

## Inverse model for pullout determination of steel fibers

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**Abstract.** Fiber-reinforced concrete (FRC) is a material with increasing application in civil engineering. Here it is assumed that the material consists of a great number of rather small fibers embedded into the concrete matrix. It would be advantageous to predict the mechanical properties of FRC using nondestructive testing; unfortunately, many testing methods for concrete are not applicable to FRC. In addition, design methods for FRC are either inaccurate or complicated. In three-point bending tests of FRC prisms, it has been observed that fiber reinforcement does not break but simply pulls out during specimen failure. Following that observation, this work is based on an assumption that the main components of a simple and rather accurate FRC model are mechanical properties of the concrete matrix and fiber pullout force. Properties of the concrete matrix could be determined from measurements on samples taken during concrete production, and fiber pullout force could be measured on samples with individual fibers embedded into concrete. However, there is no clear relationship between measurements on individual samples of concrete matrix with a single fiber and properties of the produced FRC. This work presents an inverse model for FRC that establishes a relation between parameters measured on individual material samples and properties of a structure made of the composite material. However, a deterministic relationship is clearly not possible since only a single beam specimen of 60 cm could easily contain over 100000 fibers. Our inverse model assumes that the probability density function of individual fiber properties is known, and that the global sample load-displacement curve is obtained from the experiment. Thus, each fiber is stochastically characterized and accordingly parameterized. A relationship between fiber parameters and global load-displacement response, the so-called forward model, is established. From the forward model, based on Levenberg-Marquardt procedure, the inverse model is formulated and successfully applied.

**Keywords:** fiber-reinforced concrete, inverse model, Levenberg-Marquardt procedure, fiber pullout, probability density function (pdf)

### 1. Introduction

Fiber-reinforced concrete consists of a large number of stochastically distributed fibers

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embedded into concrete matrix. Stochastic distribution of fibers can be described with a probability distribution function, e.g., see Sampson (2009). For fiber-reinforced concrete, the probability distribution function is assumed to be known, i.e., it could be determined based on some destructive experiments or X-ray or magnetic scans. Knowledge of the probability distribution function enables computer generation of material samples needed for numerical experiments. Chapter 2 describes computer generation of fiber positions and their orientations within the prismatic concrete beam. We have compared those computer-generated beams with laboratory specimens that were subjected to X-ray scans (described in detail in Kalinčević 2016).

The purpose of computer-generated samples is numerical analysis and comparison of results with experiments. However, numerical analysis of FRC samples is demanding in terms of time and resources, with the large number of fibers being the main obstacle. There are several approaches that address the problem; one could apply the conforming finite element model where fibers are discrete and are placed along finite element edges, like in Smolčić and Ožbolt (2017). The conforming approach results with a huge model that is very demanding on computer resources. In addition, one could apply the fiber bundle model for composites, like in Raischel (2008). Both approaches assume that the material sample comprises rather small constituents (matrix elements and fibers) that have some local properties, and under experimentation, the sample exhibits some global properties. Description of various models of fiber behavior and corresponding pullout tests could be found in Do *et al.* (2015a, 2015b), Ngo *et al.* (2014) and Imamovic *et al.* (2015).

The goal of the numerical model is connecting those local and global properties into a useful model. Achieving usefulness requires adjusting the local parameters to match the global results. Ad hoc approach is based on parametric analysis and material parameters with the most convenient results being chosen as model parameters. Besides many limitations, the result of such an approach is difficult to generalize because the parameter determination problem is not formulated as a global optimization problem, so there is no clear overview of the relation between local and global extremes in the parameter space.

The forward model presented in this work is based on the fiber bundle model. At this stage, the model does not take into account the concrete matrix; only steel fibers contribute to its behavior. The purpose of the model is to prove the concept of the inverse model formulation and parameter determination. Our forward model demonstrates how stochastic characterization of small constituents (fibers) leads to generation of the specimen load-displacement curve. Chapter 3 presents models where two common probability distributions, normal and cosine, characterize fiber behavior. Consequently, there are two possible choices of parameters. Although the model takes into account only fibers without the matrix, there are no conceptual limitations in upgrading the model so that a concrete matrix model is taken into account, like in Mishnaevsky (2011). Even without the matrix contribution, the model can produce realistic global load-displacement curves, as seen in e.g., Ožbolt and Ananiev (2003). The forward model numerically simulates laboratory experiments and can be deterministic or stochastic, and in the latter case it is usually based on Monte Carlo method. However, Monte Carlo method is not suitable for the inverse model formulation because parameters change value from calculation to calculation. Parameters could be fixed in the domain of the probability distribution function when ‘order statistics’ is used (see Rinne (2010)). The new model produces a very similar global response as the one based on Monte Carlo method.

The inverse model in Chapter 4 is based on the Levenberg-Marquardt procedure that is often used for inverse models in parameter determination, see Kožar *et al.* (2017) and Carvalho *et al.* (2011). The formulation of the inverse model is only possible when one alleviates the need for

Monte Carlo simulations; in this work, it is achieved using order statistics whose precision (compared to Monte Carlo) is controlled with a proper choice of subdivisions (histogram ‘bins’) in the domain of the probability distribution function. On the other hand, the number of measurement (control) points on the load-displacement curve controls the accuracy of the inverse model.

Sensitivity of the model regarding various parameters is assessed in Chapter 5. It comes out naturally from the cost function of the Levenberg-Marquardt procedure.

Chapter 6 presents examples of convergence in parameter estimation using the inverse procedure. Accurate results are obtained for fiber material parameters estimated from sample load-displacement curves. For now, only the simplest tensile experiments and corresponding load-displacement curves are addressed. Considering the mentioned limitations, the current inverse model can be thought of as a proof of concept, but there is no inherent limitation preventing the inverse model from being extended to include the concrete matrix into the model.

Figures throughout the paper come from results of numerical simulations, using both forward and inverse models. Throughout the paper, all lengths have unit of (m), material modulus (kN/m<sup>2</sup>) and forces (kN).

## **2. Material stochastic characterization**

The distribution of fibers within the concrete matrix is the main parameter of fiber reinforced concrete. One can take the arrangement of fibers from X-ray scans or some other imaging procedure, but in order to establish a relation between fiber distribution and loading response, it is necessary to parameterize the fiber distribution. This stochastic parameter is characterized with its probability distribution function. Furthermore, the knowledge of the probability distribution function enables computer generation of synthetic material samples suitable for numerical experimentation.

### *2.1 Fiber position distribution*

The generation of fibers within the matrix starts with the generation of points in  $x$  and  $y$  directions (for 2D models that are the only ones addressed in this work). Points are generated independently according to the uniform distribution  $U(x; a, b) = 1/(b - a)$ , where  $a$  and  $b$  represent the interval of the distribution. However, points are not uniformly distributed within the specimen, but are clustered according to the Poisson distribution (Sampson 2009)  $P(X) = \frac{\lambda^k}{k!} e^{-\lambda}$ . Poisson distribution gives the probability of a fiber being within a sample segment.

### *2.2 Fiber orientation distribution*

After the points are generated, fibers with orientation  $\theta$  are introduced at their positions. The length of the fibers is assumed to be known, and their orientation  $\theta$  is generated stochastically according to the uniform distribution within the range  $[0, 2\pi]$ . A mathematical model that relates material properties and the load-displacement diagram requires knowledge of fiber length or fiber stiffness. They are completely determined with the probability distribution function (that is different from the *pdf* of their orientation  $\theta$ ). For fibers with orientation  $\theta$  distributed uniformly within  $[0, 2\pi]$ , it is possible to calculate the probability distribution of fiber length or fiber stiffness.

The length and stiffness distribution follows  $x = \cos \theta$ , and cosine distribution belongs to the family of ‘location-scale distributions’; it is calculated according to the rule for transformation of random variables  $pdf(x) = pdf(\theta) \left| \frac{d\theta}{dx} \right|$ . The result is the cosine probability distribution

$$pdf(x) = \begin{cases} \frac{1}{\pi\sqrt{1-x^2}}; & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

This result is confirmed in numerical experiments with Monte Carlo simulation.

### 2.3 Fiber generation (and X-ray comparison)

Numerical experiments require generated material samples that are discretized with finite elements and analyzed for different loading conditions. Synthetic material samples are generated in Mathematica (Wolfram Research, 2015) according to a given probability distribution function; comparison with X-ray is presented in Fig. 1.

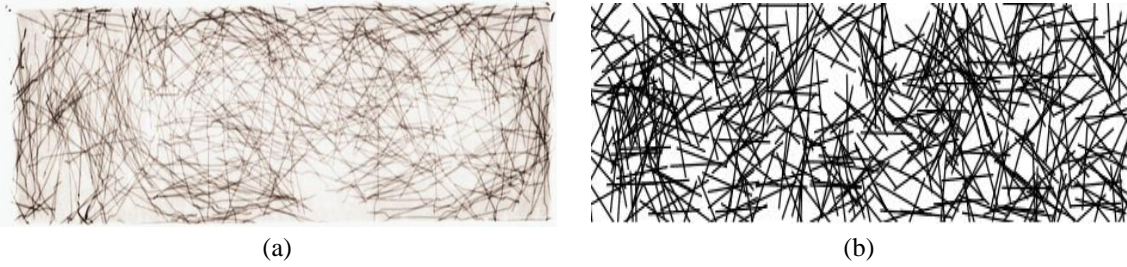


Fig. 1(a) X-ray scan of a beam sample 4×4×16 cm, (b) generated fibers

It is visible from Fig. 1 that fiber generation is quite realistic and can be used for numerical simulations.

### 3. Forward stochastic model

The main purpose of the forward model is to provide parameterization that relates material parameters and experimental results and is suitable for the formulation of an inverse model. The forward model can be easily built using the Monte Carlo method but such a model is not suitable for the inverse model formulation. Instead, the forward (and later inverse) model is based on order statistics. Order statistics is characterized with *iid* (independent identically distributed) variables  $X_i$  being arranged in ascending order  $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ .

The transition from ordinary to order statistics is essential for the success of the inverse procedure. We will explain order statistics on the normal distribution model with  $pdf(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(x - \mu)^2\right]$ . Monte Carlo simulation with mean  $\mu = 1.0$ , and standard deviation  $\sigma = 0.25$  for the length of 500 bars gives results as presented in Fig. 2(a) (each simulation produces somewhat different results).

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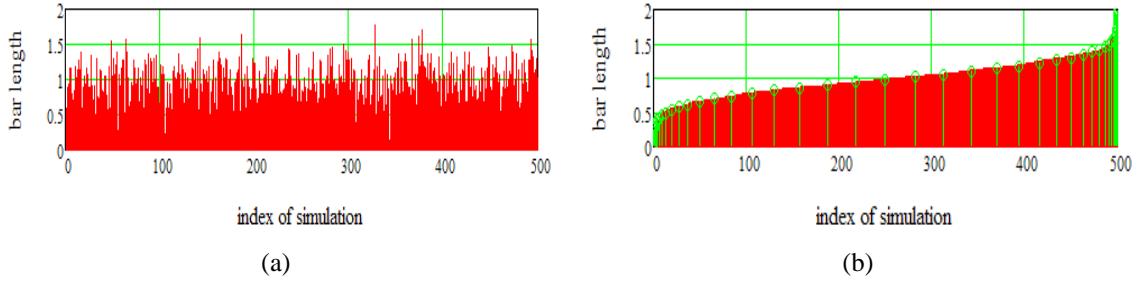


Fig. 2(a) 500 normally distributed bar lengths, (b) sorted bar lengths and their bin representatives

In Fig. 2(b) there are sorted values of randomly generated bar lengths. The simulation domain is divided into segments (bins), and a histogram is produced, see Fig. 3.

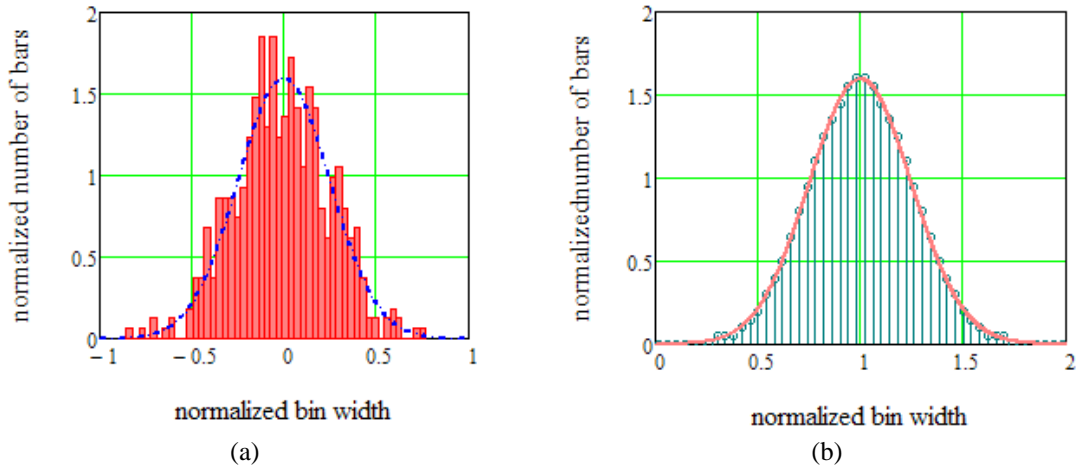


Fig. 3(a) normalized histogram of 50 bins for 500 normally distributed bar lengths generated in Monte Carlo experiment, (b) normalized histogram with bin representatives from order statistics *pdf*

In Fig. 3 normalized histograms for 50 bins are presented, i.e., the bin width is adjusted so that  $\sum_{bins} h_{bin} = N_{simulations} = 500$  (in this example). In Fig. 3(a), the normalized histogram is from Monte Carlo simulation, and in Fig. 3(b) from bin representatives for order statistics. Figs. 2-3 support the intuitive feeling that sorting of the randomly generated values does not change its statistics; mathematical proof will not be given here. In our models, we will use order statistics and bin representative values generated from the probability distribution function and not from the Monte Carlo simulation. However, Monte Carlo will be used for generation of load-displacement curves as a substitute for real experiments.

### 3.1 Stiffness vs. Length Stochastic Model

The forward model for the load-displacement curve is formulated as a sum of forces from

individual bar contributions, which can be obtained with either bar stiffness or bar length as random parameters:

$$F_{bar} = \begin{cases} \frac{EA_{\mu}}{L_{stoch.}} \delta & \text{if } \frac{\delta}{L} \leq d_t \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

$$F_{bar} = \begin{cases} \frac{EA_{stoch.}}{L_{\mu}} \delta & \text{if } \frac{EA_{stoch.}}{L} \delta \leq F_t \\ 0 & \text{otherwise} \end{cases}$$

These equations represent a linear-elastic material with sudden rupture, where  $\delta$  is displacement,  $EA$  is bar stiffness,  $L$  is bar length,  $d_t$  is damage threshold in strain and  $F_t$  is damage threshold in stress. The influence of the above equations on the load-displacement curve can be seen from Fig. 4.

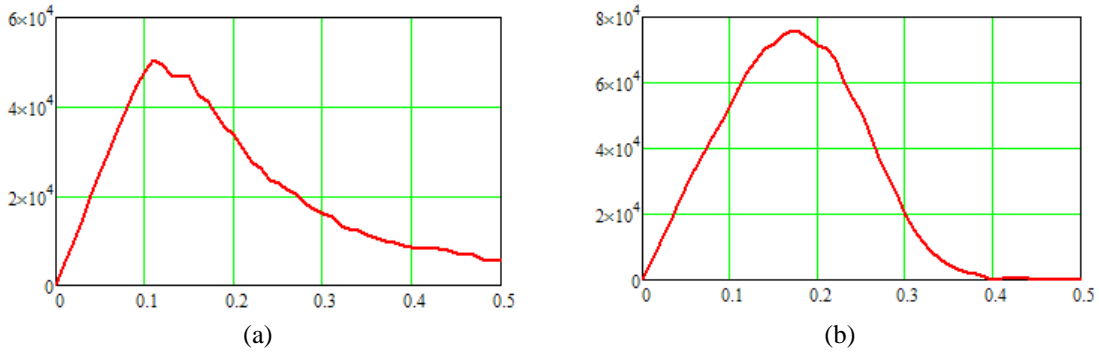


Fig. 4(a) load-displacement curve for the stiffness bar model, (b) load-displacement curve for the length bar model

In Fig. 4 we could compare results for models with stiffness random generation vs. length random generation. Parameters for the length stochastic model were  $L_{\mu} = 1.0$ ,  $\sigma_L = 0.25$ ,  $d_t = 0.25$  and for the stiffness stochastic model  $EA_{\mu} = 1000.0$ ,  $\sigma_{EA} = 500$ ,  $F_t = 200$ . The stiffness stochastic model has been chosen for numerical experiments since it is considered more realistic; it is the directional stiffness that varies according to fiber orientation.

### 3.2 Normal distribution model

Fiber stiffness is generated using Monte Carlo method and normal distribution, as shown in Eq. (2). An experiment with 500 generated bars produced mean bar stiffness  $EA_{mean} = 942.7$  with standard deviation for bar stiffness  $EA_{\sigma} = 495.5$ , and the load-displacement curve in Fig. 4(a).

In Fig. 5(a), a force-displacement curve for some bars is shown, it is linear with sudden failure. Since the distribution of stiffness in bars is stochastic, failures happen at different displacement levels (and some bars do not fail at all). In Fig. 5(b) there is a sorted contribution of all the bars at different displacement levels; at the displacement level  $\delta = 0.1$  almost all the bars contribute to

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the load-displacement curve, at the displacement level  $\delta = 0.3$  somewhat above 100 bars contribute to the load-displacement curve, etc.

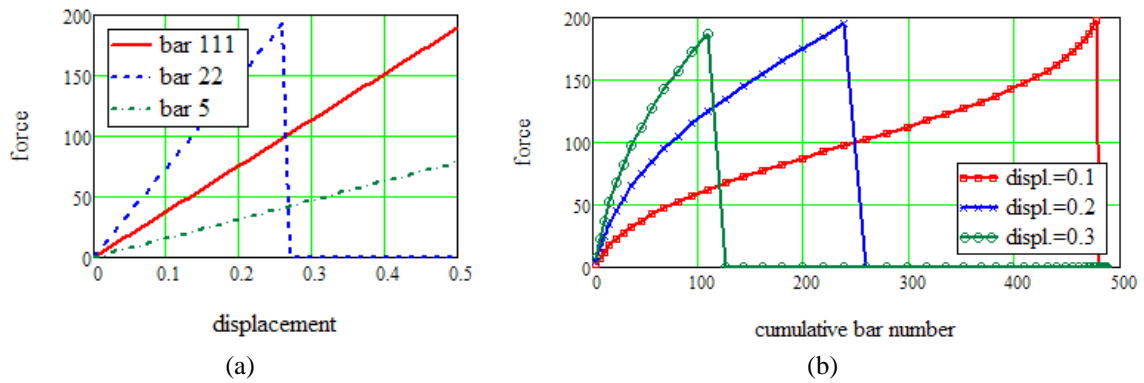


Fig. 5(a) force-displacement curve for some bars, (b) force-displacement curve for bars at different displacement levels

### 3.3 Cosine distribution model

Fiber angles are generated using Monte Carlo method with uniform *pdf*; for orientation, an experiment produced mean angle  $\theta_{mean} = 3.103$  and the analytic mean is  $\theta_{mean} = \pi = 3.146$ . Bar stiffness is generated from uniformly distributed angles using *pdf* from Eq. (1); for orientation, an experiment with 1500 generated bars produced mean bar stiffness  $EA_{mean} = 794.3$  and the maximum possible stiffness was given  $EA_{max} = 1250$ .

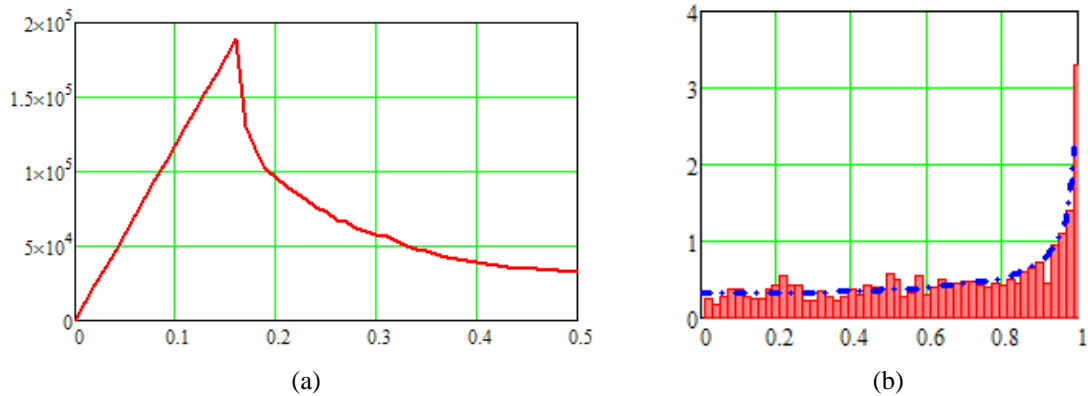


Fig. 6(a) load-displacement curve for stiffness bar model for cosine distribution, (b) comparison of normalized histograms of generated bar stiffness and cosine distribution

In Fig. 6(a) a load-displacement curve for the cosine distribution model is shown and in Fig. 6(b) there is a normalized histogram for generated stiffness and its comparison with cosine

distribution. There are not many parameters in this distribution that could serve to fine-tune the load-displacement curve. There are some propositions for how to introduce an additional variability, like considering the influence of the granulometric diagram of concrete on angle distribution (there could be no fibers in the place where the aggregate is), but that will not be addressed here.

#### 4. Inverse stochastic model

The main purpose of the inverse model is to enable a reliable calculation of model parameters from experimental results. That is enabled with the introduction of order statistics that alleviates the need for Monte Carlo simulations. The parameter estimation is formulated as a global optimization problem that is solved by applying Levenberg-Marquardt procedure. The objective function that is minimized in the Levenberg-Marquardt procedure can easily be applied for sensitivity analysis, it only needs to be treated as a function in parametric space.

Inverse parameter estimation can be formulated as a global optimization problem by applying the minimization procedure to a suitable objective function. The objective function that is minimized in Levenberg-Marquardt procedure reads

$$S_{err} = \sum_{im=1}^{nm} [F\delta_{im} - Fu(\delta_{im}, \sigma, \mu, d_t)]^2 \quad (3)$$

where  $S_{err}$  is the cumulative error,  $F\delta$  are measured values at measuring points  $im$  (from the experimentally determined load-displacement curve), and  $Fu(\delta_{im}, \sigma, \mu, d_t)$  are expected values from the stochastic model that are function of the parameters we would like to determine (mean, variance, damage threshold).

The minimization procedure leads to iterative explicit equations for each parameter calculation, e.g., for variance

$$\Delta\sigma = \frac{\sum_{im} [F\delta_{im} - Fu(\delta_{im}, \sigma, \mu, d_t)] X\sigma_{im}}{\sum_{im} (X\sigma_{im})^2} \quad (4)$$

where  $X\sigma$  is the sensitivity parameter calculated at each measuring point. The procedure is iterative, and parameter update is additive,  $\sigma_{i+1} = \sigma_i + \Delta\sigma$ . Other parameters are calculated in a similar manner.

The objective function and the minimization procedure for cosine distribution differ only in the number of parameters and the model function.

#### 5. Sensitivity analysis

The objective function for Levenberg-Marquardt procedure is analyzed in parameter space. Sensitivity analysis is performed through numerical experiments performed for a range of parameter values of the objective function. There are 21 values for each parameter and all are chosen so that they include the optimal value.

In Fig. 7(a), graphical representation of the sensitivity analysis is shown. Figures present parameter sensitivity of the normal distribution model; in order to emphasize the extremes, the error function is in semi-logarithmic scale.



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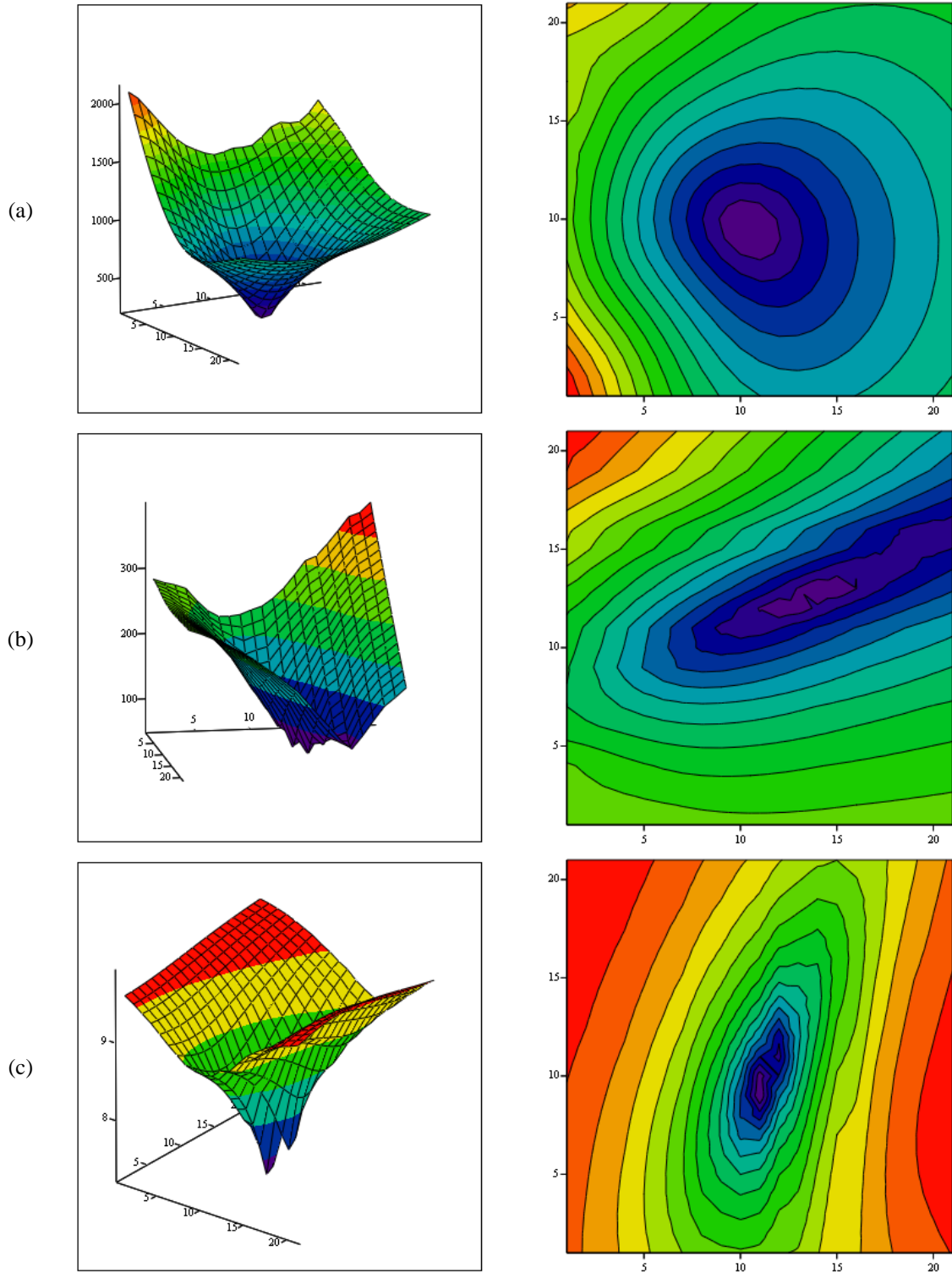


Fig. 7 Sensitivity analysis for (a) stiffness variance, (b) force damage threshold, (c) stiffness mean

From Fig. 7, it is visible that the inverse procedure based on the Levenberg-Marquardt model is well posed and stable and all minima clearly visible. The existence of some local minima for force damage threshold and stiffness mean value is evident. Consequently, one may conclude that sensitivity analysis is necessary in parameter identification based on the above inverse procedure. However, careful selection of measuring points along the load-displacement curve can reduce or eliminate the appearance of local minima.

Cosine distribution (in this work) does not have parameters, and sensitivity analysis can be assessed from parameter sensitivity presented in Fig. 8.

## 6. Parameter estimation-examples

First, the forward model based on the Monte Carlo method is employed to produce a global load-displacement curve. Material parameters in the forward model are  $EA_{mean} = 100$ ,  $\sigma_{EA} = 500$ ,  $F_{threshold} = 200$ . Those are the parameters to be retrieved from "wrong guesses" using the inverse model. Second, the order statistics model is used to generate the global response that is compared with Monte Carlo simulation; the difference in results between the two models represents the maximal possible precision of the inverse procedure. Figs. 8-9 illustrate the precision of the order statistics (and maximum possible accuracy of the inverse model).

Fig. 8 presents a comparison between experimental and model load-displacement curve for cosine distribution and change in global response for different values of the force damage threshold ("model exact" is for  $F_{threshold} = 200$ ).

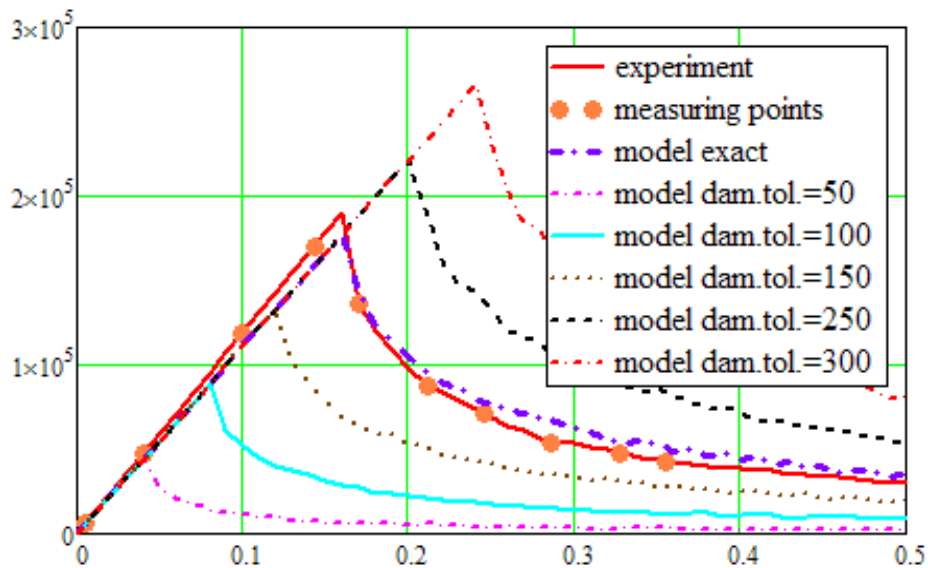


Fig. 8 Comparison of experimental and model load-displacement curve for cosine distribution

In Fig. 9, there is a comparison of experimental and model load-displacement curves for normal distribution, and changes in global response for various values of model parameters.

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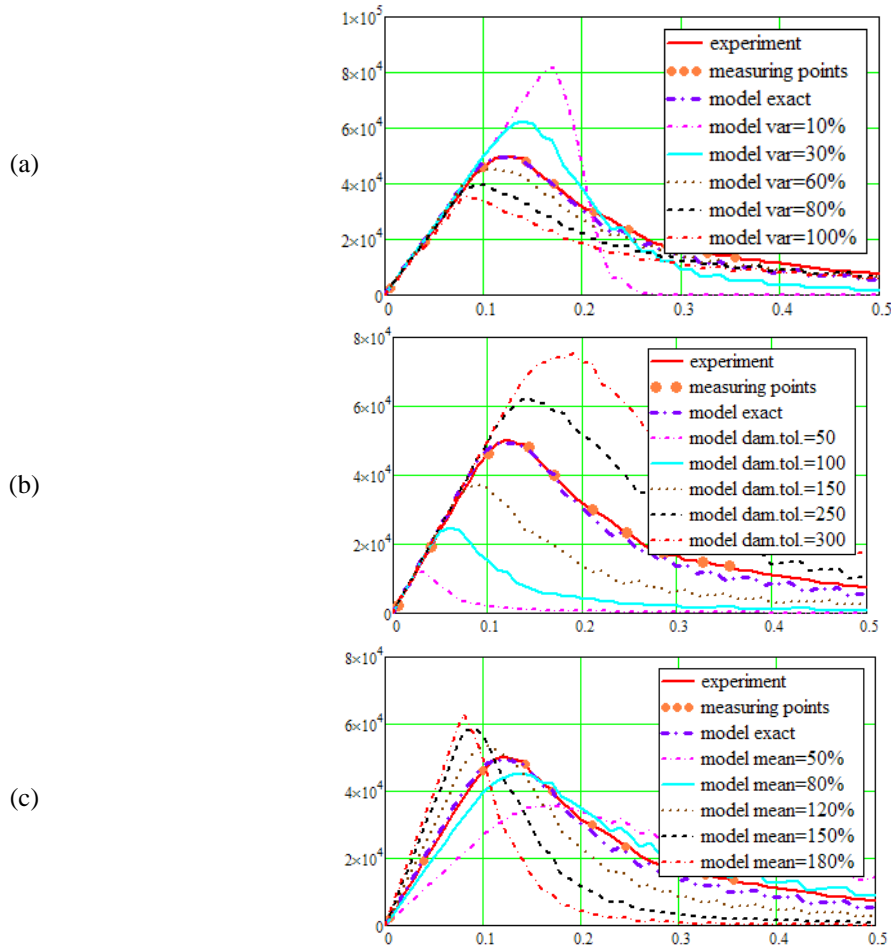


Fig. 9 Comparison of experimental and model load-displacement curve for different (a) stiffness variance, (b) force damage threshold, (c) stiffness mean

In Fig. 9, “model exact” stands for exact values of parameters as used in the forward model to produce the global load-displacement curve.

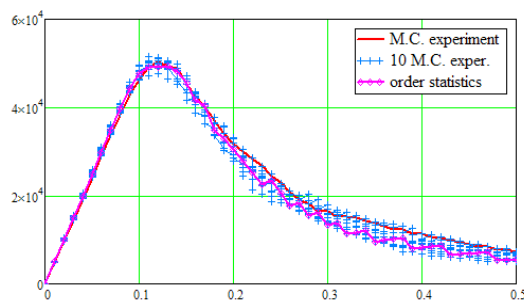


Fig. 10 Comparison of 10 Monte Carlo simulations and order statistics for 50 bins

Fig. 10 illustrates the precision of order statistics; it is evident that the order statistics is within the variance of the Monte Carlo experiment. Number of intervals ("bins") in order statistics in the example is 10; a higher number of intervals increases the precision. A detailed precision analysis will not be performed here.

Afterwards, measurement points are selected; in the example there are 10 points along the load-displacement curve. Points should be taken along the whole load-displacement curve; they are visible in Figs. 8-9 as orange dots. Points in only one part of the curve tend to produce local minima that could lead to poor accuracy of the inverse procedure.

An illustration of convergence and accuracy is given by selecting wrong parameters,  $EA_{mean} = 900$ ,  $\sigma_{EA} = 400$ ,  $F_{threshold} = 200$ . After 3 iterations, increment errors dropped below 0.1% and estimated parameters were  $EA_{mean} = 926.$ ,  $\sigma_{EA} = 475.8$ ,  $F_{threshold} = 198.7$ . The errors between estimated and exact parameter values are  $\Delta EA_{mean} = 4.8\%$ ,  $\Delta \sigma_{EA} = 7.3\%$ ,  $\Delta F_{threshold} = 0.6\%$ . The comparison of the error function calculated from estimated parameters with their value from the forward simulation is presented in Fig. 11. The presented inverse procedure cannot achieve better accuracy unless the number of intervals in order statistics is increased. The errors in parameters for a division into 150 intervals are  $\Delta EA_{mean} = 3.7\%$ ,  $\Delta \sigma_{EA} = 5.7\%$ ,  $\Delta F_{threshold} = 0.1\%$ . For a division into 250 intervals, and after 11 iterations, the errors are  $\Delta EA_{mean} = 1.8\%$ ,  $\Delta \sigma_{EA} = -0.2\%$ ,  $\Delta F_{threshold} = -0.3\%$ . Fig. 11 illustrates the change in error with bin number increase.

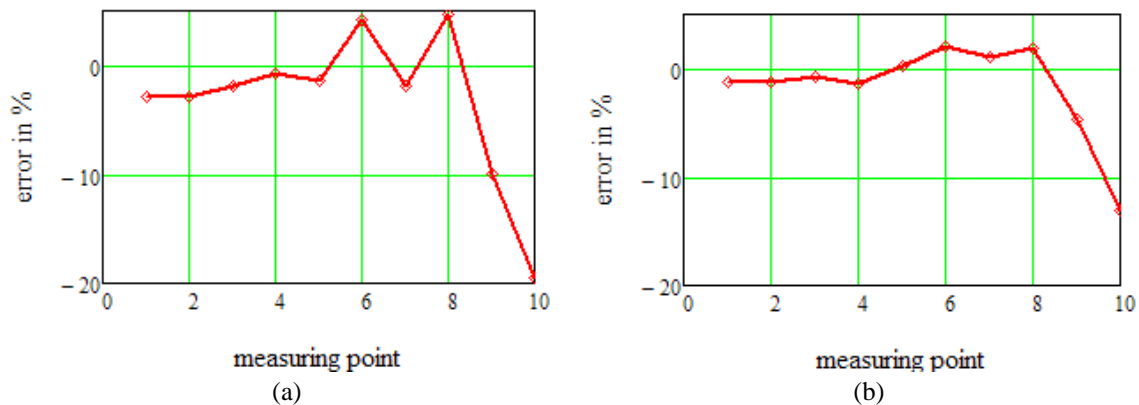


Fig. 11 Comparison of error in load-displacement at measuring points for the exact and estimated values of parameters for order statistics for (a) 50 bins, (b) 150 bins

Examples presented throughout the paper are programmed and calculated in Mathcad (PTC Mathcad 2007).

## 7. Conclusions

This work is a proof of concept showing that parameters of fiber reinforced concrete could be obtained using an inverse procedure that is formulated as a global optimization problem. The forward model is formulated using a stochastic approach and order statistics. From the forward

model, applying Levenberg-Marquardt method, an inverse model is produced. Stable convergence has been obtained for all parameters: variability of fiber stiffness variance, variation in fiber mean stiffness and variation in damage threshold. However, there are some local minima for certain parameters and it is advisable to perform the sensitivity analysis to assure that the global minimum has been reached and that the optimal value of parameters has been determined.

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