement of the previous results as anisotropic orders are allowed.
Of course, at this level of generality it cannot be expected to get an optimal order of the kernel distribution in part (b) of the previous theorem. Indeed, starting with $K \in \mathcal{D}'_{l,m}(X \times Y)$ by Theorem 5(a) K_φ (in the notation of the theorem) is continuous from $\mathrm{C}_c^l(X)$ to $\mathcal{D}'_m(Y)$, and applying Theorem 5(b) we get that the kernel (which coincides with the starting distribution K) belongs to $\mathcal{D}'_{l,r(m+2)}(X \times Y)$, which is obviously not optimal. Even if we consider only situations in which it is known that for continuous $A : \mathrm{C}_c^l(X) \to \mathcal{D}'_m(Y)$ the order of the kernel distribution is not (l, m), i.e. that some increase in order is necessary, our result is still not optimal. This could be seen on the example from the previous section as by our result $u \in \mathcal{D}'_{0,2}(\mathbf{R} \times \mathbf{R}) \cap \mathcal{D}'_{2,0}(\mathbf{R} \times \mathbf{R})$, while it has been shown that the order of u is (0,1) and (1,0). However, our result has a nice feature that preserves the order in one variable. ∎

Remark 3. If one used a more constructive proof of the Schwartz kernel theorem, for example [36, Theorem 1.3.4], one would end up increasing the order with respect to both variables $\mathbf{x}$ and $\mathbf{y}$. In this case, increasing the order with respect to both variables occurs naturally because one needs to secure the integrability of the function which is used to define the kernel function.

Let us remark in passing that another interesting approach to the kernel theorem is given in [39, Chapter 51]. This approach is based on deep results of functional analysis on tensor products of nuclear spaces of Alexander Grothendieck. However, such programme would require more elaborate investigation.

Our Theorem 5 is not particularly suitable for the purposes of article [26]. Namely, in that case we have a linear operator whose domain is C_c^∞ and range $\mathcal{D}'_m$. Theorem 5 would increase the order with respect to the $\mathbf{y}$ variable. Luckily, a special case of Grothendieck's result has been available in the work of Bogdanowicz [10], who gave simpler proofs in the case of Fréchet and

(LF)-spaces. By casting these results to the case of C^l function spaces, an appropriate version of the kernel theorem was obtained. An important detail is to notice that in order to use these results, we had to have a nuclear space, which is the case with C^∞_c. ∎

The proof of the previous theorem applies in the case of C^r smooth manifolds as well (see Remark 1). One just needs to be careful that the smoothness of underlying functions does not exceed the smoothness of the manifold. Thus, for the convenience of the reader, let us restate the kernel theorem also in this generalised form.

Theorem 6. *Let X be $\mathrm{C}^{\bar l}$ smooth manifold of dimension d, and Y $\mathrm{C}^{\bar m}$ smooth manifold of dimension r, where $\bar l, \bar m \in \mathbf{N}\cup\{\infty\}$. Let $l, m \in \mathbf{N}_0\cup\{\infty\}$ satisfy $l \leqslant \bar l$ and $m \leqslant \bar m$. Then the following statements hold:*

a) If $K \in \mathcal{D}'_{l,m}(X\times Y)$, then for each $\varphi \in \mathrm{C}^l_c(X)$ the linear form K_φ, defined by $\psi \mapsto \langle K, \varphi\boxtimes\psi\rangle$, is a distribution of order not more than m on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathrm{C}^l_c(X)$ with its inductive limit topology to $\mathcal{D}'_m(Y)$ with weak $*$ topology, is linear and continuous.

b) Let $A : \mathrm{C}^l_c(X) \to \mathcal{D}'_m(Y)$ be a continuous linear operator, in the pair of topologies as in (a) above, and let in addition $r(m+2) \leqslant \bar m$. Then there exists a unique distribution of anisotropic order $K \in \mathcal{D}'_{l,r(m+2)}(X\times Y)$ such that for any $\varphi \in \mathrm{C}^l_c(X)$ and $\psi \in \mathrm{C}^{r(m+2)}_c(Y)$

$$\langle K, \varphi\boxtimes\psi\rangle = \langle K_\varphi, \psi\rangle = \langle A\varphi, \psi\rangle \ .$$

∎

Let us conclude this section with a simple example (compare with [21, Chapter V]).

Let $X \subseteq \mathbf{R}^d$, $Y \subseteq \mathbf{R}^r$, $f \in \mathrm{C}(Y;X)$, and $A : \mathrm{C}^l_c(X) \to \mathrm{C}(Y) \hookrightarrow \mathcal{D}'_0(Y)$ defined by $A\varphi = \varphi\circ f$. Its kernel K has support contained in the graph of f and is given by

$$\langle K, \Phi\rangle = \int_Y \Phi(f(\mathbf{y}),\mathbf{y})\,d\mathbf{y} \ , \qquad \Phi \in \mathrm{C}^{l,2r}_c(X\times Y) \,.$$

In particular, for $r = d$, $Y = X$ and f identity map, we get that the support of K is contained in the diagonal $\{(\mathbf{x},\mathbf{x}) : \mathbf{x}\in X\} \subseteq X\times X$, and

$$\langle K, \Phi\rangle = \int_X \Phi(\mathbf{x},\mathbf{x})\,d\mathbf{x} \ , \qquad \Phi \in \mathrm{C}^{l,2d}_c(X\times X) \,.$$

4. H-distributions

Existence

An important result that was used in the proof of existence of H-measures was the First commutation lemma [38], which stated that the commutator of multiplication and the Fourier multiplier operator was compact on $\mathrm{L}^2(\mathbf{R}^d)$. We shall need a variant of this result for the $\mathrm{L}^p(\mathbf{R}^d)$ spaces, which was shown in [6]. It is a consequence of the following Krasnosel'skij type lemma (for details and proofs, see [6]):

Lemma 3. *Assume that linear operator A is compact on $\mathrm{L}^2(\mathbf{R}^d)$ and bounded on $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 1,\infty\rangle\setminus\{2\}$. Then A is also compact on $\mathrm{L}^p(\mathbf{R}^d)$, for any p between 2 and r (i.e. such that $1/p = \theta/2 + (1-\theta)/r$, for some $\theta\in\langle 0,1\rangle$).* ∎

With this result in hand, one just needs to use Tartar's First commutation lemma on $\mathrm{L}^2(\mathbf{R}^d)$ for compactness [38, Lemma 1.7], and the Hörmander-Mihlin theorem [18, Theorem 5.2.7; 4] for boundedness of Fourier multipliers on $\mathrm{L}^p(\mathbf{R}^d)$, for any $p \in \langle 1,\infty\rangle$, to conclude the following (recall that we took $\kappa := \lfloor d/2\rfloor + 1$):

Corollary 3. *If $b \in \mathrm{C}_0(\mathbf{R}^d)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, then the commutator $\mathcal{A}_\psi b - b\mathcal{A}_\psi$ is compact on $\mathrm{L}^p(\mathbf{R}^d)$, for any $p \in \langle 1, \infty \rangle$.* ∎

We are now ready to reprove the theorem on existence of H-distributions [7], in addition showing that they are actually distributions of order 0 (Radon measures) in $\mathbf{x}$, and of finite order $Q := (d-1)(\kappa+2)$ in $\boldsymbol{\xi}$, where $\kappa := \lfloor d/2 \rfloor + 1$, as required by the Hörmander-Mihlin theorem. Of course, we take $d \geqslant 2$, in order to exclude the trivial case.

Theorem 7. *If $u_n \longrightarrow 0$ in $\mathrm{L}^p_{\rm loc}(\mathbf{R}^d)$ and $v_n \overset{*}{\longrightarrow} v$ in $\mathrm{L}^q_{\rm loc}(\mathbf{R}^d)$ for some $p \in \langle 1, \infty \rangle$ and $q \geqslant p'$, then there exist subsequences $(u_{n'})$, $(v_{n'})$ and a complex valued anisotropic distribution $\mu \in \mathcal{D}'_{0,Q}(\mathbf{R}^d \times \mathrm{S}^{d-1})$, such that, for any $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ and $\psi \in \mathrm{C}^Q(\mathrm{S}^{d-1})$, one has:*

$$\begin{aligned}\lim_{n'\to\infty} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'})(\mathbf{x}) \overline{(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x} &= \lim_{n'\to\infty} \int_{\mathbf{R}^d} (\varphi_1 u_{n'})(\mathbf{x}) \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x} \\ &= \langle \mu, \varphi_1 \overline{\varphi}_2 \boxtimes \psi \rangle,\end{aligned}$$

where $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ is the Fourier multiplier operator with symbol $\psi \in \mathrm{C}^Q(\mathrm{S}^{d-1})$.

Remark 4.
a) Of course, for $q \in \langle 1, \infty \rangle$, weak and weak-$*$ convergence above coincide since $\mathrm{L}^q_{\rm loc}(\mathbf{R}^d)$ is reflexive.
b) The Theorem only gives us an upper bound for the order in $\boldsymbol{\xi}$. To illustrate that, consider the case $p = q = 2$, the H-distribution is actually an H-measure, which is of order 0 in $\boldsymbol{\xi}$. However, in general we cannot expect to get a distribution of order 0 in $\boldsymbol{\xi}$, as it will be illustrated in the next section. ∎

Dem. (of Theorem 7) The first equality above is clear, as the adjoint of $\mathcal{A}_\psi$ is $\mathcal{A}_{\overline{\psi}}$. Without loss of generality, we may assume that $p \leqslant 2$ (if $p > 2$, we would use the first equality in the statement of the theorem and proceed as in the case $p \leqslant 2$).

The rest of the proof follows along the same lines as in [7], after noting that

$$\lim_n {}_{\mathrm{L}^{p'}}\Big\langle \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v)}, \varphi_1 u_n \Big\rangle_{\mathrm{L}^p} = 0 \,.$$

Indeed, as $q \geqslant p'$, we have that $\varphi_2 v \in \mathrm{L}^{p'}(\mathbf{R}^d)$, thus $\mathcal{A}_{\overline{\psi}}(\varphi_2 v) \in \mathrm{L}^{p'}(\mathbf{R}^d)$ as well, and we can pass to the limit in the product.

Take ϑ_l and K_l as in the definition of metric d_p on $\mathrm{L}^p_{\rm loc}(\mathbf{R}^d)$ in the Introduction; therefore $\operatorname{supp}\varphi_2 \subseteq K_l \subseteq \operatorname{supp}\vartheta_l$ for some $l \in \mathbf{N}$, and we have:

$$\begin{aligned}\lim_n {}_{\mathrm{L}^{p'}}\Big\langle \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_n)}, \varphi_1 u_n \Big\rangle_{\mathrm{L}^p} &= \lim_n {}_{\mathrm{L}^{p'}}\Big\langle \overline{\mathcal{A}_{\overline{\psi}}\Big(\varphi_2 \vartheta_l (v_n - v)\Big)}, \varphi_1 u_n \Big\rangle_{\mathrm{L}^p} \\ &= \lim_n {}_{\mathrm{L}^{p'}}\Big\langle \overline{\mathcal{A}_{\overline{\psi}}(\vartheta_l (v_n - v))}, \varphi_1 \overline{\varphi}_2 \vartheta_l u_n \Big\rangle_{\mathrm{L}^p} \\ &= \lim_n {}_{\mathrm{L}^{p'}}\Big\langle \overline{\mathcal{A}_{\overline{\psi}}(\vartheta_l v_n)}, \varphi_1 \overline{\varphi}_2 \vartheta_l u_n \Big\rangle_{\mathrm{L}^p} \quad =: \lim_n \mu_{n,l}(\varphi_1 \overline{\varphi}_2, \psi) \,.\end{aligned} \tag{9}$$

In the second equality we have used Corollary 3 (a version of the First commutation lemma for $\mathrm{L}^p(\mathbf{R}^d)$ spaces). The final expression shows that each integral is indeed a bilinear functional depending on $\varphi = \varphi_1 \overline{\varphi}_2$ and ψ.

Furthermore, by the Hölder inequality and continuity of the Fourier multiplier operator, we have

$$|\mu_{n,l}(\varphi, \psi)| \leqslant \|\varphi \vartheta_l u_n\|_{\mathrm{L}^p} \|\mathcal{A}_{\overline{\psi}}(\vartheta_l v_n)\|_{\mathrm{L}^{p'}} \leqslant \tilde{C} \|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})} \|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} \,,$$

where the constant $\tilde{C}$ is given by $\tilde{C} = C|u_n|_{\vartheta_l,p}|v_n|_{\vartheta_l,p'}$, C depending only on d and p, as a consequence of the continuity of Fourier multiplier $\mathcal{A}_\psi$.

For $l \in \mathbf{N}$, we can bound $|u_n|_{\vartheta_l,p}$ and $|v_n|_{\vartheta_l,p'}$ by constants independent of n and apply [7, Lemma 3.2], obtaining operators $D^l \in \mathcal{L}(\mathrm{C}_{K_l}(\mathbf{R}^d); (\mathrm{C}^\kappa(\mathrm{S}^{d-1}))')$, defined by

$$\langle D^l \varphi, \psi \rangle := \lim_{n'} \mu_{n',l}(\varphi, \psi) \,, \tag{10}$$

such that D^l is an extension of D^{l-1}.

This allows us to define an operator $D : \mathrm{C}_c(\mathbf{R}^d) \longrightarrow \Big(\mathrm{C}^\kappa(\mathrm{S}^{d-1})\Big)'$; namely, for $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$ we take $l \in \mathbf{N}$ such that $\operatorname{supp}\varphi \subseteq K_l$, and set $D\varphi := D^l\varphi$, which satisfies:

$$\|D\varphi\|_{(\mathrm{C}^\kappa(\mathrm{S}^{d-1}))'} \leqslant C_{K_l}\|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} .$$

As this operator D is continuous when restricted to each $\mathrm{C}_{K_l}(\mathbf{R}^d)$, D is continuous on the strict inductive limit of these spaces as well, i.e. on $\mathrm{C}_c(\mathbf{R}^d)$.

Now we can apply Theorem 5, which gives us the unique $\mu \in \mathcal{D}'_{0,Q}(\mathbf{R}^d \times \mathrm{S}^{d-1})$. By (9) and (10), this μ satisfies the required equality $\mu(\varphi, \psi) = \langle D\varphi, \psi\rangle$.

Q.E.D.

Remark 5. Note that in the proof we actually have shown that μ is a continuous bilinear form in the product topology of $\mathrm{C}_c(\mathbf{R}^d) \times \mathrm{C}^Q(\mathrm{S}^{d-1})$. Therefore, by Theorem 5 (see also Remark 2(b)) we can conclude that μ belongs to the space $\mathcal{D}'_{2d,\kappa}(\mathbf{R}^d \times \mathrm{S}^{d-1})$ as well, hence $\mu \in \mathcal{D}'_{0,(d-1)(\kappa+2)}(\mathbf{R}^d \times \mathrm{S}^{d-1}) \cap \mathcal{D}'_{2d,\kappa}(\mathbf{R}^d \times \mathrm{S}^{d-1})$. ∎

Remark 6. One can use Marcinkiewicz's theorem instead of Hörmander-Mihlin's for continuity of the Fourier multipliers. This approach was used in [22, 25] where they had a variant of H-measures and H-distributions on manifolds different than unit sphere. However, in this way one requires a higher regularity of test functions in $\boldsymbol{\xi}$ ($\mathrm{C}^d(\mathrm{S}^{d-1})$). ∎

We shall say that (u_n) and (v_n) form a *pure pair of sequences* if the associated H-distribution is unique for all subsequences.

If (u_n) and (v_n) are L^p and L^q sequences, respectively, defined on an open set $\Omega \subseteq \mathbf{R}^d$, extending them by zero to the whole space, we would still retain weak and weak-$*$ convergence of corresponding sequences to corresponding limits. Then, applying the preceding theorem, we get that the corresponding H-distribution is supported on $\mathrm{Cl}\Omega \times \mathrm{S}^{d-1}$. Indeed, the claim follows easily if one takes test functions supported within the complement of the closure $\mathrm{Cl}\Omega$. The analogous statement holds for $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$ and $\mathrm{L}^q_{\mathrm{loc}}(\Omega)$ sequences, if we can extend them by zero to the whole space, which is not always possible (for example, take an $\mathrm{L}^1_{\mathrm{loc}}(\langle 0, 1\rangle)$ function $\mathbf{x} \mapsto \frac{1}{|\mathbf{x}|}$, which can not be extended to an $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^d)$ function).

Similar reasoning leads to the following result:

Corollary 4. *Let (u_n) and (v_n) be sequences from the preceding theorem. If there exist closed sets F_1 and F_2 of $\mathbf{R}^d$ such that u_n keep their support in F_1 and v_n in F_2, then the support of any H-distribution corresponding to subsequences of (u_n) and (v_n) is included in $(F_1 \cap F_2) \times \mathrm{S}^{d-1}$.* ∎

Basic properties

One of the useful features of H-measures is that they can determine whether a weakly converging $\mathrm{L}^2_{\mathrm{loc}}$ sequence converges strongly in the same space. Namely, an $\mathrm{L}^2_{\mathrm{loc}}$ sequence will converge to zero strongly if and only if the corresponding H-measure is zero. In this section we prove an analogous property for H-distributions.

Canonical choice of the $\mathrm{L}^{p'}$ sequence corresponding to an L^p sequence (u_n) is given by $v_n = \Phi_p(u_n)$, where Φ_p is an operator from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^{p'}(\mathbf{R}^d)$ defined by $\Phi_p(u) = |u|^{p-2}u$.

Before we proceed, let us state some properties of operator Φ_p, for $p \in \langle 1, \infty\rangle$, which we shall need later. First of all, Φ_p is a nonlinear Nemyckij operator, i.e. it is continuous from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^{p'}(\mathbf{R}^d)$ and additionally we have the following bound

$$\|\Phi_p(u)\|_{\mathrm{L}^{p'}(\mathbf{R}^d)} \leqslant \|u\|^{p/p'}_{\mathrm{L}^p(\mathbf{R}^d)}.$$

In the following lemma we generalise this to the spaces of locally integrable functions.

Lemma 4. *The Nemyckij operator Φ_p is continuous from $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ to $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$, for any $p \in \langle 1, \infty\rangle$.*

In addition, we have the following bound for the respective seminorms:

$$|\Phi_p(u)|_{\vartheta_k,p'} \leqslant |u|^{p/p'}_{\vartheta_{k+1},p} \ .$$

Dem. Take an arbitrary seminorm $|\cdot|_{\vartheta_k,p'}$ from the definition of the metric $d_{p'}$ and any $u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ in order to get the estimate

$$\begin{aligned}|\Phi_p(u)|^{p'}_{\vartheta_k,p'} &= \|\vartheta_k \Phi_p(u)\|^{p'}_{\mathrm{L}^{p'}(\mathbf{R}^d)}\\ &= \int_{\mathbf{R}^d} |\vartheta_k(\mathbf{x})|^{p'} |u(\mathbf{x})|^{(p-1)p'} d\mathbf{x}\\ &= \int_{\mathbf{R}^d} |\vartheta_k(\mathbf{x})|^{p'} |u(\mathbf{x})|^{p} d\mathbf{x}\\ &\leqslant \int_{\mathbf{R}^d} |\vartheta_{k+1}(\mathbf{x})|^{p} |u(\mathbf{x})|^{p} d\mathbf{x}\\ &= \|\vartheta_{k+1} u\|^p_{\mathrm{L}^p(\mathbf{R}^d)} \qquad = |u|^p_{\vartheta_{k+1},p} \ .\end{aligned}$$

For the inequality above we have used the fact that $|\vartheta_k(\mathbf{x})|^{p'} \leqslant 1 = |\vartheta_{k+1}(\mathbf{x})|^p$ when $\mathbf{x} \in K_{k+1}$.

From here we conclude that Φ_p maps bounded sets in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ topology to bounded sets in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$ topology (cf. [28, 6.1]). Moreover, if (u_n) is a bounded sequence in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$, then from the above estimate we get

$$|\Phi_p(u_n)|_{\vartheta_k,p'} \leqslant |u_n|^{p/p'}_{\vartheta_{k+1},p} < C^{p/p'}_{\vartheta_{k+1},p} \ ,$$

from which follows that $(\Phi_p(u_n))$ is a bounded sequence in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$, which is a (semi-)reflexive space. This implies that $(\Phi_p(u_n))$ is weakly precompact in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$ (see [28, Theorem 15.2.4]).

To conclude, take arbitrary $\varepsilon > 0$, $u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ and $k \in \mathbf{N}$. The continuity of $\Phi_p : \mathrm{L}^p(\mathbf{R}^d) \to \mathrm{L}^{p'}(\mathbf{R}^d)$ guarantees the existence of $\delta = \delta(u, k, \varepsilon) > 0$ such that

$$(\forall v \in \mathrm{L}^p(\mathbf{R}^d)) \ \Big(\|v - \vartheta_{k+1} u\|_{\mathrm{L}^p(\mathbf{R}^d)} < \delta \Longrightarrow \|\Phi_p(v) - \Phi_p(\vartheta_{k+1}u)\|_{\mathrm{L}^{p'}(\mathbf{R}^d)} < \varepsilon \Big).$$

Take $v \in \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ such that $|v - u|_{\vartheta_{k+1},p} < \delta$. We have the following:

$$\begin{aligned}|\Phi_p(v) - \Phi_p(u)|_{\vartheta_k,p'} &= \Big\|\vartheta_k \Big(\Phi_p(v) - \Phi_p(u)\Big)\Big\|_{\mathrm{L}^{p'}(\mathbf{R}^d)}\\ &= \Big\|\vartheta_k\vartheta_{k+1} \Big(\Phi_p(v) - \Phi_p(u)\Big)\Big\|_{\mathrm{L}^{p'}(\mathbf{R}^d)}\\ &= \Big\|\vartheta_k \Big(\Phi_p(\vartheta_{k+1}v) - \Phi_p(\vartheta_{k+1}u)\Big)\Big\|_{\mathrm{L}^{p'}(\mathbf{R}^d)}\\ &\leqslant \|\Phi_p(\vartheta_{k+1}v) - \Phi_p(\vartheta_{k+1}u)\|_{\mathrm{L}^{p'}(\mathbf{R}^d)} \qquad < \varepsilon.\end{aligned}$$

Since k was arbitrary, we conclude that $\Phi_p : \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d) \longrightarrow \mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$ is continuous.

Q.E.D.

Now we can state the main result of this section:

Lemma 5. *For a sequence (u_n) in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$, $p \in \langle 1, \infty\rangle$, the following are equivalent*

a) $u_n \to 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$,

b) for every sequence (v_n) satisfying conditions of the existence theorem, (u_n) and (v_n) form a pure pair and the corresponding H-distribution is zero.

Dem. For the first implication, it is enough to notice that due to the compact support of test function φ_1 and boundedness properties of the Fourier multiplier operator $\mathcal{A}_\psi$, we get that $\mathcal{A}_\psi(\varphi_1 u_n) \to 0$ in $\mathrm{L}^p(\mathbf{R}^d)$, thus

$$\lim_n \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_n)(\mathbf{x})\overline{\varphi_2 v_n}(\mathbf{x})d\mathbf{x} = 0 \ .$$

Take a sequence $(\Phi_p(u_n))$. We have already concluded that it is bounded and weakly precompact in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$. Taking symbol ψ to be equal to one (so that $\mathcal{A}_\psi$ is the identity) and test functions φ_1 and φ_2, we get

$$0 = \lim_n \int_{\mathbf{R}^d} (\varphi_1 u_n)(\mathbf{x})\overline{(\varphi_2 v_n)(\mathbf{x})}d\mathbf{x} = \lim_n \int_{\mathbf{R}^d} \varphi_1(\mathbf{x})\overline{\varphi_2(\mathbf{x})}|u_n(\mathbf{x})|^p d\mathbf{x} \ ,$$

which implies $u_n \to 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$, so we have a pure pair and the whole sequence (u_n) converges to zero strongly in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$.

Q.E.D.

Remark 7. Let us notice that in the previous Lemma, we have proved that the two claims in (a) and (b) are equivalent to
b') u_n and $(\Phi_p(u_n))$ form a pure pair and the corresponding H-distribution is zero. ∎

Remark 8. It is easy to see that claim in (b) does not imply strong convergence to zero in $\mathrm{L}^p(\mathbf{R}^d)$ of the sequence (u_n) in $\mathrm{L}^p(\mathbf{R}^d)$. Indeed, take a nontrivial $u \in \mathrm{L}^p_c(\mathbf{R}^d)$ and a unit vector $\mathsf{e} \in \mathrm{S}^{d-1}$. Define a sequence $u_n(\mathbf{x}) = u(\mathbf{x} - n\mathsf{e})$ which weakly converges to zero in $\mathrm{L}^p(\mathbf{R}^d)$. The support of u_n goes to infinity so the corresponding H-distribution is zero, while u_n does not converge to zero strongly in $\mathrm{L}^p(\mathbf{R}^d)$. ∎

Let (u_n) be a sequence weakly converging to 0 in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$. Then the sequence $(|u_n|^p)$ is bounded in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^d)$, so $|u_n|^p \xrightarrow{*} \nu$ in $\mathcal{D}'(\mathbf{R}^d)$ (up to a subsequence). Since all elements of the sequence $(|u_n|^p)$ are positive (in terms of distributions), the limit ν is a positive distribution, hence an (unbounded) Radon measure.

On the other hand, let μ be any H-distribution corresponding to the above chosen subsequence of (u_n) and $(\Phi_p(u_n))$. Taking ψ to be equal to one and test functions φ_1, φ_2 such that φ_2 is equal to one on the support of φ_1, we get the following connection between μ and ν:

$$\langle \mu, \varphi_1 \boxtimes 1 \rangle = \lim_n \int_{\mathbf{R}^d} \varphi_1 |u_n|^p d\mathbf{x} = \langle \nu, \varphi_1 \rangle.$$

We summarise this observation in the following

Corollary 5. *Let (u_n) converge weakly to zero in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$, for some $p \in \langle 1, \infty \rangle$, and let $(|u_n|^p)$ converge weakly-$*$ to a measure ν in the space of unbounded Radon measures $\mathcal{M}(\mathbf{R}^d)$ (i.e. distributions of order zero). Then for any $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$, it holds*

$$\langle \mu, \varphi \boxtimes 1 \rangle = \lim_n \int_{\mathbf{R}^d} \varphi |u_n|^p d\mathbf{x} = \langle \nu, \varphi \rangle,$$

where μ is any H-distribution corresponding to some subsequences of (u_n) and $(\Phi_p(u_n))$. ∎

At this point the reader might wonder if there is a connection between H-distributions and microlocal compactness forms [33]. The answer is positive, but it would take us to far to precisely describe it here. Instead, we point the interested reader to a recent paper [8] (in particular, cf. Theorem 10 in the aforementioned reference).

Let us see what happens when we change the positions of sequences (u_n) and (v_n) in a pair of pure sequences:

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbf{R}^d}(\varphi_1 v_n)(\mathbf{x})\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 u_n)(\mathbf{x})}d\mathbf{x} &= \lim_{n\to\infty}\int_{\mathbf{R}^d}\overline{(\varphi_2 u_n)(\mathbf{x})}\mathcal{A}_{\psi}(\varphi_1 v_n)(\mathbf{x})d\mathbf{x} = \\ &= \overline{\lim_{n\to\infty}\int_{\mathbf{R}^d}(\varphi_2 u_n)(\mathbf{x})\overline{\mathcal{A}_{\psi}(\varphi_1 v_n)(\mathbf{x})}d\mathbf{x}} = \\ &= \overline{\left\langle \mu, \overline{\varphi_1}\varphi_2\boxtimes\overline{\psi}\right\rangle} = \left\langle \overline{\mu}, \varphi_1\overline{\varphi_2}\boxtimes\psi\right\rangle,\end{aligned}$$

where μ is the H-distribution corresponding to sequences (u_n) and (v_n).

Next, we turn our attention to the relation between H-distributions corresponding to conjugated sequences. First we shall prove some auxiliary results. It is easy to see that for any $v\in\mathcal{S}(\mathbf{R}^d)$ we have:

$$\hat{\overline{v}}(\boldsymbol{\xi}) = \overline{\overline{\int_{\mathbf{R}^d}e^{-2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}\overline{v}(\mathbf{x})d\mathbf{x}}} = \overline{\int_{\mathbf{R}^d}e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}v(\mathbf{x})d\mathbf{x}} = \overline{\check{v}}(\boldsymbol{\xi})\ ,$$

and analogously $\check{\overline{v}} = \overline{\hat{v}}$. Using these relations, we arrive at the following chain of equalities valid for any $\psi\in\mathrm{C}^\kappa(\mathrm{S}^{d-1})$:

$$\mathcal{A}_\psi(\overline{v}) = (\psi\hat{\overline{v}})^\vee = (\psi\overline{\check{v}})^\vee = \left(\overline{\overline{\psi}\check{v}}\right)^\vee = \overline{\widehat{\overline{\psi}\check{v}}}.$$

Let us rewrite the last term above:

$$\begin{aligned}\widehat{\overline{\psi}\check{v}}(\mathbf{x}) &= \int_{\mathbf{R}^d}e^{-2\pi i\boldsymbol{\xi}\cdot\mathbf{x}}\overline{\psi}(\boldsymbol{\xi})\check{v}(\boldsymbol{\xi})d\boldsymbol{\xi} = \int_{\mathbf{R}^d}e^{2\pi i\boldsymbol{\eta}\cdot\mathbf{x}}\overline{\psi}(-\boldsymbol{\eta})\check{v}(-\boldsymbol{\eta})d\boldsymbol{\eta} = \int_{\mathbf{R}^d}e^{2\pi i\boldsymbol{\eta}\cdot\mathbf{x}}\overline{\tilde{\psi}}(\boldsymbol{\eta})\hat{v}(\boldsymbol{\eta})d\boldsymbol{\eta} \\ &= \left(\overline{\tilde{\psi}}\hat{v}\right)^\vee(\mathbf{x}) = \mathcal{A}_{\overline{\tilde{\psi}}}(v)(\mathbf{x}),\end{aligned}$$

where we have used the change of variables $\boldsymbol{\eta} = -\boldsymbol{\xi}$ and the notation $\tilde{v}(\mathbf{x}) = v(-\mathbf{x})$. Since $\mathcal{A}_\psi$ (and $\mathcal{A}_{\overline{\tilde{\psi}}}$) are continuous on $\mathrm{L}^p(\mathbf{R}^d)$, while $\mathcal{S}(\mathbf{R}^d)$ is dense in $\mathrm{L}^p(\mathbf{R}^d)$, we have shown that for any $v\in\mathrm{L}^p(\mathbf{R}^d)$ the equality $\mathcal{A}_\psi(\overline{v}) = \overline{\mathcal{A}_{\overline{\tilde{\psi}}}(v)}$ holds. Now, we can write

$$\begin{aligned}\lim_n\langle\mathcal{A}_\psi(\varphi_1\overline{u_n}),\overline{\varphi_2\overline{v_n}}\rangle &= \lim_n\langle\mathcal{A}_\psi(\overline{\overline{\varphi_1}u_n}),\overline{\varphi_2}v_n\rangle = \lim_n\left\langle\overline{\mathcal{A}_{\overline{\tilde{\psi}}}(\overline{\varphi_1}u_n)},\overline{\varphi_2}v_n\right\rangle \\ &= \overline{\lim_n\langle\mathcal{A}_{\overline{\tilde{\psi}}}(\overline{\varphi_1}u_n),\overline{\overline{\varphi_2}v_n}\rangle} = \overline{\left\langle\mu,\overline{\varphi_1}\varphi_2\boxtimes\overline{\tilde{\psi}}\right\rangle} \\ &= \langle\overline{\mu},\varphi_1\overline{\varphi_2}\boxtimes\tilde{\psi}\rangle = \langle\tilde{\overline{\mu}},\varphi_1\overline{\varphi_2}\boxtimes\psi\rangle,\end{aligned}$$

where μ is the H-distribution corresponding to subsequences of (u_n) and (v_n) and in the last step the tilde operation is taken only with respect to the $\boldsymbol{\xi}$ variable.

We have thus proven the following

Lemma 6. *Let (u_n) and (v_n) form a pure pair of sequences and let μ be the corresponding H-distribution. Then the following holds:*

a) The pair (v_n) and (u_n) is also pure, and the H-distribution corresponding to (v_n) and (u_n) is $\overline{\mu}$.

b) $(\overline{u_n})$ and $(\overline{v_n})$ is a pure pair and the corresponding H-distribution is $\tilde{\overline{\mu}}$, where the tilde operation is taken only with respect to the dual variable.

∎

5. Examples and applications

An example with concentration

Vitali's convergence theorem gives sufficient and necessary conditions under which a sequence of L^p functions converges strongly to a measurable function in L^p. One of them is uniform integrability, which implies that there are no concentration effects in the sequence. Hence, it is of interest to consider concentration effects in weakly converging sequences.

For $p \in \langle 1, \infty\rangle$, $\mathbf{z} \in \mathbf{R}^d$, and $n \in \mathbf{N}$, let us define a linear operator $\zeta_{p,n}$ on $\mathrm{L}^p(\mathbf{R}^d)$ by

$$\zeta_{p,n} u(\mathbf{x}) = n^{\frac{d}{p}} u(n(\mathbf{x}-\mathbf{z})) \,. \tag{11}$$

A simple change of variables shows that $\zeta_{p,n}$ is a linear isometry on $\mathrm{L}^p(\mathbf{R}^d)$, i.e. $\|\zeta_{p,n}u\|_{\mathrm{L}^p(\mathbf{R}^d)} = \|u\|_{\mathrm{L}^p(\mathbf{R}^d)}$. Moreover, for any $u \in \mathrm{L}^p(\mathbf{R}^d)$ the sequence $(\zeta_{p,n}u)$ weakly converges to 0 in $\mathrm{L}^p(\mathbf{R}^d)$.

Let us first show this convergence under an additional assumption that function u has a compact support. Since $(\zeta_{p,n}u)$ is bounded sequence in $\mathrm{L}^p(\mathbf{R}^d)$, it is sufficient to test it on a continuous test function φ with compact support (as such functions form a dense set in $\mathrm{L}^{p'}(\mathbf{R}^d)$). Thus, we get

$$\begin{aligned}
\int_{\mathbf{R}^d} \zeta_{p,n}u(\mathbf{x})\varphi(\mathbf{x})\,d\mathbf{x} &= \int_{\mathbf{R}^d} n^{d/p} u(n(\mathbf{x}-\mathbf{z}))\varphi(\mathbf{x})\,d\mathbf{x} \\
&= \int_{\mathbf{R}^d} n^{d/p-d} u(\mathbf{y})\varphi(\mathbf{y}/n+\mathbf{z})\,d\mathbf{y} \\
&= \frac{1}{n^{d/p'}} \int_{\operatorname{supp} u} u(\mathbf{y})\varphi(\mathbf{y}/n+\mathbf{z})\,d\mathbf{y} \\
&\leqslant \Big(\frac{\mathsf{vol}(\operatorname{supp} u)}{n^d}\Big)^{1/p'} \|u\|_{\mathrm{L}^p(\mathbf{R}^d)} \max_{\mathbf{R}^d} |\varphi| \,,
\end{aligned}$$

where we have used the change of variables $\mathbf{y} = n(\mathbf{x}-\mathbf{z})$ in the second equality and the Hölder inequality in the last step. Passing to the limit $n \to \infty$, we get our claim.

In a general situation one just needs to approximate u with a sequence of functions with compact support in the space $\mathrm{L}^p(\mathbf{R}^d)$. Indeed, let $u_m \longrightarrow u$ in $\mathrm{L}^p(\mathbf{R}^d)$; then for any $m \in \mathbf{N}$ one has

$$|\langle \zeta_{p,n}u, \varphi\rangle| = \Big|\langle \zeta_{p,n}u_m, \varphi\rangle + \langle \zeta_{p,n}(u-u_m), \varphi\rangle\Big| \leqslant |\langle \zeta_{p,n}u_m, \varphi\rangle| + \|u-u_m\|_{\mathrm{L}^p}\|\varphi\|_{\mathrm{L}^{p'}} \,.$$

Passing to the limit in n we get

$$\limsup_{n\to\infty} |\langle \zeta_{p,n}u, \varphi\rangle| \leqslant \|u-u_m\|_{\mathrm{L}^p}\|\varphi\|_{\mathrm{L}^{p'}} \,,$$

which can be made arbitrary small by choosing large enough m.

For arbitrary $u \in \mathrm{L}^p(\mathbf{R}^d)$ and $v \in \mathrm{L}^{p'}(\mathbf{R}^d)$, $1/p + 1/p' = 1$, we shall show that the H-distribution corresponding to sequences $(\zeta_{p,n}u)$ and $(\zeta_{p',n}v)$ is given by $\delta_{\mathbf{z}} \boxtimes \nu$, where ν is a distribution on $\mathrm{C}^\kappa(\mathrm{S}^{d-1})$ defined for $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ by

$$\langle \nu, \psi\rangle = \int_{\mathbf{R}^d} u(\mathbf{x})\overline{\mathcal{A}_{\bar\psi} v(\mathbf{x})}\,d\mathbf{x} \,.$$

Since the Nemyckij operator Φ_p and $\zeta_{p,n}$ commute in the following sense: for $u \in \mathrm{L}^p(\mathbf{R}^d)$

$$\begin{aligned}
\Phi_p(\zeta_{p,n}u)(\mathbf{x}) &= |n^{\frac{d}{p}} u(n(\mathbf{x}-\mathbf{z}))|^{p-2} n^{\frac{d}{p}} u(n(\mathbf{x}-\mathbf{z})) \\
&= n^{\frac{d(p-1)}{p}} |u(n(\mathbf{x}-\mathbf{z}))|^{p-2} u(n(\mathbf{x}-\mathbf{z})) = \zeta_{p',n}\Phi_p(u) \,,
\end{aligned}$$

by taking $v = \Phi_p(u)$ we reveal the canonical choice of the $\mathrm{L}^{p'}$ sequence corresponding to $(\zeta_{p,n}u)$, i.e. $\zeta_{p',n}v = \Phi_p(\zeta_{p,n}u)$.

Before we proceed, we shall need the following two lemmata:

Lemma 7. *Let $p \in \langle 1, \infty\rangle$ and $\mathbf{z} \in \mathbf{R}^d$. For any $u \in \mathrm{L}^p(\mathbf{R}^d)$ and $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$ it holds*

$$\varphi\zeta_{p,n}u - \varphi(\mathbf{z})\zeta_{p,n}u \longrightarrow 0 \text{ in } \mathrm{L}^p(\mathbf{R}^d)\,.$$

Dem. Using the change of variables $\mathbf{y} = n(\mathbf{x} - \mathbf{z})$, we get

$$\int_{\mathbf{R}^d} |\varphi(\mathbf{x})\zeta_{p,n}u(\mathbf{x}) - \varphi(\mathbf{z})\zeta_{p,n}u(\mathbf{x})|^p \, d\mathbf{x} = \int_{\mathbf{R}^d} |\varphi(\mathbf{y}/n + \mathbf{z}) - \varphi(\mathbf{z})|^p |u(\mathbf{y})|^p \, d\mathbf{y}\,,$$

which goes to 0 as n tends to infinity, by the Lebesgue dominated convergence theorem.

Q.E.D.

Lemma 8. *For any $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, $p \in \langle 1, \infty\rangle$, $\mathbf{z} \in \mathbf{R}^d$, and $n \in \mathbf{N}$, the operators $\mathcal{A}_\psi$ and $\zeta_{p,n}$ commute on $\mathrm{L}^p(\mathbf{R}^d)$.*

Dem. For $v \in \mathcal{S}(\mathbf{R}^d)$, we have

$$\begin{aligned}
\mathcal{A}_\psi(\zeta_{p,n}v)(\mathbf{x}) &= n^{\frac{d}{p}} \bar{\mathcal{F}}\Big(\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|) \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{y}\cdot\boldsymbol{\xi}} v(n(\mathbf{y} - \mathbf{z}))\, d\mathbf{y}\Big)(\mathbf{x}) \\
&= n^{\frac{d}{p}} \bar{\mathcal{F}}\Big(n^{-d} e^{-2\pi i \mathbf{z}\cdot\boldsymbol{\xi}} \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|) \int_{\mathbf{R}^d} e^{-2\pi i \frac{\mathbf{w}}{n}\cdot\boldsymbol{\xi}} v(\mathbf{w})\, d\mathbf{w}\Big)(\mathbf{x}) \\
&= n^{\frac{d}{p}} n^{-d} \bar{\mathcal{F}}\Big(e^{-2\pi i \mathbf{z}\cdot\boldsymbol{\xi}} \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\hat{v}(\boldsymbol{\xi}/n)\Big)(\mathbf{x}) \\
&= n^{\frac{d}{p}} n^{-d} \int_{\mathbf{R}^d} e^{2\pi i (\mathbf{x}-\mathbf{z})\cdot\boldsymbol{\xi}} \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\hat{v}(\boldsymbol{\xi}/n)\, d\boldsymbol{\xi} \\
&= n^{\frac{d}{p}} \int_{\mathbf{R}^d} e^{2\pi i \boldsymbol{\eta}\cdot(n(\mathbf{x}-\mathbf{z}))} \psi(\boldsymbol{\eta}/|\boldsymbol{\eta}|)\hat{v}(\boldsymbol{\eta})\, d\boldsymbol{\eta} = \zeta_{p,n}(\mathcal{A}_\psi v)(\mathbf{x})\,,
\end{aligned}$$

where we have used the change of variables $n(\mathbf{y} - \mathbf{z}) = \mathbf{w}$ in the second equality and $n\boldsymbol{\eta} = \boldsymbol{\xi}$ in the penultimate one. Since $\mathcal{S}(\mathbf{R}^d)$ is dense in $\mathrm{L}^p(\mathbf{R}^d)$, while $\mathcal{A}_\psi$ and $\zeta_{n,p}$ are continuous on $\mathrm{L}^p(\mathbf{R}^d)$, we get the claim.

Q.E.D.

Now, let us go back to the construction of H-distributions corresponding to sequences $(\zeta_{p,n}u)$ and $(\zeta_{p',n}v)$, where $u \in \mathrm{L}^p(\mathbf{R}^d)$ and $v \in \mathrm{L}^{p'}(\mathbf{R}^d)$ are arbitrary, and $\zeta_{p,n}$ is given by (11). Moreover, we shall show that $(\zeta_{p,n}u)$ and $(\zeta_{p',n}v)$ form a pure pair.

Taking test functions φ_1 and φ_2 to be continuous with compact supports, and $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, we get the following

$$\begin{aligned}
\lim_n \int_{\mathbf{R}^d} \varphi_1(\mathbf{x})(\zeta_{p,n}u)(\mathbf{x})\overline{\mathcal{A}_{\bar\psi}(\varphi_2\zeta_{p',n}v)(\mathbf{x})}\, d\mathbf{x} &= \varphi_1(\mathbf{z})\bar\varphi_2(\mathbf{z}) \lim_n \int_{\mathbf{R}^d} (\zeta_{p,n}u)(\mathbf{x})\overline{\mathcal{A}_{\bar\psi}(\zeta_{p',n}v)(\mathbf{x})}\, d\mathbf{x} \\
&= \varphi_1(\mathbf{z})\bar\varphi_2(\mathbf{z}) \lim_n \int_{\mathbf{R}^d} (\zeta_{p,n}u)(\mathbf{x})\overline{\zeta_{p',n}\mathcal{A}_{\bar\psi}(v)(\mathbf{x})}\, d\mathbf{x} \\
&= \varphi_1(\mathbf{z})\bar\varphi_2(\mathbf{z}) \lim_n \int_{\mathbf{R}^d} n^d u(n(\mathbf{x}-\mathbf{z}))\overline{\mathcal{A}_{\bar\psi}(v)(n(\mathbf{x}-\mathbf{z}))}\, d\mathbf{x} \\
&= \varphi_1(\mathbf{z})\bar\varphi_2(\mathbf{z}) \lim_n \int_{\mathbf{R}^d} u(\mathbf{y})\overline{\mathcal{A}_{\bar\psi}(v)(\mathbf{y})}\, d\mathbf{y} \\
&= \Big\langle \overline{\mathcal{A}_{\overline{\varphi_1(\mathbf{z})\bar\varphi_2(\mathbf{z})\psi}}(v)}, u \Big\rangle\,,
\end{aligned}$$

where we have used the preceding lemmata in the first and second equalities, and the change of variables $\mathbf{y} = n(\mathbf{x} - \mathbf{z})$ in the fourth one. In the last step we have noticed that the expression on the right hand side does not depend on n anymore.

The last expression can be extended by density to the whole $\mathrm{C}_c^{0,\kappa}(\mathbf{R}^d\times \mathrm{S}^{d-1})$, thus we finally get that $(\zeta_{p,n}u)$ and $(\zeta_{p',n}v)$ form a pure pair, and the H-distribution is given by

$$\langle \mu, \Psi\rangle = \left\langle \overline{\mathcal{A}_{\bar\Psi(\mathbf{z},\cdot)}(v)}, u\right\rangle\,, \quad \Psi\in \mathrm{C}_c^{0,\kappa}(\mathbf{R}^d\times \mathrm{S}^{d-1})\,. \tag{12}$$

Remark 9. For a given $p\in\langle 1,\infty\rangle$, if we had chosen $u\in \mathrm{L}^r_c(\mathbf{R}^d)$, where $r\geqslant \max\{2,2p-2\}$ (case $r=\infty$ included), and $v=\Phi_p(u)$, we would have been able to use Plancherel's theorem and rewrite the integral in polar coordinates to get

$$\begin{aligned}\langle \mu, \varphi_1\boxtimes\psi\rangle &= \varphi_1(\mathbf{z})\int_{\mathbf{R}^d}\hat u(\boldsymbol{\xi})\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)(\overline{|u|^{p-2}u})^\wedge(\boldsymbol{\xi})d\boldsymbol{\xi}\\ &= \varphi_1(\mathbf{z})\int_{\mathrm{S}^{d-1}}\int_0^\infty \hat u(t\boldsymbol{\eta})\psi(\boldsymbol{\eta})(\overline{|u|^{p-2}u})^\wedge(t\boldsymbol{\eta})t^{d-1}dt\; d\boldsymbol{\eta},\end{aligned}$$

since the given bound on r and the compactness of $\operatorname{supp} u$ imply that both functions u and v are contained in $\mathrm{L}^2(\mathbf{R}^d)$. ∎

In this example we got better (lower) orders than those that are provided by Theorem 7. In fact, the order $(0,\kappa)$, which is achieved in this example, is the best we can hope for H-distributions as we have that they are continuous bilinear forms on $\mathrm{C}_c(\mathbf{R}^d)\times \mathrm{C}^\kappa(\mathrm{S}^{d-1})$. However, the question of the optimal value of κ, which is dictated by the Hörmander-Mihlin theorem, remains open.

Concerning the value of κ, the case $\kappa=0$ is particularly desirable since in that case we would have that H-distributions are Radon measures. Unfortunately, already by this example we can refute the claim that all H-distributions are Radon measures, i.e. there exist $u\in \mathrm{L}^p(\mathbf{R}^d)$ and $v\in \mathrm{L}^{p'}(\mathbf{R}^d)$ such that the H-distribution for concentrating sequences as above is not a Radon measure. This is the content of the following subsection.

Not all H-distributions are Radon measures

In this subsection we shall see that for any $p\in\langle 1,\infty\rangle$ there exist $u\in \mathrm{L}^p(\mathbf{R}^d)$ and $v\in \mathrm{L}^{p'}(\mathbf{R}^d)$ such that the H-distribution associated to sequences $(\zeta_{p,n}u)$ and $(\zeta_{p',n}v)$ (see (11)) is not a Radon measure. By (12) it is sufficient to find a sequence of functions $\psi_n\in \mathrm{C}^\infty(\mathrm{S}^{d-1})$, bounded in $\mathrm{C}(\mathrm{S}^{d-1})$, but for which $\Big(|\langle \mathcal{A}_{\bar\psi_n}v,u\rangle|\Big)$ is unbounded. Of course, it is neccesary that (ψ_n) is unbounded in $\mathrm{C}^\kappa(\mathrm{S}^{d-1})$.

Take $\psi\in \mathrm{C}^\infty(\mathrm{S}^{d-1})$ such that $\|\psi\|_{\mathrm{L}^\infty(\mathrm{S}^{d-1})}=1$. Then for any $n\in\mathbf{N}$, for the n-th power of ψ we have that $\psi^n\in \mathrm{C}^\infty(\mathrm{S}^{d-1})$ and $\|\psi^n\|_{\mathrm{L}^\infty(\mathrm{S}^{d-1})}=1$. By the Banach-Steinhaus theorem the uniform boundedness in n of $(\mathcal{A}_{\bar\psi^n})$ in $\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d);\mathrm{L}^p(\mathbf{R}^d))$ (here we assume that $\bar\psi^n$ is extended to $\mathbf{R}^d\setminus\{0\}$ along rays through the origin, as usual) is equivalent to the property that for any $u\in \mathrm{L}^p(\mathbf{R}^d)$ and $v\in \mathrm{L}^{p'}(\mathbf{R}^d)$ the sequence $\Big(|\langle \mathcal{A}_{\bar\psi^n}v,u\rangle|\Big)$ is bounded. Indeed, the first implication is trivial, while to prove the latter we first apply the Banach-Steinhaus theorem to $v\mapsto\langle \mathcal{A}_{\bar\psi^n}v,u\rangle$ (for an arbitrary u), and then to $\mathcal{A}_{\bar\psi^n}$.

Thus, it is sufficient to find $\psi\in \mathrm{C}^\infty(\mathrm{S}^{d-1})$, $\|\psi\|_{\mathrm{L}^\infty(\mathrm{S}^{d-1})}=1$, such that $(\mathcal{A}_{\bar\psi^n})$ is not uniformly bounded in $\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d);\mathrm{L}^p(\mathbf{R}^d))$ as then, by the above equivalence, the mapping $\psi\mapsto\langle\mathcal{A}_{\bar\psi}v,u\rangle$ cannot be continuous on $\mathrm{C}(\mathrm{S}^{d-1})$, implying that the H-distribution above is not a Radon measure.

For an example of such ψ in two space dimensions ($d=2$) one can consider the symbol of the Ahlfros-Beurling operator (see e.g. [11]) which is given by $\psi_0(\xi_1,\xi_2):=\xi_1+i\xi_2$ since it is known that for any $p\in\langle 1,\infty\rangle$ the sequence of real numbers $\|\mathcal{A}_{\bar\psi_0^n}\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^2);\mathrm{L}^p(\mathbf{R}^2))}$ goes to infinity as n tends to infinity [11, Theorem 1.1 and Theorem 1.4].

Concerning the example in higher dimensions, we shall generalise the previous one by the method of dilations (see e.g. [30, Section 4.2]). For $k\in\mathbf{N}$, let us define $\psi_k:\mathbf{R}^2\times\mathbf{R}^k\to\mathbf{R}$ by $\psi_k(\boldsymbol{\xi},\boldsymbol{\eta}):=\psi_0(\boldsymbol{\xi})=\xi_1+i\xi_2$, and suppose that for some $p\in\langle 1,\infty\rangle$ the sequence of operators $(\mathcal{A}_{\bar\psi_k^n})_n$ is uniformly bounded on $\mathrm{L}^p(\mathbf{R}^{2+k})$, i.e. there exists $C>0$ (depending on k, but independent of n) such that for any $f\in\mathrm{L}^p(\mathbf{R}^{2+k})$ we have

$$\|\mathcal{A}_{\bar\psi_k^n}f\|_{\mathrm{L}^p(\mathbf{R}^{2+k})}\leqslant C\|f\|_{\mathrm{L}^p(\mathbf{R}^{2+k})}\,.$$

For $t>0$ we define $\mathcal{T}_{n,t}:=\delta_t^{-1}\circ\mathcal{A}_{\bar\psi_k^n}\circ\delta_t$, where δ_t is a dilation operator given by the following: for $g:\mathbf{R}^2\times\mathbf{R}^k\to\mathbf{R}$ we let $\delta_t g(\boldsymbol{\xi},\boldsymbol{\eta}):=g(\boldsymbol{\xi},t\boldsymbol{\eta})$. It is easy to see that the family of operators $(\mathcal{T}_{n,t})_{n\in\mathbf{N},\,t>0}$ is uniformly bounded (in both n and t) as well. Indeed, we have

$$\|\mathcal{T}_{n,t}f\|_{\mathrm{L}^p(\mathbf{R}^{2+k})}=t^{k/p}\|\mathcal{A}_{\bar\psi_k^n}(\delta_t f)\|_{\mathrm{L}^p(\mathbf{R}^{2+k})}\leqslant t^{k/p}C\|\delta_t f\|_{\mathrm{L}^p(\mathbf{R}^{2+k})}=C\|f\|_{\mathrm{L}^p(\mathbf{R}^{2+k})}\,.$$

Further on, one can easily check that the Fourier transform $\mathcal{F}$ and δ_t satisfy the identity $\mathcal{F}=t^k\delta_t\circ\mathcal{F}\circ\delta_t$. Thus, for an arbitrary $f\in\mathrm{L}^2(\mathbf{R}^{2+k})\cap\mathrm{L}^p(\mathbf{R}^{2+k})$ it holds

$$\begin{aligned}\widehat{\mathcal{T}_{n,t}f}(\boldsymbol{\xi},\boldsymbol{\eta})&=(\delta_t\circ\bar\psi_k^n\circ\pi)(\boldsymbol{\xi},\boldsymbol{\eta})\hat f(\boldsymbol{\xi},\boldsymbol{\eta})\\&=\left(\frac{\xi_1-i\xi_2}{\sqrt{|\boldsymbol{\xi}|^2+t^2|\boldsymbol{\eta}|^2}}\right)^n\hat f(\boldsymbol{\xi},\boldsymbol{\eta})\end{aligned}$$

(here projection π appears due to the extension of $\bar\psi_k^n$ to $\mathbf{R}^{2+k}\setminus\{0\}$). Let us fix one such f. Now the Lebesgue dominated convergence theorem and the Plancherel theorem imply that for any $n\in\mathbf{N}$

$$\lim_{t\to0+}\mathcal{T}_{n,t}f=\mathcal{T}_{n,0}f$$

in $\mathrm{L}^2(\mathbf{R}^{2+k})$, where $\widehat{\mathcal{T}_{n,0}f}(\boldsymbol{\xi},\boldsymbol{\eta}):=\left(\frac{\xi_1-i\xi_2}{|\boldsymbol{\xi}|}\right)^n\hat f(\boldsymbol{\xi},\boldsymbol{\eta})$. This implies existence of a sequence of positive numbers (t_m) converging to 0 for which $(\mathcal{T}_{n,t_m}f)_m$ converges almost everywhere to $\mathcal{T}_{n,0}f$. Finally, by Fatou's lemma we get

$$\|\mathcal{T}_{n,0}f\|_{\mathrm{L}^p(\mathbf{R}^{2+k})}\leqslant\liminf_m\|\mathcal{T}_{n,t_m}f\|_{\mathrm{L}^p(\mathbf{R}^{2+k})}\leqslant C\|f\|_{\mathrm{L}^p(\mathbf{R}^{2+k})}\,.$$

Since $f\in\mathrm{L}^2(\mathbf{R}^{2+k})\cap\mathrm{L}^p(\mathbf{R}^{2+k})$ was arbitrary, by density the above estimate holds for any $f\in\mathrm{L}^p(\mathbf{R}^{2+k})$.

For arbitrary $f\in\mathrm{L}^p(\mathbf{R}^2)$ and $g\in\mathrm{L}^p(\mathbf{R}^k)$, $g\neq0$, we have $\mathcal{T}_{n,0}(f\boxtimes g)=(\mathcal{A}_{\bar\psi_0^n}f)\boxtimes g$, hence

$$\|\mathcal{A}_{\bar\psi_0^n}f\|_{\mathrm{L}^p(\mathbf{R}^2)}=\frac{\|\mathcal{T}_{n,0}(f\boxtimes g)\|_{\mathrm{L}^p(\mathbf{R}^{2+k})}}{\|g\|_{\mathrm{L}^p(\mathbf{R}^k)}}\leqslant C\|f\|_{\mathrm{L}^p(\mathbf{R}^2)}\,.$$

However, this is in contradiction with the unboundedness of $\mathcal{A}_{\bar\psi_0^n}$. Thus, $\mathcal{A}_{\bar\psi_k^n}$ must be unbounded as well.

To conclude, with the argument above we have proved that there exist H-distributions which are not Radon measures. Therefore, our kernel theorem (Theorem 5) is really meaningful when applied on H-distributions. Nevertheless, one could still think of whether the order can be improved when for the $\mathrm{L}^{p'}$ sequence (v_n) one takes precisely the canonical choice $(\Phi_p(u_n))$. Although the above counterexample does not say anything for this specific case, the authors believe that even for such (v_n)-s in general H-distributions are not Radon measures.

Perturbations and approximations of symbols

Let (u_n) be a sequence weakly converging to 0 in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ for some $p\in\langle1,\infty\rangle$ and $v_n\xrightarrow{*}v$ in $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ for $q\geqslant p'$. Consider a sequence (d_n) strongly converging to zero in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$. Then $u_n+d_n\longrightarrow0$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ and we may ask ourselves if there exists a connection between the H-distribution μ corresponding to subsequences of (u_n) and (v_n) and the H-distribution μ_d

corresponding to subsequences of (u_n+d_n) and (v_n). It is easy to see that these two H-distributions are the same:

$$\int_{\mathbf{R}^d} \varphi_1(u_n+d_n)\overline{\mathcal{A}_{\bar\psi}(\varphi_2 v_n)}d\mathbf{x} = \int_{\mathbf{R}^d} \varphi_1 u_n\overline{\mathcal{A}_{\bar\psi}(\varphi_2 v_n)}d\mathbf{x} + \int_{\mathbf{R}^d} \varphi_1 d_n\overline{\mathcal{A}_{\bar\psi}(\varphi_2 v_n)}d\mathbf{x}\,.$$

The left hand side goes to $\langle \mu_d, \varphi_1\bar\varphi_2\boxtimes\psi\rangle$, the second term on the right hand side goes to 0, as the Fourier multiplier $\mathcal{A}_{\bar\psi}$ is a bounded operator on $\mathrm{L}^q(\mathbf{R}^d)$, while the first term goes to $\langle \mu, \varphi_1\bar\varphi_2\boxtimes\psi\rangle$.

In the above, we could have perturbed sequence (v_n) by a sequence strongly converging to zero in $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ as well. We would still get the same conclusion. In particular, by choosing $p=q=2$ we obtain a similar result for H-measures.

Now, let us turn our attention to generating sequences. We would like to know if we could use a smoother sequence to obtain the same H-distribution. Assume that we are given a family of sequences (u_n^m) in $\mathrm{C}^\infty_c(\mathbf{R}^d)$ such that $u_n^m \longrightarrow u_n$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$, for each fixed n, as $m\to\infty$. We can always find such approximating sequences since the space $\mathrm{C}^\infty_c(\mathbf{R}^d)$ is dense in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ for $p<\infty$. By using the Cantor diagonal procedure, we can extract a subsequence $w_k = u_k^{m(k)}$ such that $d_p(w_k,u_k)\leqslant 1/k$. It is straightforward to see that w_k weakly converges to zero in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$: for every $\varphi\in\mathrm{L}^{p'}_c(\mathbf{R}^d)$ it holds

$$\int_{\mathbf{R}^d} w_k\varphi = \int_{\mathbf{R}^d}(w_k-u_k)\varphi + \int_{\mathbf{R}^d} u_k\varphi\,.$$

The claim follows from the strong convergence of (w_k-u_k) and the weak convergence of (u_k).

Lemma 9. *The above defined sequences (w_n) and (u_n) generate the same H-distribution. In other words, for any $\varphi_1,\varphi_2\in\mathrm{C}_c(\mathbf{R}^d)$, $\psi\in\mathrm{C}^\kappa(\mathrm{S}^{d-1})$ and any sequence $v_n \stackrel{*}{\longrightarrow} v$ in $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ for $q\geqslant p'$, it holds*

$$\lim_n \int_{\mathbf{R}^d} \varphi_1 w_n\overline{\mathcal{A}_{\bar\psi}(\varphi_2 v_n)}d\mathbf{x} = \lim_n \int_{\mathbf{R}^d} \varphi_1 u_n\overline{\mathcal{A}_{\bar\psi}(\varphi_2 v_n)}d\mathbf{x}\,.$$

Dem. By the Hölder inequality, we get

$$\begin{aligned}\lim_n \int_{\mathbf{R}^d} |\varphi_1(w_n-u_n)\overline{\mathcal{A}_{\bar\psi}(\varphi_2 v_n)}d\mathbf{x}| &\leqslant \lim_n ||\varphi_1(w_n-u_n)||_{\mathrm{L}^p(\mathbf{R}^d)}||\mathcal{A}_{\bar\psi}(\varphi_2 v_n)||_{\mathrm{L}^{p'}(\mathbf{R}^d)}\\ &\leqslant C_\psi \lim_n ||\varphi_1(w_n-u_n)||_{\mathrm{L}^p(\mathbf{R}^d)}||\varphi_2 v_n||_{\mathrm{L}^{p'}(\mathbf{R}^d)}\,,\end{aligned}$$

where we have used the boundedness of the Fourier multiplier operator $\mathcal{A}_{\bar\psi}$ in the second inequality. Since the sequence $(\varphi_2 v_n)$ is bounded in $\mathrm{L}^{p'}$, we get that the right hand side goes to zero.

Q.E.D.

If $q<\infty$, we can approximate v_n by sequences of smooth functions in $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ as well. Analogously as we did with (u_n), we would arrive at a smooth sequence (g_n) such that $d_q(g_k,v_k)\leqslant 1/k$. We have the following:

Corollary 6. *If $q\in[p',\infty\rangle$, the pairs (u_n), (v_n) and (w_n), (g_n) generate the same H-distribution. In other words, for any $\varphi_1,\varphi_2\in\mathrm{C}_c(\mathbf{R}^d)$ and $\psi\in\mathrm{C}^\kappa(\mathrm{S}^{d-1})$, it holds:*

$$\lim_n \int_{\mathbf{R}^d} \varphi_1 w_n\overline{\mathcal{A}_{\bar\psi}(\varphi_2 g_n)}d\mathbf{x} = \lim_n \int_{\mathbf{R}^d} \varphi_1 u_n\overline{\mathcal{A}_{\bar\psi}(\varphi_2 v_n)}d\mathbf{x}\,.$$

∎

As a consequence, we get a similar statement in the case of H-measures:

Corollary 7. *If a sequence (u_n) in $\mathrm{L}^2(\mathbf{R}^d)$ generates an H-measure, then there exists a smooth sequence which generates the same H-measure.*

∎

Corollary 6 also covers the case when we use $(\Phi_p(u_n))$, the canonical $\mathrm{L}^{p'}_{\mathrm{loc}}$-sequence associated with (u_n). We could approximate it with some other sequence in $\mathrm{L}^{p'}_{\mathrm{loc}}$. However, it would be convenient if we could choose a smooth sequence (w_n) approximating (u_n) such that $(\Phi_p(w_n))$ approximates $(\Phi_p(u_n))$. This can be achieved because $\Phi_p : \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d) \longrightarrow \mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$ is continuous: repeating the construction from the beginning of this section, we need to choose $m(k) \in \mathbf{N}$ such that for $w_k = u_k^{m(k)}$ it holds both $d_p(w_k, u_k) \leqslant 1/k$ and $d_{p'}(\Phi_p(w_k), \Phi_p(u_k)) \leqslant 1/k$.

Corollary 8. *Two pairs of sequences (u_n), $(\Phi_p(u_n))$ and (w_n), $(\Phi_p(w_n))$, where (w_n) is chosen as above, generate the same H-distribution. In other words, for any $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, it holds:*

$$\lim_n \int_{\mathbf{R}^d} \varphi_1 w_n \overline{\mathcal{A}_{\bar\psi}(\varphi_2 \Phi_p(w_n))} d\mathbf{x} = \lim_n \int_{\mathbf{R}^d} \varphi_1 u_n \overline{\mathcal{A}_{\bar\psi}(\varphi_2 \Phi_p(u_n))} d\mathbf{x} \,.$$

∎

Our next step is to show that we can improve the regularity of symbol $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$. Actually, we have the following lemma:

Lemma 10. *Let $u_n \longrightarrow 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ for some $p \in \langle 1, \infty\rangle$ and $v_n \xrightarrow{*} v$ in $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ for $q \geqslant p'$. Let $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ and $\psi_k \in \mathrm{C}^\infty(\mathrm{S}^{d-1})$ be as above. Then for any $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\mathbf{R}^d)$ it holds*

$$\lim_n \int_{\mathbf{R}^d} \varphi_1 u_n \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_n)} d\mathbf{x} = \lim_k \lim_n \int_{\mathbf{R}^d} \varphi_1 u_n \overline{\mathcal{A}_{\overline{\psi_k}}(\varphi_2 v_n)} d\mathbf{x} \,.$$

Dem. Similarly as we did for the bound of $\mu_{n,l}$ in the proof of the existence of H-distributions (Theorem 6), we arrive at the following

$$\Bigl|\lim_n \int_{\mathbf{R}^d} \varphi_1 u_n \overline{\mathcal{A}_{\overline{\psi - \psi_k}}(\varphi_2 v_n)} d\mathbf{x}\Bigr| \leqslant C_{d,p} \|\varphi_1\varphi_2\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} \|\psi - \psi_k\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})} \leqslant \frac{C_{d,p}}{k} \|\varphi_1\varphi_2\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} \,.$$

Passing to the limit $k \to \infty$ we get the conclusion.

Q.E.D.

Remark 10. Throughout this paper we have used symbols associated to functions $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ by composing ψ with projection π from $\mathbf{R}^d \backslash \{0\}$ to S^{d-1}. As it is well known in the theory of pseudo-differential calculus, we can replace such symbols by $\tilde\psi \in \mathrm{C}^\kappa(\mathbf{R}^d)$ functions which are identically equal to $\psi\circ\pi$ only for large $|\boldsymbol{\xi}|$. Indeed, one needs to notice that $\eta(\boldsymbol{\xi}) := \tilde\psi(\boldsymbol{\xi}) - (\psi\circ\pi)(\boldsymbol{\xi})$ is a bounded $\mathrm{C}^\kappa(\mathbf{R}^d \setminus \{0\})$ function with compact support. Thus, $\mathcal{A}_\eta$ is a compact operator from $\mathrm{L}^p_c(\mathbf{R}^d)$ to $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ which, for $u_n \longrightarrow 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$, implies that

$$\begin{aligned} 0 = \lim_n \int_{\mathbf{R}^d} \mathcal{A}_\eta(\varphi_1 u_n)(\mathbf{x}) \overline{(\varphi_2 v_n)(\mathbf{x})} d\mathbf{x} = \lim_n \int_{\mathbf{R}^d} \mathcal{A}_{\tilde\psi}(\varphi_1 u_n)(\mathbf{x}) \overline{(\varphi_2 v_n)(\mathbf{x})} d\mathbf{x} - \\ - \lim_n \int_{\mathbf{R}^d} \mathcal{A}_{\psi\circ\pi}(\varphi_1 u_n)(\mathbf{x}) \overline{(\varphi_2 v_n)(\mathbf{x})} d\mathbf{x}. \end{aligned}$$

Hence,

$$\lim_n \int_{\mathbf{R}^d} \mathcal{A}_{\tilde\psi}(\varphi_1 u_n)(\mathbf{x}) \overline{(\varphi_2 v_n)(\mathbf{x})} d\mathbf{x} = \Bigl\langle \mu,\ \varphi_1 \overline{\varphi_2} \boxtimes \psi \Bigr\rangle \,,$$

where we understand the right hand side in the sense of existence theorem on H-distributions (Theorem 7).

∎

Compactness by compensation

It is well-known that weak convergences are ill behaved under nonlinear transformations (in contrast to their good behaviour under linear transformations). However, under some additional conditions, namely some control on the derivatives of functions in the sequence, we can pass to the limit in the weak (∗) sense, the corresponding limit being the nonlinear transformation of the weak limit. The theory of compactness by compensation (or compensated compactness) studies products, or quadratic terms of weakly converging sequences. A typical example of this theory is Tartar-Murat's div-curl lemma [27], which was soon after generalised to the case of constant coefficients, and then to the $\mathrm{L}^p - \mathrm{L}^{p'}$ setting. Further generalisation to variable coefficients in the L^2 case was derived using the localisation principle for H-measures in [38], and much later in [3] for situations with a scaling parameter. Similarly, a generalisation to the L^p case was obtained by the localisation principle for H-distributions [7], where a significant drawback was the fact that all the coefficients had to be smooth (i.e. C^∞), as we had to multiply distributions by them. One of the main motivations for this project was precisely the need to bypass this shortcoming, which is done by establishing a finer version of the existence theorem for H-distributions (see Theorem 7).

Indeed, by following closely the proof of [7, Theorem 4.1] and having in mind that H-distributions are anisotropic distributions in $\mathcal{D}'_{0,Q}$, we get the following generalisation of the localisation principle for H-distributions.

Theorem 8. *Assume that* $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ *and* $f_n \longrightarrow 0$ *in* $\mathrm{W}^{-1,q}_{\mathrm{loc}}(\mathbf{R}^d)$ *for some* $p \in \langle 1, \infty\rangle$ *and* $q \in \langle 1, d\rangle$*, such that they satisfy*

$$\sum_{i=1}^{d} \partial_i(a_i(\mathbf{x})u_n(\mathbf{x})) = f_n(\mathbf{x})\,,$$

where $a_i \in \mathrm{C}(\mathbf{R}^d)$*. Take an arbitrary sequence* (v_n) *bounded in* $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbf{R}^d)$*, and by* μ *denote the H-distribution corresponding to some subsequences of sequences* (u_n) *and* (v_n)*. Then,*

$$\sum_{i=1}^{d} a_i(\mathbf{x})\xi_i\mu(\mathbf{x},\boldsymbol{\xi}) = 0$$

in the sense of anisotropic distributions $\mathcal{D}'_{0,Q}(\mathbf{R}^d \times \mathrm{S}^{d-1})$*, where* $Q = (d-1)(\kappa+2)$*, while* $\kappa = \lfloor d/2 \rfloor + 1$*.* ∎

Let us emphasise once more that the main improvement of the above result is in having continuous coefficients of the differential operator, while before one needed to require smooth coefficients.

The same applies for other known results on H-distributions of this type [22, 25].

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