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Leading terms of relations for standard modules of the affine Lie algebras $C_n^{(1)}$

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Abstract
In this paper, we give a combinatorial parametrization of leading terms of defining relations for the vacuum level k standard modules for the affine Lie algebra of type $C_n^{(1)}$. Using this parametrization, we conjecture colored Rogers–Ramanujan type combinatorial identities for $n \geq 2$ and $k \geq 2$; the identity in the case $n = k = 1$ is equivalent to one of Capparelli's identities.

Keywords Affine (Kac–Moody)Lie algebras · Vertex operator algebras · Integrable highest weight representations · Combinatorial bases of standard modules · Leading terms of defining relations · Rogers–Ramanujan type identities

Mathematics Subject Classification 17B67 · 17B69 · 05A19

1 Introduction

The famous Rogers–Ramanujan identities are two analytic identities, for $a = 0$ or 1,

$$\prod_{m\geq 0} \frac{1}{(1-q^{5m+1+a})(1-q^{5m+4-a})} = \sum_{m\geq 0} \frac{q^{m^2+am}}{(1-q)(1-q^2)\cdots(1-q^m)}. \tag{1}$$

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If, for $a = 0$, we expand both sides in Taylor series, then the coefficient of q^m obtained from the product side can be interpreted as a number of partitions of m with parts congruent $\pm 1 \bmod 5$. On the other side, the coefficient of q^m obtained from the sum side can be interpreted as a number of partitions of m such that a difference between two consecutive parts is at least two. We can write a partition as

$$\sum_{j\geq 1} jf_j \qquad \text{or} \qquad 1^{f_1}2^{f_2}3^{f_3}\ldots,$$

meaning that the part j appears f_j times in the partition, and $f_j = 0$ for all but finitely many j. With this notation, the difference two condition between two consecutive parts can be written as

$$f_j + f_{j+1} \leq 1, \qquad j \geq 1, \tag{2}$$

and two ways of expressing the coefficient of q^m in Taylor series of (1) can be stated as combinatorial Rogers–Ramanujan identity

$$\#\Big\{m = \sum jf_j \mid j \equiv \pm 1 (\mathrm{mod}\ 5)\Big\} = \#\Big\{m = \sum jf_j \mid f_j + f_{j+1} \leq 1\Big\}.$$

The analytic Rogers–Ramanujan identities have Gordon–Andrews–Bressoud's generalization (cf. [3,4,12,13,28]). These identities also have a combinatorial interpretation, but in general it is not so easy to interpret the sum sides of analytic identities as generating functions for a number of partitions satisfying certain difference conditions among parts.

In 1980s Rogers–Ramanujan type identities appeared in statistical physics and in representation theory of affine Kac–Moody Lie algebras. This led to two lines of intensive research and numerous generalizations of both analytic and combinatorial identities; the reader may consult, for example, the papers [6,8,9,14,17,19–22,26,27,29,52] and the references therein.

Lepowsky and Milne discovered in [32] that the product sides of Gordon–Andrews–Bressoud identities, multiplied with certain fudge factor F, are principally specialized characters of standard modules for the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. Lepowsky and R. L. Wilson realized that the factor F is a character of the Fock space for the principal Heisenberg subalgebra of $\widehat{\mathfrak{sl}}_2$, and that the sum sides of Gordon–Andrews–Bressoud identities are the principally specialized characters of the vacuum spaces of standard modules for the action of principal Heisenberg subalgebra. In a series of papers (see [34,35] and the references therein), Lepowsky and Wilson discovered vertex operators in the principal picture on standard $\widehat{\mathfrak{sl}}_2$-modules and constructed bases of vacuum spaces for the principal Heisenberg subalgebra parametrized by partitions satisfying certain difference 2 conditions. Very roughly speaking, in the Rogers–Ramanujan case, the vacuum space Ω is spanned by monomial vectors of the form

$$Z(-s)^{f_s}\ldots Z(-2)^{f_2}Z(-1)^{f_1}v_0, \quad s \geq 0,\ f_j \geq 0, \tag{3}$$

where $v_0 \in \Omega$ is a highest weight vector and $Z(j)$ are certain $\mathcal{Z}$-operators. The degree of monomial vector (3) is

$$-m = -\sum_{j=1}^{s} jf_j.$$

Since $\mathcal{Z}$-operators $Z(j)$ on Ω satisfy certain relations, roughly of the form

$$\begin{aligned} Z(-j)Z(-j) + 2\sum_{i>0} Z(-j-i)Z(-j+i) \approx 0, \\ Z(-j-1)Z(-j) + \sum_{i>0} Z(j-1-i)Z(-j+i) \approx 0 \end{aligned} \tag{4}$$

(see [34,35] and [38] for the precise formulation), we may replace the *leading terms*

$$Z(-j)Z(-j) \quad \text{and} \quad Z(-j-1)Z(-j) \tag{5}$$

of relations (4) with "higher terms" $Z(-j-i)Z(-j+i)$ and $Z(-j-1-i)Z(-j+i)$, $i > 0$, and reduce the spanning set (3) of Ω to a spanning set

$$Z(-s)^{f_s} \dots Z(-2)^{f_2} Z(-1)^{f_1} v_0, \quad s \geq 0, \qquad f_j + f_{j+1} \leq 1 \qquad \text{for all } j \geq 1. \tag{6}$$

By invoking the product formula for principally specialized character of Ω and the Rogers–Ramanujan identities, we see that vectors in the spanning set (6) are in fact a basis of Ω. In such a way Gordon–Andrews–Bressoud identities also appear for low-level representations of different affine Lie algebras or for representations of low-rank affine Lie algebras (see [10,37,42–45,53]), and S. Capparelli has found new combinatorial identities (see [5,15] and [46]). The analogous construction in the homogeneous picture is obtained in [33].

The results of this paper are closely related to a similar construction of combinatorial bases for the standard $\widehat{\mathfrak{sl}}_2$-modules obtained in [39] and [40] by A. Meurman and the first author, and independently in [18] by B. Feigin, R. Kedem, S. Loktev, T. Miwa, and E. Mukhin. The starting point is a basis of $\widehat{\mathfrak{sl}}_2$

$$c \quad \text{and} \quad x(j), h(j), y(j), \quad j \in \mathbb{Z}, \tag{7}$$

where $\{x, h, y\}$ is the standard basis of $\mathfrak{sl}_2$, and the corresponding Poincaré–Birkhoff–Witt monomial spanning set of level k standard $\widehat{\mathfrak{sl}}_2$-module $L(k\Lambda_0)$

$$y(-s)^{c_s} \dots y(-2)^{c_2} h(-2)^{b_2} x(-2)^{a_2} y(-1)^{c_1} h(-1)^{b_1} x(-1)^{a_1} v_0, \quad s \geq 0, \tag{8}$$

with $a_j, b_j, c_j \geq 0$. The spanning set (8) is analogous to the spanning set (3). We have a relation

$$\sum_{j_1+\dots+j_{k+1}=m} x(j_1)\cdots x(j_{k+1}) = 0 \tag{9}$$

with the leading term

$$x(-j-1)^b x(-j)^a \tag{10}$$

with $a+b=k+1$ and $(-j-1)b+(-j)a=m$. This is analogous to (4) and (5), and we can reduce the spanning set (8) to a smaller spanning set satisfying the difference condition

$$a_{j+1}+a_j \leq k,$$

but this spanning set is not a basis of $L(k\Lambda_0)$. All the relations needed to reduce (8) to a basis of $L(k\Lambda_0)$ are obtained from (9) by the adjoint action of $\mathfrak{sl}_2$, and all the leading terms are obtained by the adjoint action of $\mathfrak{sl}_2$ on (10)

$$\begin{aligned}
&x(-j-1)^b h(-j)^{a_2} x(-j)^{a_1}, \quad a_1+a_2=a,\\
&x(-j-1)^b y(-j)^{a_2} h(-j)^{a_1}, \quad a_1+a_2=a,\\
&h(-j-1)^{b_1} x(-j-1)^{b_2} y(-j)^{a}, \quad b_1+b_2=b,\\
&y(-j-1)^{b_1} h(-j-1)^{b_2} y(-j)^{a}, \quad b_1+b_2=b
\end{aligned} \tag{11}$$

(see (33)). By using these relations and their leading terms, we can reduce a spanning set (8) to a smaller spanning set of $L(k\Lambda_0)$ satisfying the difference conditions

$$\begin{aligned}
a_{j+1}+b_j+a_j &\leq k,\\
a_{j+1}+c_j+b_j &\leq k,\\
b_{j+1}+a_{j+1}+c_j &\leq k,\\
c_{j+1}+b_{j+1}+c_j &\leq k.
\end{aligned} \tag{12}$$

In [18] and [40] it is proved, by different methods, that this spanning set is a basis of $L(k\Lambda_0)$. This is analogous to the difference conditions for the basis (6). The degree of monomial vector (8) satisfying the difference conditions (12) is

$$-m=-\sum_{j\geq 1} ja_j-\sum_{j\geq 1} jb_j-\sum_{j\geq 1} jc_j,$$

so we are naturally led to interpret monomial basis vectors (8) in terms of colored partitions with parts j in three colors: x, h, and y (cf. [1,2,29]).

In this paper, we give a combinatorial description of (some) leading terms of relations for the level $k\geq 1$ standard modules $L(k\Lambda_0)$ of the affine Lie algebras $C_n^{(1)}$ for all $n\geq 2$. For the set of indices $\{1,2,\cdots,n,\underline{n},\cdots,\underline{2},\underline{1}\}$, we parametrize a basis of the Lie algebra of type C_n as

$$B=\Big\{X_{ab}\mid b\in\{1,2,\cdots,n,\underline{n},\cdots,\underline{2},\underline{1}\},\ a\in\{1,\cdots,b\}\Big\}.$$

We visualize B as a triangle, for $n = 3$ we have

$$\begin{array}{llllll}
11 \\
12 & 22 \\
13 & 23 & 33 \\
1\underline{3} & 2\underline{3} & 3\underline{3} & \underline{33} \\
1\underline{2} & 2\underline{2} & 3\underline{2} & \underline{32} & \underline{22} \\
1\underline{1} & 2\underline{1} & 3\underline{1} & \underline{31} & \underline{21} & \underline{11}.
\end{array}$$

For any point rr on the diagonal, we observe two triangles with this point in common. For example, for $r = \underline{3}$, we have the triangles

$$\begin{array}{llllll}
11 \\
12 & 22 \\
13 & 23 & 33 \\
1\underline{3} & 2\underline{3} & 3\underline{3} & \underline{33} \\
 & & & \underline{32} & \underline{22} \\
 & & & \underline{31} & \underline{21} & \underline{11}.
\end{array}$$

In each of these triangles, we observe a cascade, i.e., a "staircase" going downwards from the right to the left with a given multiplicity at each point. For example, we have two cascades, $\mathcal{B}$ and $\mathcal{A}$,

$$\begin{array}{cccccc}
\cdot \\
\cdot & \cdot \\
\cdot & 3 & 2 \\
5 & 0 & \cdot & \cdot \\
 & & & \cdot & 1 \\
 & & & 1 & 2 & \cdot
\end{array}$$

in the triangles with the common point $\underline{33}$. For two such cascades and some $j \in \mathbb{Z}$, we can write the leading term of a relation — in our example it is the monomial

$$X_{33}(-j-1)^2 X_{23}(-j-1)^3 X_{2\underline{3}}(-j-1)^0 X_{1\underline{3}}(-j-1)^5\, X_{\underline{22}}(-j)^1 X_{\underline{21}}(-j)^2 X_{\underline{31}}(-j)^1.$$

This monomial is the leading term of some relation for

$$k + 1 = (2 + 3 + 0 + 5) + (1 + 2 + 1),$$

i.e., for the standard module $L(13\Lambda_0)$. By using these relations and their leading terms, we can reduce the PBW spanning set

$$\prod_{ab \in B,\ j>0} X_{ab}(-j)^{m_{ab;j}}\, v_0$$

of $L(k\Lambda_0)$ to a smaller spanning set satisfying difference conditions

$$\sum_{ab\in\mathcal{B}} m_{ab;j+1} + \sum_{ab\in\mathcal{A}} m_{ab;j} \leq k \tag{13}$$

for any two cascades as above. *We conjecture that this spanning set is in fact a basis.*

If we interpret $\mathfrak{sl}_2$ as type C_1 Lie algebra, then our list of leading terms coincides with (11), and our difference conditions (13) coincide with difference conditions (12), so by [18] and [40] the spanning set is a basis. For level $k = 1$ and $C_2^{(1)}$ the spanning set is a basis by [50], and we show in [49] that the spanning set is a basis for $k = 1$ and all $n \geq 2$. These are the only cases in which our conjecture is proved. Numerical evidence supports our conjecture in the case $n = k = 2$ (see Example 3 in the last section).

It took us quite a while to understand the combinatorial parametrization of leading terms even for the $B_2^{(1)} = C_2^{(1)}$ type affine Lie algebra, especially in level one case I. Siladić enumerated leading terms by using a computer. G. Trupčević in [51] first encountered the "combinatorics of cascades" in his construction of combinatorial bases of the Feigin–Stoyanovsky's type subspaces of standard modules for the affine Lie algebras $\widehat{\mathfrak{sl}}_n$, a very special case of which are the "admissible configurations" in [17] and [47]. In [7], combinatorial bases of the Feigin–Stoyanovsky's type subspaces of all standard modules for the affine Lie algebras of type $C_\ell^{(1)}$ were constructed. Since the combinatorial parametrization of the leading terms in [7] formally coincides with the one described above for $n = 2\ell$, we feel that this formal similarity might also support our conjecture (cf. [48]). In the last section, we formulate conjectured colored Rogers–Ramanujan type combinatorial identities.

2 Vertex algebras for affine Lie algebras

Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and $\langle\,,\,\rangle$ a symmetric invariant bilinear form on $\mathfrak{g}$. Via this form we identify $\mathfrak{h}$ and $\mathfrak{h}^*$ and we assume that $\langle\theta,\theta\rangle = 2$ for the maximal root θ (with respect to some fixed basis of the root system). We fix a root vector x_θ in $\mathfrak{g}$. Set

$$\hat{\mathfrak{g}} = \coprod_{j\in\mathbb{Z}} \mathfrak{g}\otimes t^j + \mathbb{C}c, \qquad \tilde{\mathfrak{g}} = \hat{\mathfrak{g}} + \mathbb{C}d.$$

Then $\tilde{\mathfrak{g}}$ is the associated untwisted affine Kac–Moody Lie algebra (cf. [30]) with the commutator

$$[x(i), y(j)] = [x, y](i + j) + i\delta_{i+j,0}\langle x, y\rangle c.$$

Here, as usual, $x(i) = x \otimes t^i$ for $x \in \mathfrak{g}$ and $i \in \mathbb{Z}$, c is the canonical central element, and $[d, x(i)] = ix(i)$. We identify $\mathfrak{g}$ and $\mathfrak{g}\otimes 1$. Set

$$\tilde{\mathfrak{g}}_{<0} = \coprod_{j<0} \mathfrak{g}\otimes t^j, \qquad \tilde{\mathfrak{g}}_{\leq 0} = \coprod_{j\leq 0} \mathfrak{g}\otimes t^j + \mathbb{C}d, \qquad \tilde{\mathfrak{g}}_{\geq 0} = \coprod_{j\geq 0} \mathfrak{g}\otimes t^j + \mathbb{C}d.$$

For $k \in \mathbb{C}$ denote by $\mathbb{C}v_k$ the one-dimensional $(\tilde{\mathfrak{g}}_{\geq 0} + \mathbb{C}c)$-module on which $\tilde{\mathfrak{g}}_{\geq 0}$ acts trivially and c as the multiplication by k. The affine Lie algebra $\tilde{\mathfrak{g}}$ gives rise to the vertex operator algebra (see [24] and [23], here we use the notation from [40])

$$N(k\Lambda_0) = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_{\geq 0}+\mathbb{C}c)} \mathbb{C}v_k$$

for level $k \neq -g^\vee$, where $g^\vee$ is the dual Coxeter number of $\mathfrak{g}$; it is generated by the fields

$$x(z) = \sum_{m\in\mathbb{Z}} x_m z^{-m-1}, \qquad x \in \mathfrak{g}, \tag{14}$$

where we set $x_m = x(m)$ for $x \in \mathfrak{g}$. As usual, for $v \in N(k\Lambda_0)$ we denote the associated vertex operator by $Y(v,z) = \sum_{m\in\mathbb{Z}} v_m z^{-m-1}$, and the vacuum vector by $\mathbf{1}$. By the state-field correspondence, we have

$$x(z) = Y(x(-1)\mathbf{1}, z) \quad \text{for } x \in \mathfrak{g}.$$

The $\mathbb{Z}$-grading is given by $L_0 = -d$. From now on, we fix the level $k \in \mathbb{Z}_{>0}$.

3 Annihilating fields of standard modules

For the fixed positive integer level k, the generalized Verma $\tilde{\mathfrak{g}}$-module $N(k\Lambda_0)$ is reducible, and we denote by $N^1(k\Lambda_0)$ its maximal $\tilde{\mathfrak{g}}$-submodule. By [30, Corollary 10.4] the submodule $N^1(k\Lambda_0)$ is generated by the singular vector $x_\theta(-1)^{k+1}\mathbf{1}$. Set

$$R = U(\mathfrak{g})x_\theta(-1)^{k+1}\mathbf{1}, \qquad \bar{R} = \mathbb{C}\text{-span}\{r_m \mid r \in R, m \in \mathbb{Z}\}.$$

Then $R \subset N^1(k\Lambda_0)$ is an irreducible $\mathfrak{g}$-module, and $\bar{R}$ is the corresponding loop $\tilde{\mathfrak{g}}$-module for the adjoint action. We have the following theorem (see [16,25,36,40]):

Theorem 1 *Let M be a highest weight $\tilde{\mathfrak{g}}$-module of level k. The following are equivalent:*

1. *M is a standard module,*
2. *$\bar{R}$ annihilates M.*

This theorem implies that for a dominant integral weight Λ of level $\Lambda(c) = k$ we have

$$\bar{R}M(\Lambda) = M^1(\Lambda),$$

where $M^1(\Lambda)$ denotes the maximal submodule of the Verma $\tilde{\mathfrak{g}}$-module $M(\Lambda)$. Furthermore, since R generates the vertex algebra ideal $N^1(k\Lambda_0) \subset N(k\Lambda_0)$, the vertex operators $Y(v,z)$, $v \in N^1(k\Lambda_0)$, annihilate all standard $\tilde{\mathfrak{g}}$-modules

$$L(\Lambda) = M(\Lambda)/M^1(\Lambda)$$

of level k. We shall call the elements $r_m \in \bar{R}$ *relations* (for standard modules), and $Y(v, z)$, $v \in N^1(k\Lambda_0)$, *annihilating fields* (of standard modules). The field

$$Y(x_\theta(-1)^{k+1}\mathbf{1}, z) = x_\theta(z)^{k+1}$$

generates all annihilating fields. We also write

$$Y(x_\theta(-1)^{k+1}\mathbf{1}, z) = \sum_{m\in\mathbb{Z}} r_{(k+1)\theta}(m)z^{-m-k-1}.$$

4 Leading terms

Set $\bar{\mathfrak{g}} = \hat{\mathfrak{g}}/\mathbb{C}c$. The associative algebra $\mathcal{U} = U(\hat{\mathfrak{g}})/(c-k)$ inherits from $U(\hat{\mathfrak{g}})$ the filtration $\mathcal{U}_\ell$, $\ell \in \mathbb{Z}_{\geq 0}$; let us denote by $\mathcal{S} \cong S(\bar{\mathfrak{g}})$ the corresponding commutative graded algebra. Let B be a basis of $\mathfrak{g}$. We fix the basis $\bar{B}$ of $\bar{\mathfrak{g}}$,

$$\bar{B} = \bigcup_{j\in\mathbb{Z}} B \otimes t^j.$$

Let $\preceq$ be a linear order on $\bar{B}$ such that

$$i < j \quad \text{implies} \quad x(i) \prec y(j).$$

The symmetric algebra $\mathcal{S}$ has a basis $\mathcal{P}$ consisting of monomials in basis elements $\bar{B}$. Elements $\Pi \in \mathcal{P}$ are finite products of the form

$$\Pi = \prod_{i=1}^{\ell} X_i(j_i), \quad X_i(j_i) \in \bar{B},$$

and we shall say that Π is a colored partition of degree $|\Pi| = \sum_{i=1}^{\ell} j_i \in \mathbb{Z}$ and length $\ell(\Pi) = \ell$, with parts $X_i(j_i)$ of degree j_i and color X_i. We shall usually assume that parts of Π are indexed so that

$$X_1(j_1) \preceq X_2(j_2) \preceq \cdots \preceq X_\ell(j_\ell). \tag{15}$$

We associate with a colored partition Π its shape sh Π, the "plain" partition

$$j_1 \leq j_2 \leq \cdots \leq j_\ell.$$

We call the basis element $1 \in \mathcal{P}$ the colored partition of degree 0 and length 0. Note that $\mathcal{P} \subset \mathcal{S}$ is a monoid with the unit element 1, the product of monomials Φ and Ψ is denoted by $\Phi\Psi$. For colored partitions Φ, Ψ, and $\Pi = \Phi\Psi$, we shall write $\Phi = \Pi/\Psi$ and $\Psi \subset \Pi$. We shall say that $\Psi \subset \Pi$ is an embedding (of Ψ in Π), notation suggesting that Π "contains" all the parts of Ψ.

The set of all colored partitions of degree m and length ℓ is denoted as $\mathcal{P}^\ell(m)$. The set of all colored partitions with parts $X_i(j_i)$ of degree $j_i < 0$ is denoted as $\mathcal{P}_{<0}$. We shall fix a monomial basis

$$X(\Pi) = X_1(j_1)X_2(j_2)\dots X_\ell(j_\ell), \quad \Pi \in \mathcal{P},$$

of the enveloping algebra $\mathcal{U}$ such that (15) holds. Then, by Poincaré–Birkhoff–Witt theorem, we have a basis

$$X(\Pi)\,\mathbf{1}, \qquad \Pi \in \mathcal{P}_{<0}, \tag{16}$$

of $N(k\Lambda_0)$, and on the quotient $L(k\Lambda_0)$ a PBW spanning set of the form (16).

Clearly $\bar{B} \subset \mathcal{P}$, viewed as colored partitions of length 1. We assume that on $\mathcal{P}$ we have a linear order $\preceq$ which extends the order $\preceq$ on $\bar{B}$. Moreover, we assume that order $\preceq$ on $\mathcal{P}$ has the following properties:

- $\ell(\Pi) > \ell(\Phi)$ implies $\Pi \prec \Phi$.
- $\ell(\Pi) = \ell(\Phi)$, $|\Pi| < |\Phi|$ implies $\Pi \prec \Phi$.
- Let $\ell(\Pi) = \ell(\Phi)$, $|\Pi| = |\Phi|$. Let Π be a partition $b_1(j_1) \preceq b_2(j_2) \preceq \cdots \preceq b_\ell(j_\ell)$ and Φ a partition $a_1(i_1) \preceq a_2(i_2) \preceq \cdots \preceq a_\ell(i_\ell)$. Then $\Pi \preceq \Phi$ implies $j_\ell \le i_\ell$.
- Let $\ell \ge 0$, $m \in \mathbb{Z}$, and let $S \subset \mathcal{P}$ be a nonempty subset such that all Π in S have length $\ell(\Pi) \le \ell$ and degree $|\Pi| = m$. Then S has a minimal element.
- $\Phi \preceq \Psi$ implies $\Pi\Phi \preceq \Pi\Psi$.
- The relation $\Pi \prec \Phi$ is a well order on $\mathcal{P}_{<0}$.

The first four listed properties guarantee the existence of leading terms defined below, and the last two are needed for inductive arguments in reducing the PBW spanning set by using relations and their leading terms.

Remark 1 An order with these properties is used in [40]; colored partitions are compared first by length and degree, and then by comparing degrees of parts and colors of parts in the reverse lexicographical order. In this paper, we shall use the same order on $\mathcal{P}$ extending a chosen linear order on B.

Remark 2 Note that for elements $X_1(j_1), X_2(j_2), \dots, X_\ell(j_\ell) \in \bar{B}$ and any permutation σ we have

$$X_1(j_1)X_2(j_2)\dots X_\ell(j_\ell) - X_{\sigma(1)}(j_{\sigma(1)})X_{\sigma(2)}(j_{\sigma(2)})\dots X_{\sigma(\ell)}(j_{\sigma(\ell)}) \in \mathcal{U}_{\ell-1}. \tag{17}$$

So if (15) holds and $\Pi = X_1(j_1)\dots X_\ell(j_\ell)$, our first requirement on order $\succeq$ and (17) imply

$$X_{\sigma(1)}(j_{\sigma(1)})\dots X_{\sigma(\ell)}(j_{\sigma(\ell)}) = X(\Pi) + \sum_{\Phi \succ \Pi} c_\Phi X(\Phi) \tag{18}$$

for some coefficients c_Φ for $\Phi \in \mathcal{P}$. Since we are mostly interested in the leading terms of relations, due to (18) we shall often make no distinction between $\Pi \in \mathcal{S}$ and $X(\Pi) \in \mathcal{U}$.

The relation $r_{(k+1)\theta}(m)$, a coefficient of the annihilating field $x_\theta(z)^{k+1}$, is an infinite sum

$$r_{(k+1)\theta}(m) = \sum_{j_1+\cdots+j_{k+1}=m} x_\theta(j_1)\cdots x_\theta(j_{k+1}), \tag{19}$$

and the smallest summand in this sum is proportional to

$$x_\theta(-j-1)^b x_\theta(-j)^a \tag{20}$$

for $a+b=k+1$ and $(-j-1)b+(-j)a=m$. Moreover, the shape of every other term Φ which appears in the sum is greater than the shape $(-j-1)^b(-j)^a$, so we can write

$$r_{(k+1)\theta}(m) = c\, x_\theta(-j-1)^b x_\theta(-j)^a + \sum_{\mathrm{sh}\,\Phi\succ(-j-1)^b(-j)^a} c_\Phi X(\Phi) \tag{21}$$

for some $c\neq 0$ and coefficients c_Φ for $\Phi\in\mathcal{P}^{k+1}(m)$. The adjoint action of $U(\mathfrak{g})$ on $r_{(k+1)\theta}(m)$, $m\in\mathbb{Z}$, gives all other relations in $\bar{R}$. For $u\in U(\mathfrak{g})$, the relation $r(m)=u\cdot r_{(k+1)\theta}(m)$ can be written as

$$r(m) = \sum_{\mathrm{sh}\,\Psi=(-j-1)^b(-j)^a} c_\Psi X(\Psi) + \sum_{\mathrm{sh}\,\Psi\succ(-j-1)^b(-j)^a} c_\Psi X(\Psi) + \sum_{\ell(\Psi)<k+1} c_\Psi X(\Psi). \tag{22}$$

Let c be as in (21). The actions of $u\in U(\mathfrak{g})$ in $\mathfrak{g}$-modules $\mathcal{U}$ and $\mathcal{S}$ are different, but because of (18) we have

$$u\Big(c\, x_\theta(-j-1)^b x_\theta(-j)^a\Big) = \sum_{\mathrm{sh}\,\Psi=(-j-1)^b(-j)^a} c_\Psi \Psi$$

with the same coefficients c_Ψ as in the first summand in (22). Hence $r(m)\neq 0$ if and only if $c_\Psi\neq 0$ for some Ψ. The smallest $\Psi\in\mathcal{P}^{k+1}(m)$ which appears in the first sum in (22) with $c_\Psi\neq 0$ we call *the leading term of relation* $r(m)$ and we denote it as $\ell t\, r(m)$. Hence we can rewrite (22) as

$$r(m) = c_\Phi X(\Phi) + \sum_{\Psi\succ\Phi} c_\Psi X(\Psi), \qquad \Phi = \ell t\, r(m). \tag{23}$$

Set

$$\mathcal{R} = \{\ell t\, r(m) \mid r\in R,\ m\in\mathbb{Z}\}, \quad \mathcal{D} = \mathcal{P}\setminus\{\Psi\Phi \mid \Psi\in\mathcal{P}, \Phi\in\mathcal{R}\},$$

where we think of $\Psi\Phi$ as a product of monomials in $\mathcal{S}$. In other words, $\mathcal{D}$ is the set of all colored partitions which *do not contain* any leading term from $\mathcal{R}$ or we say that colored partitions in $\mathcal{D}$ satisfy difference conditions $\mathcal{R}$. Since $r(m)=0$ on $L(k\Lambda_0)$, by using (23) we can replace monomial $X(\Phi)$ with a combination of monomials $X(\Psi)$, $\Psi\succ\Phi$, and reduce the spanning set (16) to a smaller spanning set parametrized with colored partitions satisfying difference conditions $\mathcal{R}$:

Proposition 1 *The standard module $L(k\Lambda_0)$ is spanned by the set of vectors*

$$X(\Pi)\,\mathbf{1}, \qquad \Pi \in \mathcal{D} \cap \mathcal{D}_{<0}. \tag{24}$$

Remark 3 In spite of the fact that the spanning set (24) is obtained by using all defining relations $\bar{R}$ for level k standard modules, this set need not be a basis of $L(k\Lambda_0)$! By results in [18] and [40] it is a basis for $\widehat{\mathfrak{sl}}_2$, but by [41] for $\widehat{\mathfrak{sl}}_3$ it is not—at least it is not a basis for a chosen ordered basis B of $\mathfrak{sl}_3$. And then again, by [50], it is a basis for the basic modules for affine Lie algebras of types $A_2^{(2)}$ and $B_2^{(1)}$. In [41] it is shown how it can happen that (24) is not a basis, but it is not clear "why" it happens—one way or the other.

5 Simple Lie algebra of type C_n

We fix a simple Lie algebra $\mathfrak{g}$ of type C_n, $n \geq 2$. For a given Cartan subalgebra $\mathfrak{h}$ and the corresponding root system Δ we can write

$$\Delta = \{\pm(\varepsilon_i \pm \varepsilon_j) \mid i, j = 1, ..., n\} \setminus \{0\}\,.$$

We choose simple roots as in [11]

$$\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3,\ \cdots\ \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n,\ \alpha_n = 2\varepsilon_n.$$

Then $\theta = 2\varepsilon_1$. For each $\alpha \in \Delta$ we choose a root vector X_α such that $[X_\alpha, X_{-\alpha}] = \alpha^\vee$. For root vectors X_α, we shall use the following notation:

$$\begin{array}{llll} X_{ij} & \text{or just} & ij & \text{if}\ \ \alpha = \varepsilon_i + \varepsilon_j\ ,\ i \leq j\ , \\ X_{\underline{i}\,\underline{j}} & \text{or just} & \underline{i}\,\underline{j} & \text{if}\ \alpha = -\varepsilon_i - \varepsilon_j\ ,\ i \geq j\ , \\ X_{i\underline{j}} & \text{or just} & i\,\underline{j} & \text{if}\ \ \alpha = \varepsilon_i - \varepsilon_j\ ,\ i \neq j\ . \end{array}$$

With the previous notation $x_\theta = X_{11}$. We also write for $i = 1, \ldots, n$

$$X_{i\underline{i}} = \alpha_i^\vee \text{ or just } i\underline{i}\,.$$

These vectors X_{ab} form a basis B of $\mathfrak{g}$ which we shall write in a triangular scheme. For example, for $n = 3$ the basis B is

$$\begin{array}{llllll} 11 & & & & & \\ 12 & 22 & & & & \\ 13 & 23 & 33 & & & \\ 1\underline{3} & 2\underline{3} & 3\underline{3} & \underline{3}\underline{3} & & \\ 1\underline{2} & 2\underline{2} & 3\underline{2} & \underline{3}\underline{2} & \underline{2}\underline{2} & \\ 1\underline{1} & 2\underline{1} & 3\underline{1} & \underline{3}\underline{1} & \underline{2}\underline{1} & \underline{1}\underline{1}. \end{array}$$

In general, for the set of indices $\{1, 2, \cdots, n, \underline{n}, \cdots, \underline{2}, \underline{1}\}$, we use the order

$$1 \succ 2 \succ \cdots \succ n-1 \succ n \succ \underline{n} \succ \underline{n-1} \succ \cdots \succ \underline{2} \succ \underline{1}$$

and a basis element X_{ab} we write in a^{th} column and b^{th} row,

$$B = \Big\{ X_{ab} \mid b \in \{1, 2, \cdots, n, \underline{n}, \cdots, \underline{2}, \underline{1}\},\ a \in \{1, \cdots, b\} \Big\}. \tag{25}$$

By using (25), we define on the basis B the corresponding reverse lexicographical order, i.e.,

$$X_{ab} \succ X_{a'b'} \textit{ if } b \succ b' \textit{ or } b = b' \textit{ and } a \succ a' \,.$$

In other words, X_{ab} is larger than $X_{a'b'}$ if $X_{a'b'}$ lies in a row b' below the row b, or X_{ab} and $X_{a'b'}$ are in the same row $b = b'$, but $X_{a'b'}$ is to the right of X_{ab}.

For $r \in \{1, \cdots, n, \underline{n}, \cdots, \underline{1}\}$, we introduce the notation

$$\triangle_r \quad \text{and} \quad {}^r\triangle$$

for triangles in the basis B consisting of rows $\{1, \ldots, r\}$ and columns $\{r, \ldots, \underline{1}\}$. These two triangles have vertices 11, $1r$, rr and rr, $r\underline{1}$, $\underline{11}$ with a common vertex rr. Note that $\triangle_r$ is above the vertex rr, and ${}^r\triangle$ is below it. For example, for $n = 3$ and $r = \underline{3}$, we have triangles $\triangle_{\underline{3}}$ and ${}^{\underline{3}}\triangle$

$$\begin{array}{llllll}
11 & & & & & \\
12 & 22 & & & & \\
13 & 23 & 33 & & & \\
1\underline{3} & 2\underline{3} & 3\underline{3} & \underline{33} & & \\
 & & & \underline{32} & \underline{22} & \\
 & & & \underline{31} & \underline{21} & \underline{11}.
\end{array}$$

We say that $[rs] = \operatorname{ad} X_{rs}$ is an arrow on the basis B. Let X_α and X_β be two elements in the basis B and let

$$[\alpha]X_\beta = [X_\alpha, X_\beta] = \sum_{\gamma \in B} c_{\alpha\beta\gamma} X_\gamma. \tag{26}$$

Than we say that the arrow $[\alpha]$ *moves a point* β *to a point* γ if $c_{\alpha\beta\gamma} \neq 0$ and we write

$$\beta \xrightarrow{[\alpha]} \gamma \qquad \text{or simply} \qquad \beta \longrightarrow \gamma.$$

We say that the arrow $[\alpha]$ *does not move a point* β if $[\alpha]X_\beta = 0$.

For example, for $n = 3$ and the arrow $[2\underline{1}]$ we have $11 \longrightarrow 12$ and $1r \longrightarrow 2r$ for $r \neq 1$, and $s\underline{2} \longrightarrow s\underline{1}$ for $s \neq \underline{1}$ and $\underline{21} \longrightarrow \underline{11}$, or we may also write

$$11 \xrightarrow{[2\underline{1}]} 12 \xrightarrow{[2\underline{1}]} 22 \quad \text{and} \quad 13 \xrightarrow{[2\underline{1}]} 23, \quad 1\underline{3} \xrightarrow{[2\underline{1}]} 2\underline{3}, \quad 1\underline{2} \xrightarrow{[2\underline{1}]} 2\underline{2}, \quad 1\underline{1} \xrightarrow{[2\underline{1}]} 2\underline{1},$$
$$1\underline{2} \xrightarrow{[2\underline{1}]} 1\underline{1}, \quad 2\underline{2} \xrightarrow{[2\underline{1}]} 2\underline{1}, \quad 3\underline{2} \xrightarrow{[2\underline{1}]} 3\underline{1}, \quad \underline{32} \xrightarrow{[2\underline{1}]} \underline{31} \quad \text{and} \quad \underline{22} \xrightarrow{[2\underline{1}]} \underline{21} \xrightarrow{[1\underline{1}]} \underline{11}.$$

The arrow $[\alpha] = \operatorname{ad} X_\alpha$ acts as a derivation on the symmetric algebra $S(\mathfrak{g})$, so for monomials with factors $X_\delta \in B$ we have

$$[\alpha] \prod_{\delta \in B} X_\delta^{m_\delta} = \sum_{\beta \in B} m_\beta \prod_{\delta \neq \beta} X_\delta^{m_\delta} X_\beta^{m_\beta - 1} \left([\alpha] X_\beta\right)$$

and, after inserting (26),

$$[\alpha] \prod_{\delta \in B} X_\delta^{m_\delta} = \sum_{\beta, \gamma \in B} m_\beta c_{\alpha\beta\gamma} \prod_{\delta \neq \beta} X_\delta^{m_\delta} X_\beta^{m_\beta - 1} X_\gamma. \tag{27}$$

In our (combinatorial) arguments based on this formula it will be convenient to visualize the monomial

$$\prod_{\delta \in B} X_\delta^{m_\delta}$$

as the set of points at places δ in our triangle B, with multiplicities m_δ, and the resulting monomials in (27)

$$\prod_{\delta \neq \beta} X_\delta^{m_\delta} X_\beta^{m_\beta - 1} X_\gamma$$

for $c_{\alpha\beta\gamma} \neq 0$ we visualize as a set of points in the basis B obtained by moving one point from the place β to the place γ, thus changing the multiplicities $m_\beta \mapsto m_\beta - 1$ and $m_\gamma \mapsto m_\gamma + 1$.

We prove the following lemmas by checking when $\alpha, \beta \in \Delta$ implies $\alpha + \beta \in \Delta$.

Lemma 1 *Let $r \in \{2, \ldots, n\}$. Arrow $[r\underline{1}]$ moves*

1. $1s \longrightarrow sr$ *for* $s = 1, \ldots, r$,
2. *does not move any point in* $\triangle_r \setminus \{11, 12, \ldots, 1r\}$.

Lemma 2 *Let $r \in \{3, \ldots, n\}$. Arrows $[2\underline{1}], [3\underline{1}], \ldots, [r-1, \underline{1}]$*

1. *move* 11 *to points* $12, 13, \ldots, 1(r-1)$,
2. *move points* $12, 13, \ldots, 1(r-1)$ *into* $\triangle_{r-1}$,
3. *move* $1r$ *to* sr *for* $s = 2, \ldots, r-1$,
4. *do not move any point in* $\triangle_r \setminus \{11, 12, \ldots, 1r\}$.

We prove the following lemma by expressing a dual root $\alpha^\vee$ in terms of simple coroots $\alpha_1^\vee, \ldots, \alpha_n^\vee$:

Lemma 3 *Let* $r \in \{2, \dots, n\}$. *Then*

1. $[r\underline{1}]X_{1r} = -X_{1\underline{1}} \cdots - X_{r-1\underline{r-1}} - 2X_{r\underline{r}} \cdots - 2X_{n\underline{n}}$,
2. $[r\underline{r}]X_{rr} = -X_{r\underline{r}} \cdots - X_{n\underline{n}}$.

Lemma 4 *Let* $r \in \{2, \dots, n\}$. *Arrow* $[r\underline{1}]$ *moves*

1. $1s \longrightarrow s\underline{r}$ *for* $s = 1, \dots, \underline{r}$, $s \neq r$,
2. $ir \longrightarrow i\underline{1}$ *for* $i = 2, \dots, r-1, r$,
3. $ri \longrightarrow i\underline{1}$ *for* $i = r, \dots, \underline{r}$,
4. *apart from* $1r$, *arrow* $[r\underline{1}]$ *does not move any other point in* $\triangle_{\underline{r}}$.

For $s \in \{\underline{1}, \dots, \underline{n}\}$ and $t \in \{1, \dots, n\}$ such that $s = \underline{t}$ we write $t = \underline{s}$.

Lemma 5 *Let* $r \in \{1, \dots, n\}$. *Arrow* $[s\underline{1}]$, $s \in \{2, \dots, \underline{r+1}\}$, *moves*

1. $11 \longrightarrow 1s$,
2. $1p \longrightarrow ps$ *for* $p = 2, \dots, s$, $p \neq \underline{s}$,
3. $1p \longrightarrow sp$ *for* $p = s, \dots, \underline{r+1}$, $p \neq \underline{s}$,
4. $i\underline{s} \longrightarrow i\underline{1}$ *for* $i = 2, \dots, \underline{s}$,
5. $\underline{s}i \longrightarrow i\underline{1}$ *for* $i = \underline{s}, \dots, \underline{r+1}$,
6. *apart from* $1\underline{s}$, *arrow* $[s\underline{1}]$ *does not move any other point in* $\triangle_{\underline{r}}$.

Lemma 6 *Let* $r \in \{2, \dots, n\}$. *Arrow* $[r\underline{r}]$ *moves*

1. $ir \longrightarrow i\underline{r}$ *for* $i = 1, \dots, r-1$,
2. $ri \longrightarrow i\underline{r}$ *for* $i = r+1, \dots, \underline{r}$,
3. *apart from* rr, *arrow* $[r\underline{r}]$ *does not move any other point in* $\triangle_{\underline{r}}$.

Note that for simple roots we have arrows

$$[2\underline{1}] = \operatorname{ad} X_{-\alpha_1}, \quad \dots, \quad [n, \underline{n-1}] = \operatorname{ad} X_{-\alpha_{n-1}}, \quad [n\underline{n}] = \operatorname{ad} X_{-\alpha_n} .$$

Lemma 7 *Let* $r \in \{1, \dots, n-1\}$. *Arrow* $[r+1, \underline{r}]$ *moves*

1. $ir \longrightarrow i(r+1)$ *for* $i = 1, \dots, r$,
2. $ri \longrightarrow (r+1)i$ *for* $i = r+1, \dots, \underline{1}$, $i \neq \underline{r+1}$
3. $i\underline{r+1} \longrightarrow i\underline{r}$ *for* $i = 1, \dots, \underline{r+1}$,
4. $\underline{r+1}i \longrightarrow i\underline{r}$ *for* $i = \underline{r}, \dots, \underline{1}$.

Lemma 8 *Arrow* $[\underline{n}, \underline{n}]$ *moves*

1. $in \longrightarrow i\underline{n}$ *for* $i = 1, \dots, n$,
2. $ni \longrightarrow \underline{n}i$ *for* $i = \underline{n}, \dots, \underline{1}$.

6 Leading terms of relations for $C_n^{(1)}$

With the order $\preceq$ on the basis B, we define a linear order on $\bar{B} = \{X(j) \mid X \in B, j \in \mathbb{Z}\}$ by

$$X_\alpha(i) \prec X_\beta(j) \quad \text{if} \quad i < j \quad \text{or} \quad i = j, \ X_\alpha \prec X_\beta .$$

With the order $\preceq$ on $\bar{B}$, we define a linear order on $\mathcal{P}$ by
$\Pi \prec \Phi$
if

- $\ell(\Pi) > \ell(\Phi)$ or
- $\ell(\Pi) = \ell(\Phi)$, $|\Pi| < |\Phi|$ or
- $\ell(\Pi) = \ell(\Phi)$, $|\Pi| = |\Phi|$, $\operatorname{sh}\Pi \prec \operatorname{sh}\Psi$ in the reverse lexicographical order or
- $\ell(\Pi) = \ell(\Phi)$, $|\Pi| = |\Phi|$, $\operatorname{sh}\Pi = \operatorname{sh}\Psi$, and colors of Π are smaller than the colors of Ψ in the reverse lexicographical order.

For example,

$$X_{11}(-3)^2 X_{1\underline{1}}(-2)^2 X_{11}(-2) \prec X_{\underline{1}\underline{1}}(-3) X_{11}(-3) X_{11}(-2)^3$$

because these two colored partitions have the same shape $(-3)^2(-2)^3$ with colors

$$11\ 11;\ 1\underline{1}\ 1\underline{1}\ 11 \quad \text{and} \quad \underline{1}\underline{1}\ 11;\ 11\ 11\ 11$$

and by comparing from the right, we see $11 = 11$, $1\underline{1} \prec 11$. Generally, we may visualize colored partitions $\Pi, \Psi \in \mathcal{P}_{<0}$ by the corresponding Young diagrams. For instance, the above-mentioned colored partitions are presented by

$$\begin{array}{ll} 11 & \square\square\square \\ 11 & \square\square\square \\ 1\underline{1} & \square\square \\ 1\underline{1} & \square\square \\ 11 & \square\square \end{array} \prec \begin{array}{ll} \underline{1}\underline{1} & \square\square\square \\ 11 & \square\square\square \\ 11 & \square\square \\ 11 & \square\square \\ 11 & \square\square \end{array}.$$

A sequence of basis elements $(X_{a_1b_1}, X_{a_2b_2}, \cdots, X_{a_sb_s})$ is a *cascade $\mathcal{C}$ in the basis B* if for each $i \in \{1, 2, \cdots, s-1\}$ we have $b_{i+1} \prec b_i$ and $a_{i+1} = a_i$, or $b_{i+1} = b_i$ and $a_{i+1} \succ a_i$. We can visualize a cascade $\mathcal{C}$ in the basis B as a staircase in the triangle B going downwards from the right to the left, or as a sequence of waterfalls flowing from the right to the left. Sometimes we shall think of a cascade $\mathcal{C}$ as a set of points in the basis B and write $\mathcal{C} \subset B$.

We say that $\mathcal{C}$ is a *cascade with multiplicities* if for each $X_{a_ib_i}$ in $\mathcal{C}$ a multiplicity $m_{a_ib_i} \in \mathbb{Z}_{\geq 0}$ is given. By abuse of language, we shall say that in the cascade $\mathcal{C}$ with multiplicities, $X_{a_ib_i}$ is the *place* $a_ib_i \in \mathcal{C} \subset B$ with $m_{a_ib_i}$ *points*. We shall also write a cascade with multiplicities $\mathcal{C}$ in the basis B as a monomial

$$\prod_{\alpha \in \mathcal{C}} X_\alpha^{m_\alpha}.$$

For example,

$$X_{33}^5 X_{23}^0 X_{2\underline{3}}^3 X_{2\underline{2}}^1 X_{1\underline{2}}^1 \tag{28}$$

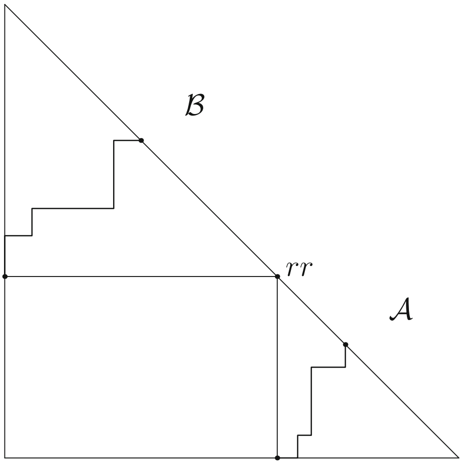

Fig. 1 Triangular scheme of admissible pair of cascades

is a cascade with multiplicities which goes from 33 one step left, then two steps down and than one step left. Along the way we have 5 points at the position 33, no points on 23, 3 points on $2\underline{3}$, and so on:

$$\begin{array}{cccccc}
\cdot & & & & & \\
\cdot & \cdot & & & & \\
\cdot & 0 & 5 & & & \\
\cdot & 3 & \cdot & \cdot & & \\
1 & 1 & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

For $j \in \mathbb{Z}$ and a cascade (with multiplicities) $\mathcal{C}$, we can replace each $X_{a_i b_i}$ in $\mathcal{C}$ with $X_{a_i b_i}(j)$—then we obtain a sequence $(X_{a_1 b_1}(j), X_{a_2 b_2}(j), \cdots, X_{a_s b_s}(j))$ in $\bar{B}$ which we call *a cascade (with multiplicities)* $\mathcal{C}(j)$ *at degree* j. Sometimes we also denote a cascade with multiplicities $\mathcal{C}(j)$ as

$$\prod_{\alpha \in \mathcal{C}} X_\alpha(j)^{m_\alpha}.$$

For example, the cascade with multiplicities $\mathcal{C}$ in (28) above gives us a cascade with multiplicities $\mathcal{C}(j)$ at degree j:

$$X_{33}(j)^5 X_{23}(j)^0 X_{2\underline{3}}(j)^3 X_{2\underline{2}}(j)^1 X_{1\underline{2}}(j)^1. \tag{29}$$

Note that in (29) factors are not written in ascending order as in (15); here we prefer a way of writing appropriate for cascade structure (see Remark 2). We say that two cascades are an admissible pair $(\mathcal{B}, \mathcal{A})$ if

$$\mathcal{B} \subset \triangle_r, \quad \text{and} \quad \mathcal{A} \subset {}^r\triangle$$

for some r. We shall also consider the case when $\mathcal{B}$ is empty and $\mathcal{A} \subset {}^1\triangle (= B)$. For general rank, we may visualize admissible pair of cascades as shown in Fig. 1.

Theorem 2 *Let*

$$(-j-1)^b(-j)^a, \quad j \in \mathbb{Z}, \quad a+b=k+1, \quad b \geq 0, \tag{30}$$

be a fixed shape and let $\mathcal{B}$ and $\mathcal{A}$ be two cascades in degree $-j-1$ and $-j$, with multiplicities $(m_{\beta,j+1}, \beta \in \mathcal{B})$ and $(m_{\alpha,j}, \alpha \in \mathcal{A})$, such that

$$\sum_{\beta\in\mathcal{B}} m_{\beta,j+1} = b, \quad \sum_{\alpha\in\mathcal{A}} m_{\alpha,j} = a. \tag{31}$$

Let $r \in \{1, \cdots, n, \underline{n}, \cdots, \underline{1}\}$. If the points of cascade $\mathcal{B}$ lie in the upper triangle $\triangle_r$ and the points of cascade $\mathcal{A}$ lie in the lower triangle ${}^r\triangle$, then

$$\prod_{\beta\in\mathcal{B}} X_\beta(-j-1)^{m_{\beta,j+1}} \prod_{\alpha\in\mathcal{A}} X_\alpha(-j)^{m_{\beta,j}} \tag{32}$$

is the leading term of a relation for level k standard module for affine Lie algebra of the type $C_n^{(1)}$.

Before we prove the theorem let us make a few remarks.

Remark 4 We believe that all leading terms of level k relations $\bar{R}$ are given by (32). In the case $k=1$ and 2, we can check this by direct calculation. On one side, by using Weyl's character formula for the simple Lie algebra C_n, we have

$$\dim L(2\theta) = \binom{2n+3}{4},$$
$$\dim L(3\theta) = \binom{2n+5}{6}.$$

On the other side, in the case $k=1$ for the shape $(-j)^2$ the number of leading terms (32) is

$$\sum_{i_1=1}^{2n}\sum_{j_1=1}^{i_1}\sum_{i_2=i_1}^{2n}\sum_{j_2=1}^{j_1} 1 = \binom{2n+3}{4},$$

and for the shape $(-j-1)(-j)$

$$\sum_{i_1=1}^{2n}\sum_{j_1=1}^{i_1}\sum_{i_2=i_1}^{2n}\sum_{j_2=i_1}^{i_2} 1 = \binom{2n+3}{4}.$$

In the case $k=2$ and the shape $(-j)^3$ the number of leading terms (32) is

$$\sum_{i_1=1}^{2n}\sum_{j_1=1}^{i_1}\sum_{i_2=i_1}^{2n}\sum_{j_2=1}^{j_1}\sum_{i_3=i_2}^{2n}\sum_{j_3=1}^{j_2} 1 = \binom{2n+5}{6},$$

for the shape $(-j-1)^2(-j)$

$$\sum_{i_1=1}^{2n}\sum_{j_1=1}^{i_1}\sum_{i_2=i_1}^{2n}\sum_{j_2=1}^{j_1}\sum_{i_3=i_2}^{2n}\sum_{j_3=i_2}^{i_3} 1 = \binom{2n+5}{6},$$

and for the shape $(-j-1)(-j)^2$

$$\sum_{i_1=1}^{2n}\sum_{j_1=1}^{i_1}\sum_{i_2=i_1}^{2n}\sum_{j_2=i_1}^{i_2}\sum_{i_3=i_2}^{2n}\sum_{j_3=i_1}^{j_2} 1 = \binom{2n+5}{6}.$$

Remark 5 The Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ may be regarded as of type C_n for $n = 1$, with the standard basis B

$$x = x_{11} \succ h = x_{1\underline{1}} \succ y = x_{\underline{1}\underline{1}}.$$

The standard basis B can be written as the triangle

$$\begin{array}{l} 11 \\ 1\underline{1}\ \underline{11} \end{array}$$

and Theorem 2 applies: for the shape $(-j-1)^b(-j)^a$, $j \in \mathbb{Z}$, $a+b = k+1$, all leading terms of relations for level k standard $\tilde{\mathfrak{g}}$-modules are monomials

$$\begin{aligned} &x(-j-1)^b h(-j)^{a_2} x(-j)^{a_1}, \quad a_1 + a_2 = a, \\ &x(-j-1)^b y(-j)^{a_2} h(-j)^{a_1}, \quad a_1 + a_2 = a, \\ &h(-j-1)^{b_1} x(-j-1)^{b_2} y(-j)^a, \quad b_1 + b_2 = b, \\ &y(-j-1)^{b_1} h(-j-1)^{b_2} y(-j)^a, \quad b_1 + b_2 = b \end{aligned} \tag{33}$$

(see Proposition 6.6.1 in [40]).

Example 1 In the case $\mathfrak{g} = \mathfrak{sl}_2$, we obtain all leading terms (33) as

$$\ell t\Big((\mathrm{ad}\, y)^c (x(-j-1)^b x(-j)^a)\Big)$$

for $c = 0, 1, \cdots, 2a+2b$. For example,

$$\begin{aligned} &\ell t\Big((\mathrm{ad}\, y)^2 (x(-j-1)^2 x(-j)^3)\Big) \\ &= \ell t\Big((\mathrm{ad}\, y)\big(-3x(-j-1)^2 h(-j) x(-j)^2 - 2h(-j-1) x(-j-1) x(-j)^3\big)\Big) \\ &= \ell t\Big(6x(-j-1)^2 h(-j)^2 x(-j) - 6x(-j-1)^2 y(-j) x(-j)^2 \\ &\quad +12h(-j-1) x(-j-1) h(-j) x(-j)^2 \\ &\quad +2h(-j-1)^2 x(-j)^3 - 4y(-j-1) x(-j-1) x(-j)^3\Big) \\ &= x(-j-1)^2 h(-j)^2 x(-j). \end{aligned}$$

We visualize these calculations by representing monomials

$$\Big(y(-j-1)^f h(-j-1)^e x(-j-1)^d\Big)\Big(y(-j)^c h(-j)^b x(-j)^a\Big)$$

as multiple points in two copies of triangle B, i.e.,

$$\begin{pmatrix} d & \\ e & f \end{pmatrix}\begin{pmatrix} a & \\ b & c \end{pmatrix},$$

and by representing the action of ad y with an arrow which moves the points from the place $x = 11$ to the place $h = 1\underline{1}$, and from the place $h = 1\underline{1}$ to the place $y = \underline{11}$. Then we can simplify and visualize previous calculation without explicitly keeping track of coefficients: After applying one arrow on

$$\begin{pmatrix} 2 & \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & \\ 0 & 0 \end{pmatrix}$$

we obtain two terms by acting on two different factors

$$\begin{pmatrix} 2 & \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & \\ 0 & 0 \end{pmatrix},$$

and the first monomial is smaller because in our reverse lexicographical order, we compare parts from the right to the left

$$h(-j) \prec x(-j), \qquad x(-j) = x(-j), \qquad x(-j) = x(-j).$$

Now we act with the second arrow on each of these monomials—from the first monomial we obtain three terms

$$\begin{pmatrix} 2 & \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & \\ 1 & 0 \end{pmatrix},$$

and from the second monomial we obtain another three terms

$$\begin{pmatrix} 1 & \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 3 & \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & \\ 0 & 0 \end{pmatrix}.$$

The smallest term is the very first monomial $x(-j-1)^2 h(-j)^2 x(-j)$ and we see that our guiding "principle" should be: *Act with an arrow on the largest part of the smallest colored partition to obtain the smallest term.*

In general we obtain all leading terms (33) in the following way: First for $c \le a$ we act with c arrows on

$$\begin{pmatrix} b & \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & \\ 0 & 0 \end{pmatrix},$$

and acting always on the largest possible part $x(-j)$, we get

$$\begin{pmatrix} b & \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a-c & \\ c & 0 \end{pmatrix}.$$

With $c = a$ the smallest possible part to act on is $h(-j)$ and with additional d arrows, $d \leq a$, we get

$$\begin{pmatrix} b & \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \\ a-d & d \end{pmatrix}.$$

Once we have obtained a factor $y(-j)^a$, the smallest part the arrow can act on non-trivially is $x(-j-1)$, and we proceed with changing the left triangle.

Example 2 Essentially the same procedure used for $\mathfrak{g} = \mathfrak{s}l_2$ can be used for $\mathfrak{g}$ of type C_n, $n \geq 2$, a difference being in a use of different "kinds" of arrows acting on

$$X_{11}(-j-1)^b X_{11}(-j)^a \,.$$

For example, by applying $2a$ arrows $[2\underline{1}]$ to the factor $X_{11}(-j)^a$ we get $X_{22}(-j)^a$. We visualize this step as moving a points from the place 11 to the place 12 by using a arrows $[2\underline{1}]$, and after that by using another a arrows $[2\underline{1}]$ we move a points from the place 12 to the place 22 in the second row (see Lemma 6). Any additional $[2\underline{1}]$ arrow will act trivially on monomial $X_{22}(-j)^a$ and will move points corresponding to the factor $X_{11}(-j-1)^b$. So with $2a + b + b' = 2a + (b - b') + 2b'$ arrows we get a relation with the leading term (with a premise that we can show it is really the leading term)

$$\begin{pmatrix} 0 & \\ b-b' & b' \\ \cdots & \end{pmatrix} \begin{pmatrix} 0 & \\ 0 & a \\ \cdots & \end{pmatrix}.$$

By following the above idea and notation, after acting of $2a + b_1 + b_2 + 2b_3$ times with arrows $[3\underline{2}]$, we get a relation with the leading term

$$\begin{pmatrix} 0 & & \\ 0 & 0 & \\ b_1 & b_2 & b_3 \\ \cdots & & \end{pmatrix} \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 0 & 0 & a \\ \cdots & & \end{pmatrix}.$$

At certain point we could start constructing a cascade on $(-j-1)^b$ part, by using, for example, only $2a + b_1$ arrows $[4\underline{3}]$ to obtain a relation with the leading term

$$\begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & b_2 & b_3 & \\ b_1 & 0 & 0 & 0 \\ \cdots & & & \end{pmatrix}\begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & a \\ \cdots & & & \end{pmatrix}.$$

After that we could use $a+a'$, $(a'<a)$, arrows $[5\underline{4}]$ to start constructing a cascade on $(-j)^a$ part and obtain a relation with the leading term

$$\begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ 0 & b_2 & b_3 & & \\ b_1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \\ \cdots & & & & \end{pmatrix}\begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & a-a' & a' \\ \cdots & & & & \end{pmatrix},$$

that is, a relation with the leading term

$$X_{14}(-j-1)^{b_1}X_{33}(-j-1)^{b_3}X_{23}(-j-1)^{b_2}X_{55}(-j)^{a'}X_{45}(-j)^{a-a'}.$$

This is a basic idea how relations with the leading terms (32) in Theorem 2 can be constructed. However, there is a slight difficulty we should mention before writing a formal proof.

Take, for example, $n=3$ and the monomial of the form (32)

$$\Pi = X_{1\underline{1}}(-j)^{n_6}X_{1\underline{2}}(-j)^{n_5}X_{1\underline{3}}(-j)^{n_4}X_{13}(-j)^{n_3}X_{12}(-j)^{n_2}X_{11}(-j)^{n_1}$$

with $n_2 \neq 0$ and $n_5 \neq 0$. Observe that we have three kinds of arrows corresponding to simple roots, moving one row to the next one:

$$11 \xrightarrow{[2\underline{1}]} 12 \xrightarrow{[3\underline{2}]} 13 \xrightarrow{[3\underline{3}]} 1\underline{3} \xrightarrow{[3\underline{2}]} 1\underline{2} \xrightarrow{[2\underline{1}]} 1\underline{1}\ .$$

So for $a = n_1+\cdots+n_6$ we start with $X_{11}(-j)^a$ and apply correct number of arrows on largest possible parts and get

$$\begin{aligned} X_{11}(-j)^a \cdots &\xrightarrow{[2\underline{1}]} X_{12}(-j)^{a-n_1}X_{11}(-j)^{n_1}\cdots \\ &\xrightarrow{[3\underline{2}]} X_{13}(-j)^{a-n_1-n_2}X_{12}(-j)^{n_2}X_{11}(-j)^{n_1}\cdots \\ &\xrightarrow{[3\underline{3}]} X_{1\underline{3}}(-j)^{a-n_1-n_2-n_3}X_{13}(-j)^{n_3}X_{12}(-j)^{n_2}X_{11}(-j)^{n_1}\cdots. \end{aligned}$$

But now we are stuck if we are to use only arrows $[2\underline{1}]$, $[3\underline{2}]$, and $[3\underline{3}]$: on one side, for moving points from $1\underline{3}$ to $1\underline{2}$ we should use arrow $[3\underline{2}]$, but in this way we will not get the smallest term unless $n_2=0$. (That is, for $n_2 \neq 0$ we will get a smaller term if we apply arrow $[3\underline{2}]$ to a larger factor $X_{12}(-j)^{n_2}$.) The way around is to use additional kind of arrows

$$11 \xrightarrow{[11]} 1\underline{1} \qquad \text{and} \qquad 12 \xrightarrow{[22]} 1\underline{2}\,.$$

With the use of these additional arrows, we can construct a relation with the leading term Π:

$$\begin{aligned}
X_{11}(-j)^a &\xrightarrow{[2\underline{1}]}\xrightarrow{[11]} \cdots X_{1\underline{1}}(-j)^{n_6} X_{12}(-j)^{a-n_1-n_6} X_{11}(-j)^{n_1} \\
\cdots &\xrightarrow{[3\underline{2}]}\xrightarrow{[22]} \quad X_{1\underline{1}}(-j)^{n_6} X_{1\underline{2}}(-j)^{n_5} X_{13}(-j)^{n_3+n_4} X_{12}(-j)^{n_2} X_{11}(-j)^{n_1} \\
\cdots &\xrightarrow{[33]} X_{1\underline{1}}(-j)^{n_6} X_{1\underline{2}}(-j)^{n_5} X_{1\underline{3}}(-j)^{n_4} X_{13}(-j)^{n_3} X_{12}(-j)^{n_2} X_{11}(-j)^{n_1}.
\end{aligned}$$

We encounter a similar problem if we want to construct a relation with the leading term

$$X_{1\underline{1}}(-j)^{b_1} X_{2\underline{1}}(-j)^{b_2} X_{3\underline{1}}(-j)^{b_3} X_{\underline{31}}(-j)^{b_4} X_{\underline{21}}(-j)^{b_5} X_{\underline{11}}(-j)^{b_6}$$

"from the previous row" by using arrow $[2\underline{1}]$,

$$\begin{array}{ccccccc}
1\underline{2} & & 2\underline{2} & 3\underline{2} & \underline{32} & \underline{22} & \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\
1\underline{1} & \rightarrow & 2\underline{1} & 3\underline{1} & \underline{31} & \underline{21} & \rightarrow \ \underline{11}\,,
\end{array}$$

because we cannot construct the factor $X_{\underline{11}}^{b_6}$ in a leading term by applying arrow on $X_{\underline{21}}^{b_5+b_6}$ unless $b_1 = 0$.

Proof (Theorem 2) We construct a relation with the leading term (32) by precisely defined application of arrows $[rs] = \operatorname{ad} X_{rs}$ on a coefficient of the relation

$$\operatorname{coeff}_{z^m} X_{11}(z)^{k+1}, \quad m = b(j+1) + aj - k - 1.$$

By previous remarks, the leading term of such relation has the shape (30) and it is enough to analyze the action of arrows $[rs]$ on the colored partition

$$Z_0 = X_{11}(-j-1)^b X_{11}(-j)^a.$$

We prove the theorem in four steps. Since the leading term (32) reminds us of waterfalls on Plitvice lakes, we title these steps as: preparation of upper barrier, construction of upper cascades, preparation of lower barrier, and construction of lower cascade.

Preparation of upper barrier. Let $t \in \{1, \cdots, n, \underline{n}, \cdots, \underline{1}\}$ be the index of uppermost row in which the upper cascade $\mathcal{B}$ in (32) has a point. For $i \in \{1, \cdots, t\}$ denote by m_i the sum of all multiplicities of $\mathcal{B}$ in the i-th column, i.e.,

$$m_i = \sum_{is \in \mathcal{B}} m_{is,j+1}, \quad m_1 + \cdots + m_t = b.$$

In this step, we construct a relation with the leading term

$$X_{1t}(-j-1)^{m_1}X_{2t}(-j-1)^{m_2}\dots X_{tt}(-j-1)^{m_t}X_{tt}(-j)^a.$$

We consider two cases: $t\in\{1,\cdots,n\}$ and $t\in\{\underline{n},\cdots,\underline{1}\}$
Case 1. Let $t=r$, $r\in\{2,\cdots,n\}$. Set

$$Z_1=[r\underline{1}]^{2a+m_1+m_r}[r-1,\underline{1}]^{m_{r-1}}\dots[3\underline{1}]^{m_3}[2\underline{1}]^{m_2}\,Z_0. \tag{34}$$

(Note that all listed arrows mutually commute.) Then for some $c\neq 0$

$$\begin{aligned}Z_1 &= c\,X_{1r}(-j-1)^{m_1+m_r}X_{1,r-1}(-j-1)^{m_{r-1}}\dots X_{12}(-j-1)^{m_2}X_{rr}(-j)^a\\ &\quad+\sum_{M\in\mathcal{P}}c_M M.\end{aligned} \tag{35}$$

We obtain the first term by applying all arrows listed in (34) on factors $X_{11}(-j)$, $X_{1r}(-j)$ and $X_{11}(-j-1)$ in the following way (see Lemma 1): $2a$ arrows $[r\underline{1}]$ move $X_{11}(-j)^a$ to $X_{rr}(-j)^a$, i.e., for some $c'\neq 0$

$$[r\underline{1}]^{2a}X_{11}(-j)^a=c'\,X_{rr}(-j)^a, \tag{36}$$

then m_2 arrows $[2\underline{1}]$ move m_2 factors $X_{11}(-j-1)$ to $X_{12}(-j-1)^{m_2}$, and so on until m_1+m_r arrows $[r\underline{1}]$ move m_1+m_r factors $X_{11}(-j-1)$ to $X_{1r}(-j-1)^{m_1+m_r}$. Multiple choice of factors on which any of the arrows may act gives a proportionality constant $c\neq 0$, and any other way that arrows act gives in (35) a combination of other monomials M.

It will be convenient to say that an arrow $[\beta]=\operatorname{ad}X_\beta$ is *misused* if it is not used to produce the first term in (35) in the way described above.

If we act with less than $2a$ arrows $[r\underline{1}]$ on $X_{11}(-j)^a$, we get (cf. Lemmas 1 and 2) a monomial M' with a factor $X_\beta(-j)$, $\beta\in\triangle_r\setminus\{rr\}$, and hence $M'\succ X_{rr}(-j)^a$. So assume that we have used $2a$ arrows $[r\underline{1}]$ on $X_{11}(-j)^a$, as in (36), but that we misused the rest of arrows by acting on some other factor different from $X_{11}(-j-1)$. Then we get a monomial M with a factor $X_{11}(-j-1)$, and this implies that M is greater then the first term in (35), that is

$$\ell t\,Z_1=X_{1r}(-j-1)^{m_1+m_r}X_{1,r-1}(-j-1)^{m_{r-1}}\dots X_{12}(-j-1)^{m_2}X_{rr}(-j)^a. \tag{37}$$

Also note (cf. Lemmas 1 and 2) that all such monomials M have all other factors $X_\beta(-j-1)$ with β in the triangle $\triangle_r$, and at least one factor $X_{lm}(-j-1)$ with $l,m\in\{2,\dots,r\}$.

Now set

$$Z_2=[r\underline{1}]^{m_2+m_3+\cdots+m_r}\,Z_1. \tag{38}$$

Then for some $c \neq 0$

$$Z_2 = c\, X_{1r}(-j-1)^{m_1} X_{2r}(-j-1)^{m_2} \dots X_{rr}(-j-1)^{m_r} X_{rr}(-j)^a + \sum_{M \in \mathcal{P}} c_M M. \tag{39}$$

The first term we obtain by applying all arrows listed in (38) on the first term in (35) in the following way: first m_2 arrows $[r\underline{1}]$ move $X_{12}(-j-1)^{m_2}$ to $X_{2r}(-j-1)^{m_2}$, and so on until m_r arrows $[r\underline{1}]$ move $X_{1r}(-j-1)^{m_r}$ to $X_{rr}(-j-1)^{m_r}$.

Note that arrows $[r\underline{1}]$ cannot move $X_{rr}(-j)^a$ and that any arrow spent on $X_{1s}(-j)$, $s \in \{2, \dots, r-1\}$, will produce a monomial M' greater than $X_{rr}(-j)^a$.

What is left to consider is the case when $2a$ arrows $[r\underline{1}]$ are used as in (36), and the rest of $m_2 + \cdots + m_r$ arrows are not used as above. Then the action of $m_2 + \cdots + m_r$ arrows $[r\underline{1}]$ on the first term in (35) will produce a monomial M with a factor $X_{1s}(-j-1)$, $s \in \{2, \dots, r\}$, or a factor $X_{1r}(-j-1)^p$, $p > m_1$, in either case a monomial greater than the first term in (39).

On the other hand, we consider three cases of the action of $m_2 + \cdots + m_r$ arrows $[r\underline{1}]$ on some M in the second summand (35):

(1) arrows will not move a factor $X_{ps}(-j-1)$ for some $p, s \in \{2, \dots, r-1\}$ in the case when M in (35) is obtained by misusing any arrow of the form $[s\underline{1}]$, $s \in \{2, \dots, r-1\}$,
(2) if some of the arrows $[r\underline{1}]$ is misused, M will have a factor $X_{1s}(-j-1)$ for some $s \in \{2, \dots, r-1\}$, or
(3) M will have a factor $X_{1r}(-j-1)^{m_1+p}$ with $p \geq 1$, we may briefly say that M has *an extra* factor $X_{1r}(-j-1)$. In either of these cases M is greater than the first term (39), i.e.,

$$\ell t\, Z_2 = X_{1r}(-j-1)^{m_1} X_{2r}(-j-1)^{m_2} \dots X_{rr}(-j-1)^{m_r} X_{rr}(-j)^a.$$

Case 2. Let $t = \underline{r}$, $r \in \{1, \cdots, n\}$. As in (34) we set

$$Z_1 = [\underline{r}\,\underline{1}]^{2a+m_1+m_{\underline{r}}} \prod_{s=2}^{\overline{r+1}} [s\underline{1}]^{m_s}\, Z_0\,; \quad r \in \{1, \dots, n-1\}$$

and for $r = n$

$$Z_1 = [\underline{n}\,\underline{1}]^{2a+m_1+m_{\underline{n}}} \prod_{s=2}^{n} [s\underline{1}]^{m_s}\, Z_0\,.$$

(Note that some listed arrows do not commute, for example $[s\underline{1}]$ and $[\underline{s}\,\underline{1}]$.) Then for some $c \neq 0$

$$Z_1 = c\, X_{1\underline{r}}(-j-1)^{m_1+m_{\underline{r}}} \prod_{s=2}^{\overline{r+1}} X_{1s}(-j-1)^{m_s} X_{r\underline{r}}(-j)^a + \sum_{M \in \mathcal{P}} c_M M \tag{40}$$

where for $r = n$ we use notation $\overline{r+1} = n$. By arguing in the same way as in Case 1, we see that the first term in (40) is the leading term $\ell t\, Z_1$ of Z_1.

From Z_1 we want to construct a relation with the leading term

$$\Pi = \prod_{s=1}^{\underline{r}} X_{s\underline{r}}(-j-1)^{m_s} X_{r\underline{r}}(-j)^a. \tag{41}$$

We will proceed as in the Case 1, by using arrows $[\underline{r}\,\underline{1}]$, except for the factor $X_{1r}(-j-1)^{m_r}$ where we encounter the following difficulty: the action

$$[\underline{r}\,\underline{1}]\, X_{1r} = [X_{-\varepsilon_1-\varepsilon_r}, X_{\varepsilon_1+\varepsilon_r}] = -(\varepsilon_1+\varepsilon_r)^\vee = -\sum_{1\le s<r} \alpha_s^\vee - 2\sum_{r\le s\le n} \alpha_s^\vee,$$

or written in our notation,

$$[\underline{r}\,\underline{1}]\, X_{1r} = -\sum_{1\le s<r} X_{s\underline{s}} - 2\sum_{r\le s\le n} X_{s\underline{s}}$$

(see Lemma 3), produce terms $X_{1\underline{1}} \prec \cdots \prec X_{r-1\,\underline{r-1}} \prec X_{r\underline{r}}$. So the action of arrow $[\underline{r}\,\underline{1}]$ on $X_{1r}(-j-1)$ produces factors $X_{s\underline{s}}(-j-1)$, $s \in \{1,\ldots,r-1\}$, which in turn may produce monomials M smaller than Π. In order to avoid this difficulty, we use the actions

$$[r\underline{1}]\, X_{1r} = c\, X_{rr}, \qquad [\underline{r}\,\underline{r}]\, X_{rr} = -(2\varepsilon_r)^\vee = -X_{r\underline{r}} - \cdots - X_{n\underline{n}},$$

(for some $c \neq 0$; see again Lemma 3). So the action of arrow $[\underline{r}\,\underline{r}]$ on $X_{rr}(-j-1)$ produces factors $X_{s\underline{s}}(-j-1)$, $s \in \{r,\ldots,n\}$, which are in the $\underline{r}$-th row in the basis B or above it. For this reason we "first act" on a factor $X_{1r}(-j-1)^{m_r}$ to "empty the place" $1r$ in the first column of the basis B:

$$[r\underline{1}]^{m_r}\, X_{1r}(-j-1)^{m_r} = c\, X_{rr}(-j-1)^{m_r}$$

(for some $c \neq 0$), and then move further with arrows

$$[\underline{r}\,\underline{r}]^{m_r}\, X_{rr}(-j-1)^{m_r} = c'\, X_{r\underline{r}}(-j-1)^{m_r} + \sum_{M\in\mathcal{P}} c_M M,$$

(for some $c' \neq 0$). After that, we shall move factors $X_{1s}(-j-1)^{m_s}$, $s \neq r$, from the first column to factors $X_{s\underline{r}}(-j-1)^{m_s}$ in the $\underline{r}$-th row by using arrows $[\underline{r}\,\underline{1}]$ (see Lemma 4). This reasoning motivates us to set

$$Z_2 = [\underline{r}\,\underline{r}]^{m_r} [r\underline{1}]^{m_r} \prod_{s=2, s\neq r}^{\underline{r}} [\underline{r}\,\underline{1}]^{m_s}\, Z_1.$$

Then

$$Z_2 = c\,\Pi + \sum_{M\in\mathcal{P}} c_M M$$

for some $c \neq 0$ and $\Pi = \ell t\, Z_2$, where Π is given by (41). By arguing as in the Case 1, we obtain Π by acting with arrows on factors in the following way: for $s \neq r$

$$[\underline{r1}]^{m_s}\, X_{1s}(-j-1)^{m_s} = c''_s\, X_{s\underline{r}}(-j-1)^{m_s} + \dots$$

for some $c''_s \neq 0$, and

$$[\underline{rr}]^{m_r}[r\underline{1}]^{m_r}\, X_{1r}(-j-1)^{m_r} = c'c\, X_{r\underline{r}}(-j-1)^{m_r} + \dots.$$

We will say that any other way of using arrows on $\ell t\, Z_1$ *is misusing arrows*.

If arrows $[r\underline{1}]$ are misused, we produce a factor $X_{sr}(-j-1)$ for $s \in \{2, \dots, r-1\}$, or $X_{rs}(-j-1)$ for $s \in \{r+1, \dots, \underline{r+1}\}$.

If arrows $[\underline{rr}]$ are misused, we produce a factor $X_{rr}(-j-1)$ or an extra factor $X_{r\underline{r}}(-j-1)$

If arrows $[\underline{r1}]$ are misused, we produce a factor $X_{rs}(-j-1)$ for some $s \neq 1, r, \underline{r}$ or an extra factor $X_{1\underline{r}}(-j-1)$. In either of these cases we obtain a monomial M greater than Π.

The analysis of how the arrows $[\underline{r1}]$, $[\underline{rr}]$, or $[r\underline{1}]$ act on monomials M in the second summand in (40) is analogous to the Case 1.

Construction of upper cascade. From the upper barrier

$$Z_2 = Z_{2;t} = \prod_{s=1}^{t} X_{st}(-j-1)^{m_s} X_{tt}(-j)^a + \sum_{M \in \mathcal{P}} c_M M \tag{42}$$

in a sequence of steps we construct a relation $Z_{2;q}$ with the leading term

$$\prod_{\beta \in \mathcal{B}} X_\beta(-j-1)^{m_{\beta,j+1}}\, X_{qq}(-j)^a, \tag{43}$$

where q is the index of the lowest row in which the upper cascade $\mathcal{B}$ has a point, and t is the index of the highest row in which $\mathcal{B}$ has a point. As above, for $i \in \{1, \cdots, t\}$ we denote by m_i the sum of all multiplicities of $\mathcal{B}$ in the i-th column, i.e.,

$$m_i = \sum_{is \in \mathcal{B}} m_{is,j+1}, \quad m_1 + \cdots + m_t = b.$$

Let $p \in \{1, \cdots, t\}$ be the index of the first column from the left for which the multiplicity in $\mathcal{B}$ is not zero, in row t

$$m_{st;j+1} = 0 \quad \text{for} \quad s \succ p, \qquad 0 < m_{pt;j+1} \leq m_p, \qquad m_{st;j+1} = m_s \quad \text{for} \quad p \succ s.$$

Let t' be the index of the first row in the basis B below t-th row. Since we can write $\underline{t} = \underline{\underline{r}} = r$ the arrow $[t'\underline{t}]$ is well defined. Then the action with arrows $[t'\underline{t}]$

corresponding to a simple root (cf. Lemmas 7 and 8), moving factors from t-th row to the next row below "like in waterfalls"

$$\begin{aligned} Z_{2;t'} &= [t'\underline{t}]^{2a+m_1+\cdots+m_p-m_{pt;j+1}} Z_{2;t} \\ &= c \prod_{s\succ p} X_{st'}(-j-1)^{m_s} X_{pt'}(-j-1)^{m_p-m_{pt;j+1}} \prod_{p\succeq s} X_{st}(-j-1)^{m_{st;j+1}} X_{t't'}(-j)^a \\ &\quad + \sum_{M\in\mathcal{P}} c_M M. \end{aligned} \tag{44}$$

It is easy to see that the first summand in (44) is obtained by using arrows $[t'\underline{t}]$ on the first summand in (42), moving first $X_{tt}(-j)^a$ to $X_{t't'}(-j)^a$, then "one-by-one" factors $X_{st}(-j-1)^{m_s}$ to $X_{st'}(-j-1)^{m_s}$, and finally, $m_p - m_{pt;j+1}$ factors $X_{pt}(-j-1)$ to $X_{pt'}(-j-1)$. It is also easy to see that (apart from the constant $c \neq 0$) the first summand in (44) is the leading term of $Z_{2;t'}$. We say that we constructed *a waterfall at p-th column.*

Note that the arrows we used correspond to the simple roots, i.e., $[t'\underline{t}] = \operatorname{ad} X_{-\alpha_r}$ if $t = r$ and $[t'\underline{t}] = \operatorname{ad} X_{-\alpha_{r-1}}$ if $t = \underline{r}$ for $r \in \{1,\ldots,n\}$. If

$$m_s = 0 \quad \text{for} \quad s \succ p,$$

then we move factors row by row by using the arrows corresponding to the simple roots, producing a relation $Z_{2;q}$ with the leading term (43):

$$\prod_{r=t}^{q} X_{pr}(-j-1)^{m_{pr;j+1}} \prod_{p\succ s} X_{st}(-j-1)^{m_{st;j+1}} X_{qq}(-j)^a.$$

If

$$m_s \neq 0 \quad \text{for some} \quad s \succ p,$$

We take the smallest such $p'' \succ p$ and we find the first row below t-th row, say t''-th row, such that

$$m_{p''t'';j+1} \neq 0.$$

Then we move, step by step, all the factors in i-th column, $i \prec p$, from t'-th row to t''-th row, and create a waterfall at p-th column:

$$\prod_{s\succ p} X_{st''}(-j-1)^{m_s} \prod_{r=t}^{t''} X_{pr}(-j-1)^{m_{pr;j+1}} \prod_{p\succ s} X_{st}(-j-1)^{m_{st;j+1}} X_{qq}(-j)^a.$$

In such a way, we proceed and construct a relation with the leading term (43).

Preparation of lower barrier. Let $t \in \{q \dots, \underline{1}\}$ be the index of the uppermost row in which the lower cascade $\mathcal{A}$ in (32) has a point. For $i \in \{q, \cdots, t\}$ denote by m_i the sum of all multiplicities of $\mathcal{A}$ in the i-th column, i.e.,

$$m_i = \sum_{is \in \mathcal{A}} m_{is,j}, \quad m_q + \cdots + m_t = a.$$

In this step, we construct a lower barrier, i.e., a relation with the leading term

$$\prod_{\beta \in \mathcal{B}} X_\beta(-j-1)^{m_{\beta,j+1}}\, X_{\mathrm{qt}}(-j)^{m_q} \dots X_{tt}(-j)^{m_t}. \tag{45}$$

In the previous steps, we have acted with arrows on leading terms of relations in such a way that "the first" $2a$ arrows were "spent" on monomials in degree $-j$, i.e.,

$$[s\underline{r}\,]^{2a} X_{rr}(-j)^a = c' X_{ss}(-j)^a,$$

and any further action of arrows did not move monomial $X_{ss}(-j)^a$. Hence with the rest of arrows we could construct the upper cascades (43).

In this step, we will obtain the leading term (45) by acting with arrows only on factors $X_\gamma(-j)$ with γ in a lower triangle ${}^q\triangle$, while the action on factors $X_\beta(-j-1)$ will produce greater terms. For this reason, here we will omit writing factors in degree $-j-1$.

Let $q' \prec q$ be the index of the first column (row) next to the q-th column (row). As before, we see that

$$X_{qt}(-j)^{m_q+m_t} \dots X_{qq'}(-j)^{m_{q'}} = \ell t\Big([t\underline{q}]^{m_q+m_t} \dots [q'\underline{q}]^{m_{q'}} X_{qq}(-j)^a\Big).$$

If the set of points $\{qt, q't, \dots, tt\}$ in the lower triangle ${}^q\triangle$ does not contain the point $\underline{t}t$, then we apply

$$[t\underline{q}]^{m_{q'}+\cdots+m_t}$$

to produce a relation with the leading term (45). On the other hand, if the set of points $\{qt, q't, \dots, tt\}$ contains the point $\underline{t}t$, then we should modify our construction (as in the Case 2 above) and, for moving the factor $X_{\underline{t}q}(-j)^{m_{\underline{t}}}$ to $X_{\underline{t}t}(-j)^{m_{\underline{t}}}$ via $X_{\underline{tt}}(-j)^{m_{\underline{t}}}$, we act with arrows

$$[tt]^{m_{\underline{t}}}[\underline{t}\underline{q}]^{m_{\underline{t}}}$$

instead of $[t\underline{q}]^{m_{\underline{t}}}$.

Construction of lower cascade. From a relation with the leading term (45), we construct a relation with the leading term

$$\prod_{\beta \in \mathcal{B}} X_\beta(-j-1)^{m_{\beta,j+1}} \prod_{\alpha \in \mathcal{A}} X_\alpha(-j)^{m_{\alpha,j}}$$

in a way described in the construction of upper cascade, except that here we act with arrows only on factors $X_\gamma(-j)$ with γ in a lower triangle ${}^q\triangle$, while the action on factors $X_\beta(-j-1)$ will produce greater terms. $\square$

7 Conjectured colored Rogers–Ramanujan type identities

By using Theorem 2, we can explicitly describe the reduced spanning set in Proposition 1. We conjecture that this spanning set is a basis:

Conjecture 1 *Let $n \geq 2$ and $k \geq 2$. We consider the standard module $L(k\Lambda_0)$ for the affine Lie algebra of type $C_n^{(1)}$ with the basis*

$$\{X_{ab}(j) \mid ab \in B,\ j \in \mathbb{Z}\} \cup \{c, d\},$$

where $B = \{ab \mid b \in \{1, 2, \cdots, n, \underline{n}, \cdots, \underline{2}, \underline{1}\},\ a \in \{1, \cdots, b\}\}$.

We conjecture that the set of monomial vectors

$$\prod_{ab\in B,\ j>0} X_{ab}(-j)^{m_{ab;j}}\, v_0, \tag{46}$$

satisfying difference conditions

$$\sum_{ab\in\mathcal{B}} m_{ab;\,j+1} + \sum_{ab\in\mathcal{A}} m_{ab;\,j} \leq k$$

for any admissible pair of cascades $(\mathcal{B}, \mathcal{A})$, is a basis of $L(k\Lambda_0)$.

As already mentioned, the conjecture is true for $n = 1$ and all $k \geq 1$ [40] and for $k = 1$ for all $n \geq 2$ [49]. We have also checked by hand the corresponding combinatorial identity below for partitions of $m = 1, \ldots, 8$ in the case $n = k = 2$.

If our conjecture is true, then we have a combinatorial Rogers–Ramanujan type identities by using Lepowsky's product formula for principally specialized characters of standard modules (see [31], cf. [40,43]). In the case of $n = 2$ and $k \geq 1$, we have product formulas for principally specialized characters of standard $C_2^{(1)}$-modules $L(k\Lambda_0)$

$$\prod_{\substack{j\geq 1\\ j\not\equiv 0 \bmod 2}} \frac{1}{1-q^j} \prod_{\substack{j\geq 1\\ j\not\equiv 0,\pm 1,\pm 2,\pm 3 \bmod 2k+6}} \frac{1}{1-q^j} \prod_{\substack{j\geq 1\\ j\not\equiv 0,\pm 1,\pm(k+1),\pm(k+2),k+3 \bmod 2k+6}} \frac{1}{1-q^j}. \tag{47}$$

This product can be interpreted combinatorially in the following way: For fixed k let $\mathcal{C}_k$ be a disjoint union of integers in three colors, say j_1, j_2, j_3 is the integer j in colors

1, 2, 3, satisfying the following congruence conditions

$$\begin{aligned}
&\{j_1 \mid j \geq 1,\, j \not\equiv 0 \bmod 2\},\\
&\{j_2 \mid j \geq 1,\, j \not\equiv 0, \pm 1, \pm 2, \pm 3 \bmod 2k+6\},\\
&\{j_3 \mid j \geq 1,\, j \not\equiv 0, \pm 1, \pm(k+1), \pm(k+2), k+3 \bmod 2k+6\}.
\end{aligned} \tag{48}$$

Set $|j_a| = j$. If we expand the product (47) in Taylor series, then the coefficient of q^m can be interpreted as a number of colored partitions of m

$$m = \sum_{j_a \in \mathcal{C}_k} j_a f_{j_a}. \tag{49}$$

(To be correct, in (49) we should write $m = \sum |j_a| f_{j_a}$.) For example, for $k = 2$ we have

$$\mathcal{C}_2 = \{1_1, 3_1, 5_1, 7_1, \ldots\} \sqcup \{4_2, 5_2, 6_2, 14_2, \ldots\} \sqcup \{2_3, 8_3, 12_3, 18_3 \ldots\};$$

all ordinary partitions of $m = 5$ are

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1,$$

and all colored partitions of 5 with colored parts in $\mathcal{C}_2$ are

$$\begin{gathered}
5_1,\ 5_2,\ \ 4_2+1_1,\ 3_1+2_3,\ 3_1+1_1+1_1,\ 2_3+2_3+1_1,\\
2_3+1_1+1_1+1_1,\ 1_1+1_1+1_1+1_1+1_1.
\end{gathered}$$

On the other hand, in the principal specialization $e^{-\alpha_i} \mapsto q^1$, $i = 0, 1, 2$, the sequence of root subspaces in $C_2^{(1)}$

$$X_{ab}(-1), ab \in B, \quad X_{ab}(-2), ab \in B, \quad X_{ab}(-3), ab \in B, \quad \ldots \tag{50}$$

obtains degrees

$$\begin{matrix}
1 & & & & 5 & & & & 9 & & & & \\
2 & 3 & & & 6 & 7 & & & 10 & 11 & & & \\
3 & 4 & 5 & & 7 & 8 & 9 & & 11 & 12 & 13 & & \cdots\\
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 &
\end{matrix} \tag{51}$$

One way or the other, in (51) we see almost two sequences of natural numbers and almost one sequence of odd numbers. In order to make numbers distinct, we consider four colors 1, 2, 3, 4, say

$$\begin{matrix}
1_1 & & & & 5_1 & & & & 9_1 & & & & \\
2_2 & 3_2 & & & 6_2 & 7_2 & & & 10_2 & 11_2 & & & \\
3_3 & 4_3 & 5_3 & & 7_3 & 8_3 & 9_3 & & 11_3 & 12_3 & 13_3 & & \ldots,\\
4_4 & 5_4 & 6_4 & 7_4 & 8_4 & 9_4 & 10_4 & 11_4 & 12_4 & 13_4 & 14_4 & 15_4 &
\end{matrix} \tag{52}$$

so that numbers in the first row have color 1, numbers in the second row have color 2, and so on. In other words, for fixed $n = 2$ we consider a disjoint union $\mathcal{D}_2$ of integers in four colors, say j_1, j_2, j_3, j_4 is the integer j in colors 1, 2, 3, 4, satisfying the congruence conditions

$$\begin{aligned} &\{j_1 \mid j \geq 1, j \equiv 1 \bmod 4\}, \\ &\{j_2 \mid j \geq 2, j \equiv 2, 3 \bmod 4\}, \\ &\{j_3 \mid j \geq 3, j \equiv 0, 1, 3 \bmod 4\}, \\ &\{j_4 \mid j \geq 4, j \equiv 0, 1, 2, 3 \bmod 4\} \end{aligned} \tag{53}$$

and arranged in a sequence of triangles (52). For adjacent triangles in (52) corresponding to

$$\ldots, \quad X_{ab}(-j), ab \in B, \quad X_{ab}(-j-1), ab \in B, \quad \ldots$$

in (50) and a fixed row r we consider the corresponding two triangles: ${}^r\triangle$ on the left and $\triangle_r$ on the right. For example, for the third row we have $r = \underline{2}$ and two triangles denoted by bullets

$$\ldots \begin{array}{cccccccccccccccc} \cdot & & & & \cdot & & & & \bullet & & & & \cdot & & & \\ \cdot & \cdot & & & \cdot & \cdot & & & \bullet & \bullet & & & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \bullet & & \bullet & \bullet & \bullet & & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \ldots \tag{54}$$

are ${}^{\underline{2}}\triangle$ on the left and $\triangle_{\underline{2}}$ on the right. We say that two cascades

$$\mathcal{A} \subset {}^r\triangle \quad \text{and} \quad \mathcal{B} \subset \triangle_r$$

form an admissible pair of cascades in the sequence (52).

If our Conjecture 1 is correct, then the coefficient of q^m in the principally specialized character of $L(k\Lambda_0)$ equals the number of basis vectors (46) of degree $-m$, i.e., equals the number of colored partitions of m

$$m = \sum_{j_a \in \mathcal{D}_2} j_a f_{j_a} \tag{55}$$

satisfying difference conditions

$$\sum_{j_a \in \mathcal{A}} f_{j_a} + \sum_{j_b \in \mathcal{B}} f_{j_b} \leq k : \tag{56}$$

for every admissible pair of cascades in the sequence (52).

Example 3 Let $n = k = 2$. Then the first nine terms of Taylor series (47) are

$$1 + q + 2q^2 + 3q^3 + 5q^4 + 8q^5 + 12q^6 + 17q^7 + 25q^8 + \cdots . \tag{57}$$

By enumerating all admissible cascades for the basis B of C_2, we made a list of $4 \times 8 = 32$ difference conditions. From the list of difference conditions and the list of ordinary partitions, direct calculation gives all colored partitions of $m = 1, 2, \cdots, 8$ with colored parts in $\mathcal{D}_2$:

$$
\begin{aligned}
&1 = 1_1\\
&2 = 2_2 = 1_1 + 1_1\\
&3 = 3_2 = 3_3 = 2_2 + 1_1\\
&4 = 4_3 = 4_4 = 3_2 + 1_1 = 3_3 + 1_1 = 2_2 + 2_2\\
&5 = 5_1 = 5_3 = 5_4 = 4_3 + 1_1 = 4_4 + 1_1 = 3_2 + 2_1 = 3_3 + 2_1 = 3_2 + 1_1 + 1_1\\
&6 = 6_2 = 6_4 = 5_1 + 1_1 = 5_3 + 1_1 = 5_4 + 1_1 = 4_3 + 2_2 = 4_4 + 2_2 = 4_3 + 1_1 + 1_1\\
&7 = 7_2 = 7_3 = 7_4 = 6_2 + 1_1 = 6_4 + 1_1 = 5_1 + 2_2 = 5_3 + 2_2 = 5_4 + 2_2 = 5_3 + 1_1 + 1_1\\
&\quad = 5_4 + 1_1 + 1_1 = 4_3 + 3_2 = 4_3 + 3_3 = 4_4 + 3_2 = 4_4 + 3_3 = 4_3 + 2_2 + 1_1\\
&\quad = 3_2 + 3_2 + 1_1 = 3_2 + 3_3 + 1_1\\
&8 = 8_3 = 8_4 = 7_2 + 1_1 = 7_3 + 1_1 = 7_4 + 1_1 = 6_2 + 2_2 = 6_4 + 2_2 = 6_2 + 1_1 + 1_1\\
&\quad = 6_4 + 1_1 + 1_1 = 5_1 + 3_2 = 5_1 + 3_3 = 5_4 + 3_2 = 5_4 + 3_3 = 5_3 + 3_2 = 5_3 + 3_3\\
&\quad = 5_3 + 2_2 + 1_1 = 5_4 + 2_2 + 1_1 = 4_3 + 4_3 = 4_3 + 4_4 = 4_4 + 4_4 = 4_3 + 3_2 + 1_1\\
&\quad = 4_3 + 3_3 + 1_1 = 4_4 + 3_2 + 1_1 = 4_3 + 2_2 + 2_2 = 3_2 + 3_2 + 1_1 + 1_1.
\end{aligned}
$$

Hence the number of partitions (55) satisfying difference conditions (56) coincides with the coefficients of above Taylor series (57) for $m = 1, 2, \cdots, 8$.

We omit the details of calculations above; we will only explain how difference conditions eliminated the colored partition $5_1 + 2_2 + 1_1$ in the case $m = 8$. First of all, notice that 5_1 belongs to the triangle $X_{ab}(-2)$, and 2_2 and 1_1 belong to the triangle $X_{ab}(-1)$ (see 52). Now we choose $r = 1$ and consider the triangles ${}^1\triangle$ and $\triangle_1$

$$
\begin{array}{cccccccc}
\bullet & & & & \bullet & & & \\
\bullet & \bullet & & & \cdot & \cdot & & \\
\bullet & \bullet & \bullet & & \cdot & \cdot & \cdot & \\
\bullet & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

One pair of admissible cascades is

$$
\begin{array}{lccccccc}
m_{11;1} & & & & m_{11;2} & & & \\
m_{12;1} & \cdot & & & \cdot & \cdot & & \\
m_{1\underline{2};1} & \cdot & \cdot & & \cdot & \cdot & \cdot & \\
m_{1\underline{1};1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

and the corresponding difference condition (one of the 32 conditions) is given by

$$
m_{11;2} + m_{11;1} + m_{12;1} + m_{1\underline{2};1} + m_{1\underline{1};1} \leq 2\,.
$$

Since $m_{11;2} + m_{11;1} + m_{12;1} + m_{1\underline{2};1} + m_{1\underline{1};1} = 1 + 1 + 1 + 0 + 0 = 3 > 2$, the observed colored partition is eliminated from the list.

Conjecture 2 *Let $n = 2$ and $k \geq 2$. We conjecture that for every $m \in \mathbb{N}$ the number of colored partitions*

$$m = \sum_{j_a \in \mathcal{C}_k} j_a f_{j_a}$$

in three colors satisfying congruence conditions (48) equals the number of colored partitions

$$m = \sum_{j_a \in \mathcal{D}_2} j_a f_{j_a}$$

in four colors satisfying congruence conditions (53) and difference conditions (56) for every admissible pair of cascades in the sequence (52).

Remark 6 It is clear how to extend the above conjecture to $C_n^{(1)}$ for $n > 2$. The product formulas for principally specialized characters of some $L(\Lambda_0)$ and $L(2\Lambda_0)$ are given in [43] and [45].

In the case $n = 1$ and $k \geq 1$, the product formulas for principally specialized characters of $L(k\Lambda_0)$ and the corresponding combinatorial identities are given in [40].

In (54) we had several choices for triangles ${}^{r}\triangle$ on the left and $\triangle_r$ on the right. For $n = 1$, we have only two choices: (i) ${}^{1}\triangle$ on the left and $\triangle_1$ on the right, and (ii) ${}^{\underline{1}}\triangle$ on the left and $\triangle_{\underline{1}}$ on the right:

$$\cdots \begin{array}{cccccccc} \cdot & & \bullet & & \bullet & & \cdot & \\ \cdot & \cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot \end{array} \cdots, \quad \cdots \begin{array}{cccccccc} \cdot & & \cdot & & \bullet & & \cdot & \\ \cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot \end{array} \cdots, \tag{58}$$

Moreover, for $n = 1$, cascades are either vertical or horizontal. Altogether this gives four conditions (12).

In the case $n = k = 1$, the corresponding identity is equivalent to one of Capparelli's identities (see [5,15,40]).

In a way analogous to (54) and (58), we can also visualize difference conditions (6) in the Rogers–Ramanujan case—it is just one difference condition (2) for two adjacent points

$$\cdots \quad \cdot \quad \bullet \quad \bullet \quad \cdot \quad \cdots.$$

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