On the extension of $D(-8k^2)$-triple $\{1, 8k^2, 8k^2+1\}$

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Problem

Let $k \in \mathbb{N}$. Triple $\{1, 8k^2, 8k^2+1\}$ has $D(-8k^2)$-property:

$$1 \cdot 8k^2 - 8k^2 = 0, \quad 1 \cdot (8k^2+1) - 8k^2 = 1, \quad 8k^2 \cdot (8k^2+1) - 8k^2 = 64k^4.$$

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$$\begin{aligned} d - 8k^2 &= x^2 \\ 8k^2 d - 8k^2 &= (y')^2 \qquad (d - 1 = 2y^2) \\ (8k^2+1)d - 8k^2 &= z^2 \end{aligned}$$

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We will show that this extension is possible if and only if $24k^2+1$ is a square (d can only be $32k^2+1$).

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$$z_{n+1} = (16k^2+1)z_n + 4k(16k^2+2)y_n, \quad z_0 = 16k^2+1,\ z_{-1} = 1$$
$$y_{n+1} = (16k^2+1)y_n + 4kz_n, \quad y_0 = 4k,\ y_{-1} = 0,$$

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and recurrence relations

$$z_{n+2} = 2(16k^2 + 1)z_{n+1} - z_n$$
$$y_{n+2} = 2(16k^2 + 1)y_{n+1} - y_n$$

with same initial conditions.

$$y_n = c_1(16k^2 + 1 + 4k\sqrt{16k^2 + 2})^n + c_2(16k^2 + 1 - 4k\sqrt{16k^2 + 2})^n$$

Sequence X_n

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$X_n := 2y_n^2 - 8k^2 + 1.$

When is $X_n = \square$?

Odd indices

$X_{2n+1} = 2y_{2n+1}^2 - 8k^2 + 1$ is never a square.

$$y_{2n+1} = 2y_n z_n$$

Odd indices

$$\begin{aligned} X_{2n+1} &= 2y_{2n+1}^2 - 8k^2 + 1 \\ &= 2(2y_n z_n)^2 - 8k^2 + 1 \\ &= 8y_n^2 z_n^2 - 8k^2 + 1 \qquad (z_n^2 = (16k^2+2)y_n^2 + 1) \\ &= 8y_n^2(1 + (16k^2+2)y_n^2) - 8k^2 + 1 \\ &= 8y_n^2 + 16(8k^2+1)y_n^4 - 8k^2 + 1 \\ \\ &= \mathbf{(4y_n^2 + 1)(32y_n^2k^2 + 4y_n^2 - 8k^2 + 1)}. \end{aligned}$$

Relatively prime factors – Principle of descent

Assume

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$$p \mid 32y_n^2k^2 + 4y_n^2 - 8k^2 + 1 - (4y_n^2 + 1) = 8k^2(4y_n^2 - 1).$$

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Hence, $p \mid k^2$, so $p \mid k$.

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Since $32y_{n-1}^2 k^2 + 4y_{n-1}^2 - 8k^2 + 1 = 8k^2(4y_{n-1}^2 - 1) + 4y_{n-1}^2 + 1$,

p divides $32y_{n-1}^2 k^2 + 4y_{n-1}^2 - 8k^2 + 1$ too.

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Further "descent" implies that
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We conclude that $4y_n^2 + 1$ and $32y_n^2k^2 + 4y_n^2 - 8k^2 + 1$ don't have prime factors in common.

Remaining case

Even indices are resolved similarly, except for 0.

When $n = 0$, i.e. $X_0 = \dfrac{z_0 - 8k^2}{8k^2+1} \cdot (z_0 + 8k^2) = 24k^2 + 1$ is a square.

$D(-8k^2)$-triple $\{1, 8k^2, 8k^2+1\}$ has at most one extension.
Extension exists when $24k^2 + 1$ is a square.
In that case $d = 32k^2 + 1$.