On the extension of D$(-8k^2)$-triple $\{1, 8k^2, 8k^2+1\}$

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Introduction

Definition *A set of m positive integers $\{a_1, a_2, \ldots, a_m\}$ is called $D(n)$-m-tuple if $a_ia_j + n$ is a perfect square for all $1 \leqslant i < j \leqslant m$.*

Extending the initial triple $\{1, 8k^2, 8k^2+1\}$ with d and then eliminating d leads to a system

$$x^2 - 2y^2 = -8k^2 + 1$$
$$z^2 - (16k^2+2)y^2 = 1$$

From Pell's equation, we get recurrent sequences y_n and z_n. Due to the first equation, the problem reduces to examining when can an element of the new sequence $X_n = 2y_n^2 - 8k^2 + 1$ be a complete square.
Using the relations between y_n and z_n, e.g. $y_{2n+1} = 2y_nz_n$, we write X_n as a product of two factors, one of which is obviously not a square. We finish the proof by showing that these factors are relatively prime via principle of descent.

The result

By elementary means, we show that the $D(-8k^2)$-triple $\{1, 8k^2, 8k^2+1\}$ can be extended to at most a quadruple (the fourth element can be only $32k^2+1$). The extension is possible if and only if $24k^2+1$ is a square.

The proof

For odd indices, we use the following identity:

$$y_{2n+1} = 2y_nz_n,$$

to show that X_{2n+1} is never a square.

$$\begin{aligned}X_{2n+1} &= 2y_{2n+1}^2 - 8k^2 + 1\\ &= 2(2y_nz_n)^2 - 8k^2 + 1\\ &= 8y_n^2z_n^2 - 8k^2 + 1 \text{ (substitute } z_n^2 \text{ from Pell's equation)}\\ &= 8y_n^2(1 + (16k^2+2)y_n^2) - 8k^2 + 1\\ &= 8y_n^2 + 16(8k^2+1)y_n^4 - 8k^2 + 1 \text{ (can be factored as)}\\ &= \mathbf{(4y_n^2+1)(32y_n^2k^2 + 4y_n^2 - 8k^2 + 1)}.\end{aligned}$$

Obviously, the first factor cannot be a square (for $y_n \in \mathbb{N}$). Therefore, it suffices to show that **two factors obtained are relatively prime. We prove this via principle of descent.**

Principle of descent

We prove that

$$p \mid 4y_n^2 + 1 \text{ and } p \mid 32y_n^2k^2 + 4y_n^2 - 8k^2 + 1 \Longrightarrow$$
$$\Longrightarrow p \mid 4y_{n-1}^2 + 1 \text{ and } p \mid 32y_{n-1}^2k^2 + 4y_{n-1}^2 - 8k^2 + 1$$

Assume now that prime p divides both $4y_n^2+1$ and $32y_n^2k^2 + 4y_n^2 - 8k^2 + 1$. Then p divides $32y_n^2k^2 + 4y_n^2 - 8k^2 + 1 - (4y_n^2 + 1) = 8k^2(4y_n^2 - 1)$ too. Since p is odd (because it divides an odd number $4y_n^2+1$), it follows that $p \mid k^2(4y_n^2-1)$. It cannot divide a second factor, because then it would divide $4y_n^2 + 1 - (4y_n^2 - 1) = 2$ as well. To conclude, p divides k^2, but then it also divides k.
Let us prove now that p divides both $4y_{n-1}^2+1$ and $32y_{n-1}^2k^2 + 4y_{n-1}^2 - 8k^2 + 1$. From recurrence relation for y_n: $y_n = (16k^2+1)y_{n-1} + 4kz_{n-1}$, we get $y_n - y_{n-1} = 16k^2y_{n-1} + 4kz_{n-1}$, so, since p divides k, it follows that it divides the right hand side, which is equal to $y_n - y_{n-1}$. Hence, $y_n \equiv y_{n-1} \pmod p$. Therefore, p divides $4y_{n-1}^2+1$. On the other hand, since $32y_{n-1}^2k^2 + 4y_{n-1}^2 - 8k^2 + 1 = 8k^2(4y_{n-1}^2 - 1) + 4y_{n-1}^2 + 1$, it is true that p divides $32y_{n-1}^2k^2 + 4y_{n-1}^2 - 8k^2 + 1$.
Further "descent" implies that p divides $4y_0^2 + 1 = 64k^2 + 1$ (and $32y_0^2k^2 + 4y_0^2 - 8k^2 + 1$). However, since p divides k, it would 1, which is a contradiction. To conclude, $4y_n^2+1$ and $32y_n^2k^2 + 4y_n^2 - 8k^2 + 1$ do not have any common prime factors, i.e. they are relatively prime.

For even indices, we use the following identity:

$$z_{2n} - 1 = \left(\frac{y_n + y_{n-1}}{4k}\right)^2,$$

to show that X_{2n} is not a square (except possibly for $n = 0$).

$$\begin{aligned}X_{2n} &= 2y_{2n}^2 - 8k^2 + 1 = 2 \cdot \frac{z_{2n}^2 - 1}{16k^2+2} - 8k^2 + 1 =\\ &= \frac{z_{2n}^2 - 1 - 64k^4 - 8k^2 + 8k^2 + 1}{8k^2+1}\\ &= \frac{z_{2n}^2 - 64k^2}{8k^2+1}\\ &= \frac{z_{2n} - 8k^2}{8k^2+1} \cdot (z_{2n} + 8k^2)\end{aligned}$$

We show that the second factor, $z_{2n} + 8k^2$ cannot be a square for $n \geqslant 1$. Due to the identity given above, we obtain

$$\left(\frac{y_n + y_{n-1}}{4k}\right)^2 < z_{2n} + 8k^2 = \left(\frac{y_n + y_{n-1}}{4k}\right)^2 + 1 + 8k^2 < \left(\frac{y_n + y_{n-1}}{4k} + 1\right)^2.$$

Last inequality holds if and only if $1 + 8k^2 < \frac{y_n + y_{n-1}}{2k} + 1$, i.e. if and only if $16k^3 < y_n + y_{n-1}$.
Since already $y_1 = 2(16k^2+1) \cdot 4k$, that inequality truly holds for $n \geqslant 1$.

In this case, it is easier to prove that the factors obtained are relatively prime. Assume prime p divides both $\frac{z_{2n} - 8k^2}{8k^2+1}$ and $z_{2n} + 8k^2$. Then it divides $z_{2n} - 8k^2$, as well as the difference $z_{2n} + 8k^2 - (z_{2n} - 8k^2) = 16k^2$. Since p divides $z_{2n} + 8k^2$, and all elements of a sequence z_n are odd, it follows that p is odd. Hence $p \mid 16k^2 \Rightarrow p \mid k^2$. It follows that $p \mid z_{2n}$. But the sequence z_n is relatively prime with k since all elements give a remainder of 1 when divided with k. Therefore, it is impossible that z_{2n} and k have a common prime factor p.

One should also check that the first factor, fraction $\frac{z_{2n} - 8k^2}{8k^2+1}$, is actually an integers. But that is easy: inductively prove that elements of z_n, when divided by $8k^2+1$, leave remainders $(1, 8k^2, 1, 8k^2, \ldots)$. Hence $z_{2n} - 8k^2$ is indeed divisible by $8k^2+1$.

We are left with the case $n = 0$, i.e. when $X_0 = \frac{z_0 - 8k^2}{8k^2+1} \cdot (z_0 + 8k^2) = 24k^2 + 1$ is a square.

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