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Interpolations between Jordanian twists, the Poincaré−Weyl algebra and dispersion relations

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We consider a two-parameter family of Drinfeld twists generated from a simple Jordanian twist further twisted by 1-cochains. Twists from this family interpolate between two simple Jordanian twists. Relations between them are constructed and discussed. It is proved that there exists a one-parameter family of twists identical to a simple Jordanian twist. The twisted coalgebra, star product and coordinate realizations of the κ-Minkowski noncommutative space–time are presented. Real forms of Jordanian deformations are also discussed. The method of similarity transformations is applied to the Poincaré–Weyl

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Hopf algebra and two types of one-parameter families of dispersion relations are constructed. Mathematically equivalent deformations, that are related to nonlinear changes of symmetry generators and linked with similarity maps, may lead to differences in the description of physical phenomena.

1. Introduction

Many proposals to resolve fundamental issues at the Planck scale involve the development of models of field theories on noncommutative (NC) spaces, most notably the κ-Minkowski space–time. The parameter κ is here usually interpreted as the Planck mass or the quantum gravity scale. One of the possible quantum symmetries of the κ-Minkowski NC space–time is the κ-Poincaré quantum group[1,2] and it constitutes one of the examples of deformed relativistic space–time symmetries and the corresponding dispersion relations. Similar models exhibit Hopf algebra symmetries.

For Hopf algebras, there is a remarkable systematic twisting procedure, invented by Drinfeld.[3,4] Namely, from a given Hopf algebra H and a twist element $\mathcal{F} \in H \otimes H$ satisfying the 2-cocycle condition, one produces a new Hopf algebra $H^{\mathcal{F}}$ with the same algebra sector, but a different coalgebra sector, with coproduct $\Delta^{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}$. If a space–time has a Hopf algebra symmetry, and the Hopf algebra is twisted, then the space–time can also be twisted using the same twist and preserving the covariance properties. Moreover, many other constructions like differential calculi and, to some extent, some basic constructions of field theories can be systematically deformed by procedures involving only a twist. Thus, the Drinfeld twist provides a laboratory for systematic deformation of space–time and for investigating its deformed relativistic symmetry, geometric and physical structures.

Drinfeld 2-cocycles can be further modified by 1-cochains.[3,4] Given a Drinfeld twist $\mathcal{F}$ and an invertible 1-cochain $\omega \in H$, the expression $\mathcal{F}_\omega = (\omega^{-1} \otimes \omega^{-1})\mathcal{F}\Delta(\omega)$ defines a new Drinfeld twist cohomologous to $\mathcal{F}$. If we start from a trivial 2-cocycle $1 \otimes 1$, we obtain a twist, $(\omega^{-1} \otimes \omega^{-1})\Delta(\omega)$, which is a 2-coboundary in the sense of non-Abelian cohomology.[4] Cohomologous 2-cocycles induce isomorphic deformed Hopf algebras and equivalent related mathematical constructions. Hence, we may say that the transformation of changing a 2-cocycle by a 1-cochain is a gauge transformation of the 2-cocycle.

In the late 1980s remarkable deformations of R-matrices and related quantum groups have been found under the name of Jordan(ian) R-matrices and Jordanian deformations.[5,6] The corresponding Drinfeld twist has been written out independently in Refs. 7 and 8. These examples involve the universal enveloping algebra of the two dimensional solvable Lie algebra (with generators H, E, $[H, E] = E$), some Hopf algebra which contains it ($U(\mathfrak{sl}(2))$, Yangian $Y(\mathfrak{gl}(2))$ etc.) or their

duals. A special example of a Jordanian twist is $\mathcal{F}_0 = \exp(\ln(1 + \alpha E) \otimes H) = 1 \otimes 1 + \alpha E \otimes H + O(\alpha^2)$ with α a deformation parameter.[8] Also, r-symmetric versions of the Jordanian twist, where $r = E \otimes H - H \otimes E$ is the classical r-matrix, were introduced in Refs. 9–11.

The Jordanian twist reappeared in the context of the κ-Minkowski NC space–time.[12–15] A relation with the symmetry of the κ-Minkowski space–time is established with the introduction of the generators of relativistic symmetries, dilatation D and momenta p_α (instead of generators H and E of the $\mathfrak{sl}(2)$ algebra) satisfying the same commutation relation, $[p_\alpha, D] = p_\alpha$. Dilatation D is included in a minimal extension of the relativistic space–time symmetry, the so-called Poincaré–Weyl algebra generated by $\{M_{\mu\nu}, p_\mu, D\}$, where $M_{\mu\nu}$ denote the Lorentz generators. One-parameter interpolation between Jordanian twists, which are generated from a simple Jordanian twist $\mathcal{F}_0$ by twisting with 1-cochains, were studied in Refs. 16–18.

Applications of Jordanian twists have been of interest in recent literature. For example, Jordanian deformations of the conformal algebra were considered in Refs. 19–22. In Refs. 20–22, deformations of the anti-de Sitter and the de Sitter algebra were investigated. Jordanian deformations have also been considered within applications in the AdS/CFT correspondence.[23–26] Integrable deformations of sigma models in relation to deformations of AdS$_5$ and supergravity were investigated.[27,28] Jordanian twists have been applied in the deformation of space–time metrics,[29] dispersion relations[14,15] and gauge theories.[30]

In this paper, we consider a two-parameter family of Drinfeld twists generated from a simple Jordanian twist by further twisting by 1-cochains. This is a generalization of results presented in Refs. 16–18. The first part of the paper is a sort of supplement to Ref. 17 with new results presented in Secs. 3–5. This two parameter generalization leads to the κ-Minkowski space–time and produces the same deformation of the Poincaré–Weyl symmetry algebra. We show that twists from this family interpolate between two simple Jordanian twists. Using the method of similarity transformations, we construct one-parameter families of dispersion relations related to each other by inverse transformation. We point out that mathematically equivalent deformations, that are related to nonlinear changes of the symmetry generators and linked with similarity maps, may lead to differences in the description of physical phenomena.

In Sec. 2, two special families of twists induced by 1-cochains are presented with results rewritten from Refs. 17 and 18. These twists interpolate between two simple Jordanian twists. In Sec. 3, we define a two-parameter family of Jordanian twists which is a generalization from Ref. 17. Relations between them are presented and discussed. In Sec. 4, we also generalize results from Ref. 17. In Subsec. 4.1, the twisted coalgebra sector and, in Subsec. 4.2, coordinate realizations and the star product are presented. Real forms of new Jordanian deformations are discussed in Subsec. 4.3. In Sec. 5, a method of similarity transformations is applied to the Poincaré–Weyl Hopf algebra and one-parameter families of dispersion relations are constructed. At the end of Sec. 5, concluding remarks are given.

2. Families $\mathcal{F}_{L,u}$ and $\mathcal{F}_{R,u}$ of Jordanian Twists and the Relation between Them

In Ref. 16, the following family of Drinfeld twists was considered:

$$\mathcal{F}_{L,u} = \exp(-u(DA \otimes 1 + 1 \otimes DA))$$

$$\times \exp(-\ln(1+A) \otimes D)\exp(\Delta(uDA)) \tag{1}$$

$$= \exp\left(\frac{u}{\kappa}(DP \otimes 1 + 1 \otimes DP)\right)\exp\left(-\ln\left(1 - \frac{1}{\kappa}P\right) \otimes D\right)$$

$$\times \exp\left(-\Delta\left(\frac{u}{\kappa}DP\right)\right), \tag{2}$$

where generators of dilatation D and momenta P satisfy $[P, D] = P$ and $A = -\frac{1}{\kappa}P$. The deformation parameter κ is of the order of the Planck mass, u is a real dimensionless parameter and Δ is the undeformed coproduct. These twists are constructed using a 1-cochain $\omega_L = \exp\left(-\frac{u}{\kappa}DP\right)$ and they satisfy the normalization and cocycle condition.

The corresponding deformed Hopf algebra is given by

$$\Delta^{\mathcal{F}_{L,u}}(p_\mu) = \mathcal{F}_{L,u}\Delta p_\mu \mathcal{F}_{L,u}^{-1} = \frac{p_\mu \otimes \left(1 + \frac{u}{\kappa}P\right) + \left(1 - \frac{1-u}{\kappa}P\right) \otimes p_\mu}{1 \otimes 1 + u(1-u)\frac{1}{\kappa^2}P \otimes P}, \tag{3}$$

$$\Delta^{\mathcal{F}_{L,u}}(D) = \mathcal{F}_{L,u}\Delta D \mathcal{F}_{L,u}^{-1}$$

$$= \left(D \otimes \frac{1}{1 + \frac{u}{\kappa}P} + \frac{1}{1 - \frac{1-u}{\kappa}P} \otimes D\right)$$

$$\times \left(1 \otimes 1 + u(1-u)\frac{1}{\kappa^2}P \otimes P\right), \tag{4}$$

$$S^{\mathcal{F}_{L,u}}(p_\mu) = \frac{p_\mu}{1 - (1-2u)\frac{1}{\kappa}P}, \tag{5}$$

$$S^{\mathcal{F}_{L,u}}(D) = -\left(\frac{1 - (1-2u)\frac{P}{\kappa}}{1 + \frac{u}{\kappa}P}\right)D\left(1 + \frac{u}{\kappa}P\right), \tag{6}$$

where p_μ, $\mu \in \{0, 1, \ldots, n-1\}$ are momenta in the Minkowski space–time and $P = v^\alpha p_\alpha$, where $v_\alpha v^\alpha \in \{1, 0, -1\}$.

For $u = 0$, $\mathcal{F}_{L,u=0}$ reduces to the Jordanian twist

$$\mathcal{F}_0 = \exp\left(-\ln\left(1 - \frac{1}{\kappa}P\right) \otimes D\right)$$

$$= \sum_{k=0}^{\infty}\left(-\frac{P}{\kappa}\right)^k \otimes \binom{-D}{k}. \tag{7}$$

For $u = 1/2$, $\mathcal{F}_{L,u=1/2}$ corresponds to the twist proposed in Refs. 9 and 10. For $u = 1$, it follows from Refs. 17 and 18 that $\mathcal{F}_{L,u=1}$ is identical to the Jordanian twist

$$\mathcal{F}_1 = \exp\left(-D \otimes \ln\left(1 + \frac{1}{\kappa}P\right)\right)$$

$$= \sum_{l=0}^{\infty} \binom{-D}{l} \otimes \left(\frac{P}{\kappa}\right)^l. \tag{8}$$

Hence, the family of twists $\mathcal{F}_{L,u}$ interpolates between twists $\mathcal{F}_0$ and $\mathcal{F}_1$.

Note that from $[P, D] = P$, it follows that $DP = P(D-1)$, and generally

$$f(D)P = Pf(D-1), \tag{9}$$

$$f(D)P^m = P^m f(D-m). \tag{10}$$

Another family of twists induced with a 1-cochain $\omega_R = \exp\left(-\frac{u}{\kappa}PD\right)$ is[17]

$$\mathcal{F}_{R,u} = \mathrm{e}^{\frac{u}{\kappa}(PD\otimes 1 + 1 \otimes PD)}\, \mathrm{e}^{-\ln\left(1 - \frac{1}{\kappa}P\right)\otimes D}\, \mathrm{e}^{-\Delta\left(\frac{u}{\kappa}PD\right)}. \tag{11}$$

These twists satisfy the normalization and cocycle condition.

The corresponding deformed Hopf algebra is given by

$$\Delta^{\mathcal{F}_{R,u}}(p_\mu) = \mathcal{F}_{R,u}\Delta p_\mu \mathcal{F}_{R,u}^{-1}$$

$$= \frac{p_\mu \otimes \left(1 + \frac{u}{\kappa}P\right) + \left(1 - \frac{1-u}{\kappa}P\right) \otimes p_\mu}{1 \otimes 1 + \frac{u(1-u)}{\kappa^2}P \otimes P}, \tag{12}$$

$$\Delta^{\mathcal{F}_{R,u}}(D) = \mathcal{F}_{R,u}\Delta D \mathcal{F}_{R,u}^{-1}$$

$$= \left(1 \otimes 1 + \frac{u(1-u)}{\kappa^2}P \otimes P\right)$$

$$\times \left(D \otimes \frac{1}{1 + \frac{u}{\kappa}P} + \frac{1}{1 - \frac{1-u}{\kappa}P}\right), \tag{13}$$

$$S^{\mathcal{F}_{R,u}}(p_\mu) = -\frac{p_\mu}{1 - \frac{1-2u}{\kappa}P}, \tag{14}$$

$$S^{\mathcal{F}_{R,u}}(D) = -\left(1 - (1-u)\frac{P}{\kappa}\right)D\left(\frac{1 - (1-2u)\frac{P}{\kappa}}{1 - (1-u)\frac{P}{\kappa}}\right). \tag{15}$$

For $u = 0$, $\mathcal{F}_{R,u=0}$ reduces to the Jordanian twist $\mathcal{F}_0$ and for $u = 1$ it was shown in Ref. 18 that $\mathcal{F}_{R,u=1} = \mathcal{F}_1$. Hence, the family of twists $\mathcal{F}_{R,u}$ interpolates between two Jordanian twists $\mathcal{F}_0$ and $\mathcal{F}_1$.

We point out that for $u = 1/2$, $\mathcal{F}_{R,u=1/2}^{-1} = \mathcal{F}_{\mathrm{GZ}}^{-1}$, where $\mathcal{F}_{\mathrm{GZ}}^{-1}$ is the twist proposed in Ref. 11, Theorem 2.20. Hence, the twist $\mathcal{F}_{R,u=1/2}$ is given by (11) and satisfies the normalization and cocycle condition automatically, as it is obtained as

the cochain twist of a normalized 2-cocycle. It was shown in Ref. 18 that the twist $\mathcal{F}_{R,u}^{-1}$ can be written as

$$\mathcal{F}_{R,u}^{-1} = \sum_{k,l=0}^{\infty} (u-1)^k \left(\frac{P}{\kappa}\right)^k \binom{D}{l} \otimes \left(\frac{u}{\kappa}P\right)^l \binom{D}{k}. \tag{16}$$

Starting with

$$\mathcal{F}_{R,u}^{-1} = \mathrm{e}^{\Delta\left(\frac{u}{\kappa}PD\right)} \mathrm{e}^{-\Delta\left(\frac{u}{\kappa}DP\right)} \mathcal{F}_{L,u}^{-1} \mathrm{e}^{\frac{u}{\kappa}(DP\otimes 1 + 1\otimes DP)} \mathrm{e}^{-\frac{u}{\kappa}(PD\otimes 1 + 1\otimes PD)}, \tag{17}$$

and using (see also Eq. (39) in Ref. 17)

$$\exp\left(\frac{u}{\kappa}PD\right) \exp\left(-\frac{u}{\kappa}DP\right) = 1 + \frac{u}{\kappa}P, \tag{18}$$

we find

$$\mathcal{F}_{R,u}^{-1} = \mathcal{F}_{L,u}^{-1} \frac{1}{1 \otimes 1 + \frac{u(1-u)}{\kappa^2}P \otimes P}. \tag{19}$$

Note that for $u = 1$, $\mathcal{F}_{L,u=1} = \mathcal{F}_{R,u=1}$.

3. Two-Parameter Family of Jordanian Twists and Relations between Them

More generally, we can define a two-parameter family of twists $\mathcal{F}_{w,u}$, with coboundary twist $\omega_{w,u} = \exp\left(-\frac{u}{\kappa}(D+w)P\right)$

$$\mathcal{F}_{w,u} = \exp\left(\frac{u}{\kappa}((D+w)P \otimes 1 + 1 \otimes (D+w)P)\right)$$

$$\times \exp\left(-\ln\left(1 - \frac{P}{\kappa}\right) \otimes D\right) \exp\left(-\Delta\left(\frac{u}{\kappa}(D+w)P\right)\right). \tag{20}$$

For $w = 1$ it coincides with $\mathcal{F}_{R,u}$ and for $w = 0$ with $\mathcal{F}_{L,u}$. Let us use relation (18). For $u = 1$, it yields

$$\mathrm{e}^{\frac{P}{\kappa}D} \mathrm{e}^{-D\frac{P}{\kappa}} = \mathrm{e}^{\frac{1}{\kappa}(DP+P)} \mathrm{e}^{-D\frac{P}{\kappa}} = \mathrm{e}^{\ln\left(1+\frac{P}{\kappa}\right)}. \tag{21}$$

Generalizing this relation for arbitrary w and using the BCH formula, it follows:

$$\exp\left(D\frac{P}{\kappa} + w\frac{P}{\kappa}\right) \exp\left(-D\frac{P}{\kappa}\right) = \exp\left(w\ln\left(1 + \frac{P}{\kappa}\right)\right) = \left(1 + \frac{P}{\kappa}\right)^w. \tag{22}$$

After rescaling $P \to uP$, $D \to D$, $w \to w$, it follows:

$$\exp\left(\frac{u}{\kappa}(D+w)P\right) \exp\left(-\frac{u}{\kappa}DP\right) = \left(1 + \frac{u}{\kappa}P\right)^w. \tag{23}$$

Using the above identity, the relation between $\mathcal{F}_{w,u}$ and $\mathcal{F}_{w=0,u} = \mathcal{F}_{L,u}$, one gets

$$\mathcal{F}_{w,u}^{-1} = \Delta\left(1 + \frac{u}{\kappa}P\right)^w \mathcal{F}_{L,u}^{-1} \frac{1}{\left(1 \otimes 1 + \frac{u}{\kappa}P \otimes 1\right)^w \left(1 \otimes 1 + 1 \otimes \frac{u}{\kappa}P\right)^w}, \quad (24)$$

$$\mathcal{F}_{L,u}\mathcal{F}_{w,u}^{-1} = \Delta^{\mathcal{F}_{L,u}}\left(1 + \frac{u}{\kappa}P\right)^w \left(\frac{1}{1 \otimes 1 + \frac{u}{\kappa}P \otimes 1}\right)^w \left(\frac{1}{1 \otimes 1 + 1 \otimes \frac{u}{\kappa}P}\right)^w$$

$$= \left(\frac{1}{1 \otimes 1 + \frac{u(1-u)}{\kappa^2}P \otimes P}\right)^w. \quad (25)$$

Hence,

$$\mathcal{F}_{w,u}^{-1} = \mathcal{F}_{L,u}^{-1}\left(\frac{1}{1 \otimes 1 + \frac{u(1-u)}{\kappa^2}P \otimes P}\right)^w. \quad (26)$$

The same relations follow from the definition of the star product and the methods in Refs. 31 and 32. In the limit $u \to 0$, $\mathcal{F}_{w,u=0}^{-1} = \mathcal{F}_0^{-1}$, and for $u = 1$

$$\mathcal{F}_{w,u=1}^{-1} = \mathcal{F}_1^{-1}, \quad \forall w \in \mathbb{R}. \quad (27)$$

Hence, the above one-parameter family of twists $\mathcal{F}_{w,u}$ is identically equal to the simple Jordanian twist $\mathcal{F}_1$, generalizing the results from Secs. 2 and 3. Note that for $u = 1/2$ twists $F_{w,u=1/2}$ are r-symmetric in the first order in $1/\kappa$ for all values of w.

The quantum R-matrices are

$$R_{w,u} = \left(1 \otimes 1 + \frac{u(1-u)}{\kappa^2}P \otimes P\right)^w R_{L,u}\left(1 \otimes 1 + \frac{u(1-u)}{\kappa^2}P \otimes P\right)^{-w}. \quad (28)$$

Note that the classical r-matrix does not depend on the parameters w and u.

$$r = \frac{1}{\kappa}(D \otimes P - P \otimes D). \quad (29)$$

4. Twisted Coalgebra, Star Product and Realizations

4.1. *Twisted coalgebra sector*

The corresponding Hopf algebra is defined with

$$\Delta^{\mathcal{F}_{w,u}}(p_\mu) = \Delta^{\mathcal{F}_{L,u}}p_\mu, \quad (30)$$

$$S^{\mathcal{F}_{w,u}}(p_\mu) = S^{\mathcal{F}_{L,u}}(p_\mu) \quad \text{for arbitrary } w, \quad (31)$$

$$\Delta^{\mathcal{F}_{w,u}}(D) = \left(1 \otimes 1 + \frac{u(1-u)}{\kappa^2}P \otimes P\right)^w \Delta^{\mathcal{F}_{L,u}}(D)$$

$$\times \left(1 \otimes 1 + \frac{u(1-u)}{\kappa^2}P \otimes P\right)^{-w}, \quad (32)$$

$$S^{\mathcal{F}_{w,u}}(D) = -\left(\frac{1 - \frac{1-2u}{\kappa}P}{1 + \frac{u}{\kappa}P}\right)^{1-w}\left(1 - \frac{1-u}{\kappa}P\right)^{w}D$$

$$\times \left(\frac{1 - \frac{1-2u}{\kappa}P}{1 - \frac{1-u}{\kappa}P}\right)^{w}\left(1 + \frac{u}{\kappa}P\right)^{1-w}. \tag{33}$$

Note that

$$S^{\mathcal{F}_{w,u=0}}(D) = -\left(1 - \frac{P}{\kappa}\right)D, \quad \forall w, \tag{34}$$

$$S^{\mathcal{F}_{w,u=1}}(D) = -D\left(1 + \frac{P}{\kappa}\right), \quad \forall w. \tag{35}$$

4.2. *Coordinate realizations of the κ-Minkowski space−time and star product*

Let us define the Heisenberg(–Weyl) algebra generated by commutative coordinates x_μ and the corresponding momenta p_μ, satisfying

$$[x_\mu, x_\nu] = 0, \quad [p_\mu, p_\nu] = 0, \quad [p_\mu, x_\nu] = -i\eta_{\mu\nu}. \tag{36}$$

The Heisenberg algebra acts on the space of functions $f(x) = f(\{x_\mu\})$ of the commutative coordinates, where x_μ act by multiplication and the action of generators p_μ and dilatation operator $D = ix_\mu p_\mu$ is given by

$$(p_\mu \triangleright f)(x) = -i\frac{\partial f(x)}{\partial x_\mu}, \tag{37}$$

$$(D \triangleright f)(x) = x_\mu \frac{\partial f(x)}{\partial x_\mu}. \tag{38}$$

The subalgebra of coordinates becomes noncommutative due to a twist deformation, replacing the usual multiplication with star product multiplication. This star product is associative, because the twist $\mathcal{F}_{w,u}$ satisfies the 2-cocycle condition. When the functions are exponentials e^{ikx} and e^{iqx}, we find their star product

$$e^{ikx} * e^{iqx} = e^{i\mathcal{D}_\mu(k,q)x_\mu}G(k,q), \tag{39}$$

where

$$\mathcal{D}_\mu(k,q) = \frac{k_\mu\left(1 + \frac{u}{\kappa}(v \cdot q)\right) + \left(1 - \frac{1-u}{\kappa}(v \cdot k)\right)q_\mu}{1 + \frac{u(1-u)}{\kappa^2}(v \cdot k)(v \cdot q)}, \tag{40}$$

$$G(k,q) = \left(1 + \frac{u(1-u)}{\kappa^2}(v \cdot k)(v \cdot q)\right)^{-w}, \tag{41}$$

and $kx = k_\alpha x_\alpha$, $qx = q_\alpha x_\alpha$, $v \cdot k = v_\alpha k_\alpha$, $v \cdot q = v_\alpha q_\alpha$. Results (39)–(41) follow also from the methods in Refs. 31 and 32.

Directly from the twist $\mathcal{F}_{w,u}$, we also calculate the realizations of the noncommutative coordinates $\hat{x}_\mu$ in terms of the Heisenberg algebra generators,

$$\hat{x}_\mu = m\left[\mathcal{F}_{w,u}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)\right]$$

$$= \left(x_\mu + iv_\mu \frac{1-u}{\kappa}D\right)\left(1 + \frac{u}{\kappa}P\right) + iw\frac{u(1-u)}{\kappa^2}v_\mu P\,, \qquad (42)$$

where m is the multiplication map $m : a \otimes b \mapsto ab$ of the Heisenberg algebra. These realizations are generalizations of those discussed in Refs. 33–37.

In the case $u = 0$, $\hat{x}_\mu = x_\mu + \frac{i}{\kappa}v_\mu D$ and in the case $u = 1$, $\hat{x}_\mu = x_\mu\left(1 + \frac{1}{\kappa}P\right)$. Note that for $u = 0$ and $u = 1$ realizations do not depend on w. Noncommutative coordinates $\hat{x}_\mu$, (42), generate the κ-Minkowski space–time and satisfy

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{\kappa}\left(v_\mu \hat{x}_\nu - v_\nu \hat{x}_\mu\right)\,, \qquad (43)$$

$$[p_\mu, \hat{x}_\nu] = \left(-i\eta_\mu^\nu + \frac{i}{\kappa}v_\nu \frac{1-u}{\kappa}p_\mu\right)\left(1 + \frac{u}{\kappa}P\right)\,. \qquad (44)$$

Note that the term in $\hat{x}_\mu$ proportional to w does not influence the commutation relations for $[\hat{x}_\mu, \hat{x}_\nu]$ and $[p_\mu, \hat{x}_\nu]$.

4.3. *Real forms of Jordanian deformations*

For physical applications it is important to address the question if the symmetry Hopf algebras can be endowed with (compatible) $*$-structures. Hopf $*$-algebras are Hopf algebras with an involution $a \mapsto a^*$ satisfying identities, needed to treat unitarity and hermiticity in physical applications. In our case, to construct the $*$-involution, we start with $P^* = P$, $D^* = -D$ and find

$$(\mathcal{F}_{w,u})^{*\otimes*} = \mathcal{F}_{w,u}^{-1}\left(1 \otimes 1 + \frac{u(1-u)}{\kappa^2}P \otimes P\right)^{2w-1} \qquad (45)$$

and

$$\left(S^{\mathcal{F}_{w,u}}(g)\right)^* = S^{\mathcal{F}_{(1-w),(1-u)}}(g^*)\big|_{-\kappa}\,, \quad \forall g\,. \qquad (46)$$

For $w = 1/2$, it follows:

$$\left(\mathcal{F}_{1/2,u}\right)^{*\otimes*} = \mathcal{F}_{1/2,u}^{-1}\,, \qquad (47)$$

$$\left(S^{\mathcal{F}_{1/2,u}}(D)\right)^* = S^{\mathcal{F}_{1/2,(1-u)}}(D^*)\big|_{-\kappa}\,. \qquad (48)$$

Hence the corresponding twist is unitary for $w = 1/2$, $u \in \mathbb{R}$ and also for $u = 0$, $u = 1$ and for arbitrary w.

Generally, the twist $\mathcal{F}_{w,u}$ is not unitary. If one wants to obtain a Hopf $*$-algebra structure on the deformation, there is a construction[4] provided the condition

$$(S \otimes S)(\mathcal{F})^{*\otimes*} = \mathcal{F}^\tau\,, \qquad (49)$$

is satisfied, where $\mathcal{F}^{\tau}$ denotes the flipped (transposed) twist $\mathcal{F}$. In our case, for $\mathcal{F}_{w,u}$, we check

$$(S \otimes S)(\mathcal{F}_{w,u})^{*\otimes*} = \mathcal{F}_{w,u}|_{-\kappa} = \mathcal{F}^{\tau}_{w,(1-u)}, \tag{50}$$

and the condition (49) is satisfied for $u = 1/2$. A whole discussion of real forms of Jordanian deformations can be found in Sec. 4 of Ref. 17.

5. Similarity Transformations, the Poincaré–Weyl Algebra and Dispersion Relations

Let us consider the twist $\mathcal{F}_{w,u}$, (20), for $u = 0$ and denote the corresponding generators with P^0, D^0, i.e.

$$\mathcal{F}_0 = \exp\left(-\ln\left(1 - \frac{P_0}{\kappa}\right) \otimes D_0\right), \tag{51}$$

leading to the following Hopf algebra

$$\Delta^{\mathcal{F}_0}\left(p^0_{\mu}\right) = p^0_{\mu} \otimes 1 + \left(1 - \frac{P_0}{\kappa}\right) \otimes p^0_{\mu}, \tag{52}$$

$$\Delta^{\mathcal{F}_0}(D_0) = D_0 \otimes 1 + \frac{1}{1 - \frac{1}{\kappa}P_0} \otimes D_0, \tag{53}$$

$$S^{\mathcal{F}_0}\left(p^0_{\mu}\right) = -\frac{p^0_{\mu}}{1 - \frac{1}{\kappa}P_0}, \tag{54}$$

$$S^{\mathcal{F}_0}(D_0) = -\left(1 - \frac{1}{\kappa}P_0\right)D_0. \tag{55}$$

We define Lorentz generators $M^0_{\mu\nu}$ generating the Poincaré–Weyl algebra together with p^0_{μ} and D^0

$$\left[M^0_{\mu\nu}, M^0_{\rho\sigma}\right] = -\left(\eta_{\mu\rho}M^0_{\nu\sigma} - \eta_{\mu\sigma}M^0_{\nu\sigma} + \eta_{\nu\rho}M^0_{\mu\sigma} - \eta_{\nu\sigma}M^0_{\mu\rho}\right),$$

$$\left[M^0_{\mu\nu}, p^0_{\lambda}\right] = -\left(\eta_{\mu\lambda}p^0_{\nu} - \eta_{\nu\lambda}p^0_{\lambda}\right), \quad \left[D^0, M^0_{\mu\nu}\right] = \left[p^0_{\mu}, p^0_{\nu}\right] = 0, \quad \left[D^0, p^0_{\mu}\right] = -p^0_{\mu}.$$

Then the coproduct and antipodes are

$$\Delta^{\mathcal{F}_0}\left(M^0_{\mu\nu}\right) = M^0_{\mu\nu} \otimes 1 + 1 \otimes M^0_{\mu\nu} + \frac{1}{\kappa}\frac{v_{\mu}p^0_{\nu} - v_{\mu}p^0_{\mu}}{1 - \frac{1}{\kappa}p^0} \otimes D^0, \tag{56}$$

$$S^{\mathcal{F}_0}\left(M^0_{\mu\nu}\right) = -M^0_{\mu\nu} + \frac{1}{\kappa}\left(v_{\mu}p^0_{\nu} - v_{\nu}p^0_{\mu}\right)D^0. \tag{57}$$

Generalizing Ref. 15, and using similarity transformations induced by $\omega_{w,u}$, we obtain new generators

$$p_{\mu} = \exp\left(-\frac{u}{\kappa}(D_0 + w)P_0\right)p^0_{\mu}\exp\left(\frac{u}{\kappa}(D_0 + w)P_0\right) = \frac{p^0_{\mu}}{1 - \frac{u}{\kappa}P_0}, \tag{58}$$

$$D = \exp\left(-\frac{u}{\kappa}(D_0 + w)P_0\right)D_0 \exp\left(\frac{u}{\kappa}(D_0 + w)P_0\right)$$

$$= D_0\left(1 - \frac{u}{\kappa}P_0\right) - \frac{u}{\kappa}wP_0 \,, \tag{59}$$

$$M_{\mu\nu} = \exp\left(-\frac{u}{\kappa}(D_0 + w)P_0\right)M_{\mu\nu}^0 \exp\left(\frac{u}{\kappa}(D_0 + w)P_0\right)$$

$$= M_{\mu\nu}^0 - \frac{u}{\kappa}(D^0 + w)\left(v_\mu p_\nu^0 - v_\nu p_\mu^0\right), \tag{60}$$

and the inverse relations

$$p_\mu^0 = \exp\left(\frac{u}{\kappa}(D_0 + w)P_0\right)p_\mu \exp\left(-\frac{u}{\kappa}(D_0 + w)P_0\right) = \frac{p_\mu}{1 + \frac{u}{\kappa}P} \,, \tag{61}$$

$$D^0 = \exp\left(\frac{u}{\kappa}(D_0 + w)P_0\right)D \exp\left(-\frac{u}{\kappa}(D_0 + w)P_0\right)$$

$$= D\left(1 + \frac{u}{\kappa}P\right) + \frac{u}{\kappa}wP \,, \tag{62}$$

$$M_{\mu\nu}^0 = \exp\left(\frac{u}{\kappa}(D_0 + w)P_0\right)M_{\mu\nu} \exp\left(-\frac{u}{\kappa}(D_0 + w)P_0\right)$$

$$= M_{\mu\nu} + \frac{u}{\kappa}(D + w)(v_\mu p_\nu - v_\nu p_\mu) \,. \tag{63}$$

Note that

$$(D + w)P = (D_0 + w)P_0 \,. \tag{64}$$

Using (58)–(60), we express $M_{\mu\nu}$, D, p_μ, in terms of $M_{\mu\nu}^0$, D_0, p_μ^0, and calculate the deformed coproducts of $M_{\mu\nu}$, D, p_μ in terms of $M_{\mu\nu}^0$, D_0 and p_μ^0, and then we use (61)–(63) to reexpress the resulting coproducts in terms of $M_{\mu\nu}$, D and p_μ, obtaining

$$\Delta^{\mathcal{F}_{w,u}}(p_\mu) = \Delta^{\mathcal{F}_0}\left(\frac{p_\mu^0}{1 - \frac{u}{\kappa}P_0}\right) = \frac{p_\mu \otimes \left(1 + \frac{u}{\kappa}P\right) + \left(1 - \frac{1-u}{\kappa}P\right) \otimes p_\mu}{1 \otimes 1 + \frac{u(1-u)}{\kappa^2}P \otimes P} \,, \tag{65}$$

$$\Delta^{\mathcal{F}_{w,u}}(D) = \Delta^{\mathcal{F}_0}\left(D_0\left(1 - \frac{u}{\kappa}P_0\right) - \frac{u}{\kappa}wP_0\right)$$

$$= \left(1 \otimes 1 + \frac{u(1-u)}{\kappa^2}P \otimes P\right)^w \left(D \otimes \frac{1}{1 + \frac{u}{\kappa}P} + \frac{1}{1 - \frac{1-u}{\kappa}P} \otimes D\right)$$

$$\times \left(1 \otimes 1 + \frac{u(1-u)}{\kappa^2}P \otimes P\right)^{1-w} \,, \tag{66}$$

$$\Delta^{\mathcal{F}_{w,u}}(M_{\mu\nu}) = \Delta^{\mathcal{F}_0}\left(M_{\mu\nu}^0 - \frac{u}{\kappa}(D^0 + w)(v_\mu p_\nu^0 - v_\nu p_\mu^0)\right)$$

$$= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}$$

$$- \frac{1}{\kappa}\left[(u-1)\frac{v_\mu p_\nu - v_\nu p_\mu}{1 + \frac{u-1}{\kappa}P} \otimes \left(D\left(1 + \frac{u}{\kappa}P\right) + \frac{u}{\kappa}wP\right)\right.$$

$$\left. + u\left(D + (D+w)\frac{u-1}{\kappa}P\right) \otimes \frac{v_\mu p_\nu - v_\nu p_\mu}{1 + \frac{u}{\kappa}P}\right], \tag{67}$$

and similarly for the antipode

$$S^{\mathcal{F}_{w,u}}(p_\mu) = S^{\mathcal{F}_0}\left(\frac{p_\mu^0}{1 - \frac{u}{\kappa}P_0}\right) = -\frac{p_\mu}{1 + \frac{2u-1}{\kappa}P}, \tag{68}$$

$$S^{\mathcal{F}_{w,u}}(D) = S^{\mathcal{F}_0}\left(D_0\left(1 - \frac{u}{\kappa}P_0\right) - \frac{u}{\kappa}wP_0\right)$$

$$= -\frac{1 + \frac{2u-1}{\kappa}P}{1 + \frac{u}{\kappa}P}D\left(1 + \frac{u}{\kappa}P\right)$$

$$+ w\frac{u(1-u)}{\kappa^2}P^2\frac{2 + \frac{2u-1}{\kappa}P}{\left(1 + \frac{u}{\kappa}P\right)\left(1 + \frac{u-1}{\kappa}P\right)}, \tag{69}$$

$$S^{\mathcal{F}_{w,u}}(M_{\mu\nu}) = S^{\mathcal{F}_0}\left(M_{\mu\nu}^0 - \frac{u}{\kappa}\left(D^0 + w\right)\left(v_\mu p_\nu^0 - v_\nu p_\mu^0\right)\right)$$

$$= -M_{\mu\nu} - \frac{u}{\kappa}(D+w)\left(v_\mu p_\nu - v_\nu p_\mu\right)$$

$$- \frac{v_\mu p_\nu - v_\nu p_\mu}{\kappa}\left[(u-1)D + u\frac{(u-1)(w-1)\frac{p}{\kappa} - w}{1 + \frac{u}{\kappa}P}\right]. \tag{70}$$

in agreement with the results (30)–(33) in Sec. 4. We point out that Eqs. (65)–(70) for $u = 1$ do not depend on the parameter w, in accordance with (27).

For the Poincaré algebra generated by Lorentz generators $M_{\mu\nu}^0$ and p_μ^0, the corresponding quadratic Casimir is $(p^0)^2 = (p^0)_\alpha(p^0)^\alpha$. It is also invariant under the deformed Poincaré algebra generated with $M_{\mu\nu}^0$ and p_μ, where p_μ is related to p_μ^0 (58) and p_μ^0 to p_μ (61). Note that $p^2 = p^\alpha p_\alpha$ is not invariant under $M_{\mu\nu}^0$. Hence

$$(p^0)^2 = \frac{p^2}{\left(1 + \frac{u}{\kappa}P\right)^2}. \tag{71}$$

Momenta p_μ correspond to a realization of $\hat{x}_\mu$ in terms of x_μ and p_μ (42). The corresponding dispersion relation is explicitly given

$$E^2 - \mathbf{p}^2 = m^2\left(1 + \frac{u}{\kappa}E\right)^2, \tag{72}$$

where m is the mass of the particle, E is the energy and $\mathbf{p}$ is the momentum. Note that m^2 is the value of the deformed mass Casimir.

On the other hand, the quadratic Casimir for the Poincaré algebra generated with $M_{\mu\nu}$ and p_μ is p^2. It is also invariant under the deformed Poincaré algebra generated with $M_{\mu\nu}$ and p_μ^0, where p_μ^0 is related to p_μ (61) and p_μ to p_μ^0 (58). Note that $(p^0)^2$ is not invariant under $M_{\mu\nu}$. Hence

$$p^2 = \frac{(p^0)^2}{\left(1 - \frac{u}{\kappa}P_0\right)^2} \, . \tag{73}$$

Momenta p_μ^0 correspond to a realization of $\hat{x}_\mu$ expressed in terms of x_μ^0, p_μ^0. The corresponding dispersion relation is explicitly given

$$E^2 - \mathbf{p}^2 = m^2\left(1 - \frac{u}{\kappa}E\right)^2 \, . \tag{74}$$

In the limit $\kappa \to \infty$ of Eqs. (72) and (74) the standard bilinear mass shell condition is obtained. These two families of dispersion relations are related to each other by inverse transformation, with a parameter change $u \mapsto -u$.

The addition of momenta $k_\mu \oplus q_\mu$ also depends on the parameter u

$$k_\mu \oplus q_\mu = \frac{k_\mu\left(1 + \frac{u}{\kappa}v \cdot q\right) + \left(1 + \frac{u-1}{\kappa}v \cdot k\right)q_\mu}{1 + \frac{u(1-u)}{\kappa^2}(v \cdot k)(v \cdot q)} \, . \tag{75}$$

Note that dispersion relations (71), (73) and addition of momenta (75) do not depend on parameter w, whereas the realization of $\hat{x}_\mu$ (42), the star product (39), $\Delta^{\mathcal{F}_{w,u}}(D)$ (32), $\Delta^{\mathcal{F}_{w,u}}(M_{\mu\nu})$ (67), $S^{\mathcal{F}_{w,u}}(D)$ (33) and $S^{\mathcal{F}_{w,u}}(M_{\mu\nu})$ (70) do depend on w.

Concluding remarks. The physical interpretation depends on the realization of the generators of the Poincaré–Weyl algebra.[15,31,38] Particularly, the spectrum of the relativistic hydrogen atom depending on the parameter u was investigated in Ref. 15. Differences in realizations of NC coordinates could also lead to different physical predictions, see e.g. Ref. 39, where dispersion relations and the time delay parameter were investigated. Mathematically equivalent deformations, that are related to nonlinear changes of symmetry generators and linked with similarity maps, may lead to differences in the description of physical phenomena.

Acknowledgments

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