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A note on symmetric orderings

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Abstract

Let $\hat{A}_n$ be the completion by the degree of a differential operator of the n-th Weyl algebra with generators $x_1, \ldots, x_n, \partial^1, \ldots, \partial^n$. Consider n elements $X_1, \ldots, X_n$ in $\hat{A}_n$ of the form

$$X_i = x_i + \sum_{K=1}^{\infty} \sum_{l=1}^{n} \sum_{j=1}^{n} x_l p_{ij}^{K-1,l}(\partial) \partial^j,$$

where $p_{ij}^{K-1,l}(\partial)$ is a degree $(K-1)$ homogeneous polynomial in $\partial^1, \ldots, \partial^n$, antisymmetric in subscripts i, j. Then for any natural k and any function $i\colon \{1, \ldots, k\} \to \{1, \ldots, n\}$ we prove

$$\sum_{\sigma \in \Sigma(k)} X_{i_{\sigma(1)}} \cdots X_{i_{\sigma(k)}} \triangleright 1 = k!\, x_{i_1} \cdots x_{i_k},$$

where $\Sigma(k)$ is the symmetric group on k letters and $\triangleright$ denotes the Fock action of the $\hat{A}_n$ on the space of (commutative) polynomials.

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1 Introduction and motivation

In an earlier article [3], we derived a universal formula for an embedding of the universal enveloping algebra $U(\mathfrak{g})$ of any Lie algebra $\mathfrak{g}$ with underlying rank n free module over a commutative ring $\boldsymbol{k}$ containing the field $\mathbb{Q}$ of rational numbers into a completion $\hat{A}_{n,\boldsymbol{k}}$ of the n-th Weyl algebra over $\boldsymbol{k}$.

Definition 1. *The n-th Weyl algebra $A_{n,\boldsymbol{k}}$ over a commutative ring $\boldsymbol{k}$ is the associative $\boldsymbol{k}$-algebra defined by generators and relations as follows:*

$$A_{n,\boldsymbol{k}} := \boldsymbol{k}\langle x_1, \ldots, x_n, \partial^1, \ldots, \partial^n\rangle / \langle [x_i, x_j], [\partial^i, \partial^j], [x_i, \partial^j] - \delta_i^j,\ i, j, = 1, \ldots, n\rangle.$$

We use the "contravariant" notation for the generators of $A_{n,\boldsymbol{k}}$ ([7, 1.1]) and δ^i_j is the Kronecker symbol. The reader should recall the usual interpretation of the Weyl algebra elements as regular differential operators [2, 3]. In other words, the elements of $A_{n,\boldsymbol{k}}$ act on the polynomial algebra $\boldsymbol{k}[x_1, \ldots, x_n]$, consisting of commutative polynomials via the physicists' Fock action here denoted by $\triangleright\colon A_{n,\boldsymbol{k}} \otimes \boldsymbol{k}[x_1, \ldots, x_n] \to \boldsymbol{k}[x_1, \ldots, x_n]$. By definition, generators x_i act as the multiplication operators by x_i and ∂^j act as partial derivatives. The unit polynomial $1 \in \boldsymbol{k}[x_1, \ldots, x_n]$ is interpreted as the vacuum state.

Complete $A_{n,\boldsymbol{k}}$ along the filtration given by the degree of differential operator ([3, 7, 8]); the completion will be denoted $\hat{A}_{n,\boldsymbol{k}}$. Thus, the elements in $\hat{A}_{n,\boldsymbol{k}}$ can be represented as arbitrary power series in $\partial^1, \ldots, \partial^n$ with coefficients (say on the left) in the polynomial ring $\boldsymbol{k}[x_1, \ldots, x_n]$.

For a fixed basis $X_1^{\mathfrak{g}}, \ldots, X_n^{\mathfrak{g}}$ of $\mathfrak{g}$, denote by $C_{ij}^k \in \boldsymbol{k}$ for $i, j, k \in \{1, \ldots, n\}$ the structure constants defined by

$$[X_i^{\mathfrak{g}}, X_j^{\mathfrak{g}}] = \sum_{k=1}^n C_{ij}^k X_k^{\mathfrak{g}}. \tag{1}$$

Constants C_{ij}^k are antisymmetric in lower indices and satisfy a quadratic relation reflecting the Jacobi identity in $\mathfrak{g}$. According to [3], there is a unique monomorphism of $\boldsymbol{k}$-algebras $\iota\colon U(\mathfrak{g}) \to \hat{A}_{n,\boldsymbol{k}}$ extending the formulas

$$X_i^{\mathfrak{g}} \mapsto \iota(X_i^{\mathfrak{g}}) = \sum_{l=1}^n x_l \sum_{N=0}^\infty \frac{(-1)^N}{N!} B_N (\mathcal{C}^N)_i^l, \tag{2}$$

where B_N is the n-th Bernoulli number and $\mathcal{C}$ is an $n \times n$ matrix with values in $\boldsymbol{k}$, defined by

$$\mathcal{C}_j^i = \sum_{k=1}^n C_{jk}^i \partial^k.$$

The monomorphism ι does not depend on the choice of the basis; over $\mathbb{R}$ and $\mathbb{C}$ the formula (2) appeared to be known much before ([1, 5]) and, suitably interpreted, corresponds to the Gutt's star product [4]. A simple differential geometric derivation of the formula (2) over $\mathbb{R}$ is explained in detail in [7, Section 1.2.]. Similarly, Sections 7–9 of [3] provide a geometrical derivation in formal geometry over any ring containing rationals. See also [6] for another point of view. Expression (2) is related to the part of Campbell-Baker-Hausdorff series linear in the first argument ([3, Sections 7–9]). Denote by

$$e^{\mathfrak{g}}\colon \boldsymbol{k}[x_1,\ldots,x_n]\to U(\mathfrak{g}),\quad x_{\alpha_1}\cdots x_{\alpha_k}\mapsto \frac{1}{k!}\sum_{\sigma\in\Sigma(k)} X^{\mathfrak{g}}_{\alpha_{\sigma 1}}\cdots X^{\mathfrak{g}}_{\alpha_{\sigma k}} \quad (3)$$

the standard symmetrization (or coexponential) map (of vector spaces), where the symmetric group on k letters is denoted $\Sigma(k)$. Via monomorphism ι, the expression on the right-hand side of (3) can be interpreted in $\hat{A}_{n,\boldsymbol{k}}$. If we apply the resulting element of $\hat{A}_{n,\boldsymbol{k}}$ on 1 using (the formal completion of) the Fock action, we recover back the left-hand side of (3). In other words ([3, 8]),

$$((\iota\circ e^{\mathfrak{g}})(q))\triangleright 1 = q,\quad q\in\boldsymbol{k}[x_1,\ldots,x_n], \quad (4)$$

where $\triangleright$ denotes the Fock action by differential operators.

In this paper, it is proven that already the tensorial form,

$$X_i\mapsto \tilde{X}_i := \sum_{l=1}^{n} x_l \sum_{N=0}^{\infty} A_N(\mathcal{C}^N)^l_i, \quad (5)$$

of the universal formula (2), with $A_1 = 1$, guarantees in characteristic 0 that precisely the *symmetrically* ordered noncommutative expressions

$$\frac{1}{k!}\sum_{\sigma\in\Sigma(k)} X_{\alpha_{\sigma(1)}}\cdots X_{\alpha_{\sigma(k)}},$$

interpreted via the embedding (5), and after acting upon the vacuum, recover back the commutative product $x_{\alpha_1}\cdots x_{\alpha_k}$. The coefficients A_N in (5) may be arbitrary for $N>0$ and $A_1=1$, instead of the choice $A_N = \frac{(-1)^N}{N!}B_N$ for all N, and X_i may be generators of an arbitrary finitely generated associative $\boldsymbol{k}$-algebra U, instead of the motivating choice $X_i = X^{\mathfrak{g}}_i\in U(\mathfrak{g})$.

Even more generally, we may replace $A_N(\mathcal{C}^N)^l_i$ in (5) by any expression of the form $p^{N-1,l}_{ij}(\partial^1,\ldots,\partial^n)\partial^j$ provided that $p^{N-1,l}_{ij} = p^{N-1,l}_{ij}(\partial^1,\ldots,\partial^n)$ is a homogeneous polynomial of degree $(N-1)$ in $\partial^1,\ldots,\partial^n$, antisymmetric under interchange of i and j. Note that the previous case involving $U(\mathfrak{g})$ may be recovered by setting

$$p^{N-1,l}_{ij} = \frac{(-1)^N B_N}{N!}\sum_{s=1}^{n}(\mathcal{C}^{N-1})^l_s C^s_{ij}.$$

We do not discuss when the correspondence (5) (or its generalization involving $p^{N-1,l}_{ij}$) extends to a homomorphism $U\to\hat{A}_{n,\boldsymbol{k}}$ of algebras (in physics literature also called a *realization* of U). If U is tautologically defined as the subalgebra of $\hat{A}_{n,\boldsymbol{k}}$ generated by the expressions $\tilde{X}_i \in \hat{A}_{n,\boldsymbol{k}}$, we alert the reader that the corresponding PBW type theorem often fails and the dimension of the space of degree $k > 1$ noncommutative polynomials in $\tilde{X}_i$ generically exceeds the dimension of the space of symmetric polynomials of degree k.

In the rest of the article below, X_i-s are defined as elements in $\hat{A}_{n,\boldsymbol{k}}$ from the start, hence we proceed without a distinction between X_i and $\tilde{X}_i$.

2 Results

Theorem 2. *Assume* $\boldsymbol{k}$ *is a field of characteristic different from* 2. *Let*

$$X_i = x_i + \sum_{l=1}^{n} x_l \sum_{N=1}^{\infty} \sum_{j=1}^{n} p_{ij}^{N-1,l}(\partial^1, \ldots, \partial^n)\partial^j, \quad i = 1, \ldots, n, \tag{6}$$

be n *distinguished elements of* $\hat{A}_{n,\boldsymbol{k}}$, *where* $p_{ij}^{N-1,l}(\partial^1, \ldots, \partial^n)$ *are arbitrary homogeneous polynomials of degree* $(N-1)$ *in* $\partial^1, \ldots, \partial^n$, *antisymmetric in lower indices* i, j. *Let* $\alpha\colon \{1, \ldots, k\} \to \{1, \ldots, n\}$ *be any function. Then, in the index notation,* $\alpha_i = \alpha(i)$,

$$\sum_{\sigma \in \Sigma(k)} X_{\alpha_{\sigma(1)}} \cdots X_{\alpha_{\sigma(k)}} \triangleright 1 = k!\, x_{\alpha_1} \cdots x_{\alpha_k}. \tag{7}$$

Proof. We prove the theorem by induction on degree k. For $k = 1$ all terms with $N \geq 1$ vanish, because we apply at least one derivative to 1.

For general k, we write the sum (7) over all permutations in $\Sigma(k)$ in a different way. We use the fact that the set of permutations of k elements $\Sigma(k)$ is in the bijection with the set of pairs (i, ρ) where $0 \leq i \leq k$ and $\rho \in \Sigma(k-1)$. This can be done in many ways, but we use this concrete simple-minded bijection

$$(i, \rho) \mapsto \sigma, \quad \sigma(k) := \begin{cases} i, & k = 1, \\ \rho(k-1), & k > 1 \text{ and } \rho(k-1) < i, \\ \rho(k-1) + 1, & k > 1 \text{ and } \rho(k-1) \geq i. \end{cases}$$

For example, $(3, (2, 3, 1, 5, 4)) \mapsto (3, 2, 4, 1, 6, 5)$.

Define a bijection $\Theta_i\colon \{1, \ldots, k-1\} \to \{1, \ldots, i-1, i+1, \ldots, k\}$ by

$$\Theta_i(j) := \begin{cases} j, & j < i, \\ j + 1, & j \geq i. \end{cases}$$

Clearly now $\sigma(j+1) = \Theta_i(\rho(j))$ for $1 \le j < k$.

We may thus renumber the sum

$$\sum_{\sigma \in \Sigma(k)} X_{\alpha_{\sigma(2)}} \cdots X_{\alpha_{\sigma(k)}}$$

as the double sum

$$\sum_{i=1}^{k} X_{\alpha(i)} \cdot \sum_{\rho \in \Sigma(k-1)} X_{(\alpha \circ \Theta_i)(\rho(1))} \cdots X_{(\alpha \circ \Theta_i)(\rho(k-1))}$$

By the assumption of induction,

$$\sum_{\rho \in \Sigma(k-1)} X_{(\alpha \circ \Theta_i)(\rho(1))} \cdots X_{(\alpha \circ \Theta_i)(\rho(k-1))} \triangleright 1$$
$$= (k-1)!\, x_{(\alpha \circ \Theta_i)(1)} \cdots x_{(\alpha \circ \Theta_i)(k-1)}$$

The function Θ_i takes all values between 1 and k except i exactly once.

Therefore, the left-hand side of (7) may be rewritten as

$$(k-1)! \sum_{i=1}^{k} X_{\alpha(i)} \triangleright (x_{\alpha(1)} \cdots x_{\alpha(i-1)} x_{\alpha(i+1)} \cdots x_{\alpha(k)}). \tag{8}$$

Substituting the expression (6) for $X_{\alpha(i)}$ in (8) we immediately observe two summands. Let δ be the Kronecker symbol. Then the first summand is

$$(k-1)! \sum_{i=1}^{k} \sum_{r=1}^{n} x_r \delta^r_{\alpha(i)} \cdot (x_{\alpha(1)} \cdots x_{\alpha(i-1)} x_{\alpha(i+1)} \cdots x_{\alpha(k)}) = k!\, x_{\alpha(1)} \cdots x_{\alpha(k)},$$

yielding the desired right-hand side for (7). Hence for the step of induction on k it is sufficient to show that the remaining summand

$$(k-1)! \sum_{i=1}^{k} \sum_{N=1}^{\infty} \sum_{l=1}^{n} x_l \sum_{s=1}^{n} p^{N-1,l}_{\alpha(i)s} \partial^s (x_{\alpha(1)} \cdots x_{\alpha(i-1)} x_{\alpha(i+1)} \cdots x_{\alpha(k)})$$

vanishes. This follows if for any $N > 0$ the contribution

$$\sum_{i=1}^{k} \sum_{s=1}^{n} p^{N-1,l}_{\alpha(i)s} \partial^s (x_{\alpha(1)} \cdots x_{\alpha(i-1)} x_{\alpha(i+1)} \cdots x_{\alpha(k)}) = 0. \tag{9}$$

Let $s \in \{1, \ldots, n\}$ and $M(s) = \{j \in \{1, \ldots, i-1, i+1, \ldots, k\} | s = \alpha(j)\}$. By elementary application of partial derivatives,

$$\partial^s (x_{\alpha(1)} \cdots x_{\alpha(i-1)} x_{\alpha(i+1)} \cdots x_{\alpha(k)}) = \sum_{j \in M(s)} \prod_{r \in \{1,\ldots,k\} \setminus \{i,j\}} x_{\alpha(r)}. \tag{10}$$

In particular, the contributions from $s \notin \{\alpha(1),\ldots,\alpha(i-1),\alpha(i+1),\ldots,\alpha(k)\}$, that is for $M(s)=\emptyset$, vanish and $\partial^s(x_{\alpha(1)}\cdots x_{\alpha(i-1)}x_{\alpha(i+1)}\cdots x_{\alpha(k)}) = 0$.

Thus, for fixed i, the overall sum over all $s\in\{1,\ldots,n\}$ becomes a new sum over all $j\in\{1,\ldots,i-1,i+1,\ldots,k\}$ and each $j\neq i$ appears precisely once, namely for $s=\alpha(j)$. For fixed pair (i,s), notice that the summands do not depend on $j\in M(s)$, but we do not use this fact. By antisymmetry, $p^{N-1,l}_{\alpha(i)\alpha(i)}=0$ if char $\boldsymbol{k}\neq 2$, hence we are free to add any terms multiplied by $p^{N-1,l}_{\alpha(i)\alpha(i)}$. For fixed i, we conclude

$$\sum_{s=1}^n p^{N-1,l}_{\alpha(i)s}\partial^s(x_{\alpha(1)}\cdots x_{\alpha(i-1)}x_{\alpha(i+1)}\cdots x_{\alpha(k)}) = \sum_{j=1}^k p^{N-1,l}_{\alpha(i)\alpha(j)}\prod_{r\in\{1,\ldots,k\}\setminus\{i,j\}} x_{\alpha(r)}.$$

Regarding that $\prod_{r\in\{1,\ldots,k\}\setminus\{i,j\}} x_{\alpha(r)}$ is a symmetric tensor in i,j, and $p^{N-1,l}_{\alpha(i)\alpha(j)}$ is antisymmetric under exchange of i and j, their contraction must be zero,

$$\sum_{i=1}^k\sum_{j=1}^k p^{N-1,l}_{\alpha(i)\alpha(j)}\prod_{r\in\{1,\ldots,k\}\setminus\{i,j\}} x_{\alpha(r)} = 0.$$

Therefore, (9) follows, and consequently the step of induction on k. □

The reader may want to understand the reindexing and cancellation arguments following formula (10) on an example where α is not injective. Suppose $n=3$, $k=4$, and α sends $1,2,3,4$ to $1,3,3,2$ respectively. Then $\sum_{i=1}^4\sum_{s=1}^4 p^{N-1,l}_{\alpha(i)s}\partial^s\left(\prod_{r\neq i}x_{\alpha(r)}\right)$ has contributions as follows: for $i=1$ one obtains $\sum_s p^{N-1,l}_{1s}\partial^s(x_3x_3x_2) = p^{N-1,l}_{12}x_3x_3+2p^{N-1,l}_{13}x_3x_2$, for $i=2$ and $i=3$ equal contributions $\sum_s p^{N-1,l}_{3s}\partial^s(x_1x_3x_2) = p^{N-1,l}_{31}x_3x_2 + p^{N-1,l}_{32}x_1x_3+p^{N-1,l}_{33}x_1x_2$, and for $i=4$ one obtains $\sum_s p^{N-1,l}_{2s}\partial^s(x_1x_3x_3) = p^{N-1,l}_{21}x_3x_3+2p^{N-1,l}_{23}x_1x_3$. By the antisymmetry of $p^{N-1,l}$, the double sum is 0.

Corollary 3. *Under the assumptions of Theorem 2, there is a well-defined* $\boldsymbol{k}$*-linear map*

$$\tilde{e}\colon \boldsymbol{k}[x_1,\ldots,x_n]\to\hat{A}_{n,\boldsymbol{k}}$$

extending the formulas

$$\tilde{e}\colon x_{\alpha_1}\cdots x_{\alpha_k}\mapsto\sum_{\sigma\in\Sigma(k)} X_{\alpha_{\sigma(1)}}\cdots X_{\alpha_{\sigma(k)}}, \tag{11}$$

for all $k \geq 0$ and for all (nonstrictly) monotone $\alpha\colon \{1,\ldots,k\} \to \{1,\ldots,n\}$. Map $\tilde{e}$ satisfies

$$\tilde{e}(P_k) \triangleright 1 = k! P_k \tag{12}$$

for all (commutative) polynomials $P_k = P_k(x_{\alpha_1},\ldots,x_{\alpha_n})$ homogeneous of degree k. In particular, $\tilde{e}$ is injective iff $\mathrm{char}\,\boldsymbol{k} = 0$*. In that case, the elements $e(x_{\alpha_1}\cdots x_{\alpha_n})$ are linearly independent. If* $\mathrm{char}\,\boldsymbol{k} = 0$*, a modified map $e\colon \boldsymbol{k}[x_1,\ldots,x_n] \to \hat{A}_{n,\boldsymbol{k}}$ with normalization on k-homogeneous elements given by*

$$e\colon x_{\alpha_1}\cdots x_{\alpha_k} \mapsto \frac{1}{k!}\sum_{\sigma\in\Sigma(k)} X_{\alpha_{\sigma(1)}}\cdots X_{\alpha_{\sigma(k)}}, \tag{13}$$

is an injection.

The map $\tilde{e}$ is well-defined because the right-hand side in (11) is symmetric in $\alpha_1,\ldots,\alpha_k$. Formula (7) can be restated as $\tilde{e}(-)\triangleright 1 = k!\,\mathrm{id}$. Note that the expressions (11) do not span an associative subalgebra, but only a subspace $e(\boldsymbol{k}[x_1,\ldots,x_n])$ of the subalgebra $\boldsymbol{k}\langle X_1,\ldots,X_n\rangle$ of $\hat{A}_{n,\boldsymbol{k}}$ generated by $X_1,\ldots,X_n$, in general. Denote by $\pi\colon \boldsymbol{k}\langle X_1,\ldots,X_n\rangle \to \boldsymbol{k}[x_1,\ldots,x_n]$ the vector space projection given by the Fock action on the vacuum vector $1\in\boldsymbol{k}[x_1,\ldots,x_n]$, that is $\pi(P) = P\triangleright 1$, $P\in\boldsymbol{k}\langle X_1,\ldots,X_n\rangle$. If $\mathrm{char}\,\boldsymbol{k} = 0$, the map e can be viewed as a $\boldsymbol{k}$-linear section of the projection map π. In particular, e is an isomorphism onto its own image and $\mathrm{Ker}\,\pi \oplus \mathrm{Im}\,e = \boldsymbol{k}\langle X_1,\ldots,X_n\rangle$.

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