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CRITERIA ENLARGEMENT FOR AN INEQUALITY BETWEEN QUASI-ARITHMETIC MEANS

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Abstract

Given criteria on differentiable functions for inequalities between a function value calculated on two quasi-arithmetic means and the quasi-arithmetic mean calculating on function values of two variables are enlarged for a case of three quasi-arithmetic means and functions of three variables. Special investigations are analyzed in additional and multiplicative cases.

1. Introduction

The quasi-arithmetic mean in discrete instance is defined for a continuous and monotone function $\varphi : J_x \subseteq \mathbb{R} \to \mathbb{R}$, sentence $(x) = (x_1, \ldots, x_n)$ and a probability weight sentence $(a) = (a_1, \ldots, a_n)$ with $\sum_{k=1}^{n} a_k = 1$ by the formula

$$M_\varphi(x;\ a) = \varphi^{-1}\left(\sum_{k=1}^{n} a_k \varphi(x_k)\right). \tag{1.1}$$

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For continuous and monotone functions $\psi : J_y \to \mathbb{R}$ and $\rho : J_z \to \mathbb{R}$ that are defined on intervals $J_y, J_z \subseteq \mathbb{R}$, for sentences $(y) = (y_1, \dots, y_n) \subset J_y$ and $(z) = (z_1, \dots, z_n) \subset J_z$, using a function $f : J_x \times J_y \times J_z \to \mathbb{R}$, we will consider the inequality between three quasi-arithmetic means and quasi-arithmetic mean defined by the values of $f(x, y, z)$:

$$f(M_\varphi(x; a), M_\psi(y; a), M_\rho(z; a)) \geq M_\chi(f(x, y, z); a). \tag{1.2}$$

Fundamental function required for assaying conditions is

$$H(s, t, r) = \chi f(\varphi^{-1}(s), \psi^{-1}(t), \rho^{-1}(r)), \tag{1.3}$$

where $s = \varphi(x)$, $t = \psi(y)$, and $r = \rho(z)$.

Relationship between inequality (1.2) and function (1.3) is given in the next lemma.

Lemma 1.1. *The inequality* (1.2) *holds in the case that function* $H(s, r, t)$ *is concave and* χ *increases or in the case that function* $H(s, r, t)$ *is convex and* χ *decreases.*

The opposite inequality is arising when H is convex and χ *increases and H is concave and* χ *decreases.*

Proof. Suppose $H(s, t, r)$ is concave and χ increases. Then χ^{-1} increases too. Concavity gives

$$H\left(\sum_{i=1}^{n} a_i s_i, \sum_{i=1}^{n} a_i t_i, \sum_{i=1}^{n} a_i r_i\right) \geq \sum_{i=1}^{n} a_i H(s_i, t_i, r_i).$$

Use definitions (1.3) and (1.1). Then apply χ^{-1} on the both sides and obtain (1.2). Other cases are similarly provable.

Taylor queue, for at least twice differentiable functions, is giving a concavity characterization considering the second differential $d^2H(ds, dt, dr)$ as a quadratic form in (ds, dt, dr) :

$$d^2H(ds, dt, dr) = H_{ss}ds^2 + H_{tt}dt^2 + H_{rr}dr^2 + 2H_{st}dsdt + 2H_{sr}dsdr + 2H_{rt}drdt. \tag{1.4}$$

The well-known facts are following in the next lemma.

Lemma 1.2. *Suppose that function defined by* (1.3) *has continuous second partial derivatives. Function* $H(s, t, r)$ *is concave if the form* (1.4) *is negative semidefinite. The form* (1.4) *is negative definite if and only if*

$$H_{ss} \leq 0, \quad \begin{vmatrix} H_{ss} & H_{st} \\ H_{st} & H_{tt} \end{vmatrix} \geq 0, \text{ and } \begin{vmatrix} H_{ss} & H_{st} & H_{sr} \\ H_{st} & H_{tt} & H_{rt} \\ H_{rs} & H_{rt} & H_{rr} \end{vmatrix} \leq 0. \tag{1.5}$$

Function $H(s, t, r)$ *is convex if the form* (1.4) *is positive semidefinite. The form is positive semidefinite if the all determinants in* (1.5) *are greater or equal than zero.*

2. Additive and Multiplicative Case

Inequality

$$f(M_{\varphi}(x; a), M_{\psi}(y; a)) \geq M_{\chi}(f(x, y); a) \tag{2.1}$$

was investigated by Beck in [9] for additive case, where $f(x, y) = x + y$ and multiplicative case with $f(x, y) = xy$. In this section, we enlarged this cases on three variables for differentiable functions considering (1.5) for (1.4).

2.1. Additive case

The observed function in (1.2) is $f(x, y, z) = x + y + z$. Given functions are $\varphi(x) = s$, $\psi(y) = t$, $\rho(z) = r$, and $\chi(u) = w$ with $u = x + y + z$. Basic research are carried out by the function (1.3) in the shape of

$$H(s, t, r) = \chi(\varphi^{-1}(s) + \psi^{-1}(t) + \rho^{-1}(r)). \tag{2.2}$$

Auxiliary functions introduced for shorter expressions are

$$F_1(x) := \frac{\varphi'(x)}{\varphi''(x)}, \quad F_2(y) := \frac{\psi'(y)}{\psi''(y)}, \quad F_3(z) := \frac{\rho'(z)}{\rho''(z)}, \quad \text{and } F_4(u) := \frac{\chi'(u)}{\chi''(u)}. \tag{2.3}$$

Appreciating (2.3), we express the next two theorems.

Theorem 2.1. *Suppose that the functions* φ, ψ, ρ, *and* χ *have second derivatives and functions* F_i, $i = 1, 2, 3$ *from* (2.3) *are definable on the intervals* J_x, J_y, J_z, *and* J_u. *For the queues* $(x) \in J_x$, $(y) \in J_y$, $(z) \in J_z$, *and* $(x) + (y) + (z) \in J_u$, *the inequality*

$$M_\varphi(x;\, a) + M_\psi(y;\, a) + M_\rho(z;\, a) \geq M_\chi(x + y + z;\, a), \tag{2.4}$$

is valid if

(i) *all* F_i, $i = 1, 2, 3, 4$ *are positive and* $F_1(x) + F_2(y) + F_3(z) \leq F_4(x + y + z)$;

(ii) *or if* F_4 *is negative, but* F_i, $i = 1, 2, 3$ *are positive.*

The inequality is opposite if

(iii) *all* F_i, $i = 1, 2, 3, 4$ *are negative and* $F_1(x) + F_2(y) + F_3(z) \geq F_4(x + y + z)$;

(iv) *or if* F_4 *is positive, but all* F_i, $i = 1, 2, 3$ *are negative.*

Proof. Examining the conditions (1.5) for the (1.3) in the case $f(x, y, z) = x + y + z$:

$$H(s, t, r) = \chi(\varphi^{-1}(s) + \psi^{-1}(t) + \rho^{-1}(r)). \tag{2.5}$$

The first condition from (1.5) gives $\frac{\partial^2 H}{\partial s^2} = \frac{\chi'}{\varphi'^2}\left(\frac{\chi''}{\chi'} - \frac{\varphi''}{\varphi'}\right) \leq 0$, which is fulfilled in the terms (2.3) if and only if

$$\chi'\left(\frac{1}{F_4} - \frac{1}{F_1}\right) \leq 0. \tag{2.6}$$

Second condition, after calculating the derivatives and determinant calculation gives

$$\frac{\chi'}{\varphi'^2}\left(\frac{\chi''}{\chi'}-\frac{\varphi''}{\varphi'}\right)\frac{\chi'}{\psi'^2}\left(\frac{\chi''}{\chi'}-\frac{\psi''}{\psi'}\right)-\frac{\chi''^2}{\varphi'^2\psi'^2}\geq 0 \;\Big|\cdot \frac{\varphi'^2\psi'^2}{\chi'^2}$$

$$\left(\frac{\chi''}{\chi'}-\frac{\varphi''}{\varphi'}\right)\left(\frac{\chi''}{\chi'}-\frac{\psi''}{\psi'}\right)-\frac{\chi''^2}{\chi'^2}\geq 0,$$

which is fulfilled in the terms (2.3) if and only if

$$\frac{1}{F_1}\frac{1}{F_2}-\frac{1}{F_4}\frac{1}{F_1}-\frac{1}{F_4}\frac{1}{F_2}\geq 0. \tag{2.7}$$

Third condition in additive case after derivatives is

$$\begin{vmatrix} \frac{\chi'}{\varphi'^2}\left(\frac{\chi''}{\chi'}-\frac{\varphi''}{\varphi'}\right) & \frac{\chi''}{\psi'\varphi'} & \frac{\chi''}{\varphi'\rho'} \\ \frac{\chi''}{\psi'\varphi'} & \frac{\chi'}{\psi'^2}\left(\frac{\chi''}{\chi'}-\frac{\psi''}{\psi'}\right) & \frac{\chi''}{\psi'\rho'} \\ \frac{\chi''}{\varphi'\rho'} & \frac{\chi''}{\psi'\rho'} & \frac{\chi'}{\rho'^2}\left(\frac{\chi''}{\chi'}-\frac{\rho''}{\rho'}\right) \end{vmatrix}\leq 0.$$

Elementary transformations and extractions gives

$$\frac{\chi'^3}{\rho'^2\psi'^2\varphi'^2}\cdot\left[\frac{\chi''}{\chi'}\frac{\psi''}{\psi'}\frac{\rho''}{\rho'}+\frac{\chi''}{\chi'}\frac{\varphi''}{\varphi'}\frac{\rho''}{\rho'}+\frac{\chi''}{\chi'}\frac{\psi''}{\psi'}\frac{\varphi''}{\varphi'}-\frac{\varphi''}{\varphi'}\frac{\psi''}{\psi'}\frac{\rho''}{\rho'}\right]\leq 0.$$

Condition in the terms of (2.3) is

$$\chi'^3\left(\frac{1}{F_4}\frac{1}{F_2}\frac{1}{F_3}+\frac{1}{F_4}\frac{1}{F_1}\frac{1}{F_3}+\frac{1}{F_4}\frac{1}{F_2}\frac{1}{F_1}-\frac{1}{F_1}\frac{1}{F_2}\frac{1}{F_3}\right)\leq 0. \tag{2.8}$$

According to the Lemma 1.1, the inequality (2.4) holds if $\chi' > 0$ and H in (2.5) is concave. It is fulfilled when (i) or (ii) appears. If $\chi' < 0$, then H must be convex. It is fulfilled again when (i) or (ii) appears, because all inequalities in (2.6), (2.7), and (2.8) must be greater than zero.

The proof of the opposite is similar, so it is wasted to reader. □

2.2. Multiplicative case

Enlargement on three variables in multiplicative case needs $f(x, y, z) = xyz$ in (1.2) and now $\chi(u) = w$ with $u = xyz$. Basic research are carried out by the function (1.3) in the shape of

$$H(s, t, r) = \chi(\varphi^{-1}(s) \cdot \psi^{-1}(t) \cdot \rho^{-1}(r)). \tag{2.9}$$

Suitable auxiliary functions are

$$D_1(x) = \frac{\varphi'(x)}{\varphi'(x) + x\varphi''(x)};\ D_2(y) = \frac{\psi'(y)}{\psi'(y) + y\psi''(y)};\ D_3(z) = \frac{\rho'(z)}{\rho'(z) + z\rho''(z)};$$

$$\text{and } D_4(u) = \frac{\chi'(u)}{\chi'(u) + u\chi''(u)}. \tag{2.10}$$

Inequality is based on concavity or convexity of the function (2.9).

Theorem 2.2. *Suppose that* x, y, z *are positive and suppose that* $\varphi, \psi, \rho,$ *and* χ *have up to second derivatives such that* D_i *from* (2.10) *are definable. Then the inequality*

$$M_\varphi(x; a) \cdot M_\psi(y; a) \cdot M_\rho(z; a) \ge M_\chi(xyz; a) \tag{2.11}$$

holds if

(i) *all* D_i, $i = 1, 2, 3, 4$ *are positive and* $D_4(xyz) \ge D_1(x) + D_2(y) + D_3(z)$;

(ii) *or if* D_4 *is negative, but all* D_i, $i = 1, 2, 3$ *are positive.*

The inequality is opposite if

(iii) *if all* D_i, $i = 1, 2, 3, 4$ *are negative and* $D_4(xyz) \le D_1(x) + D_2(y) + D_3(z)$;

(iv) *or if* D_4 *is positive, but all* D_i, $i = 1, 2, 3$ *are negative.*

Proof. The first condition from (1.5), accepting the (2.10) is

$$\frac{\partial^2 H}{\partial s^2} = \frac{yz\chi'}{x\varphi'^2}\left(xyz\,\frac{\chi''}{\chi'} - x\,\frac{\varphi''}{\varphi'}\right) = \frac{yz\chi'}{x\varphi'^2}\left(\frac{1}{D_4} - \frac{1}{D_1}\right) \le 0. \tag{2.12}$$

Second condition is positive determinant

$$\begin{vmatrix} \frac{yz\chi'}{x\varphi'^2}\left(\frac{1}{D_4}-\frac{1}{D_1}\right) & \frac{z\chi'}{\varphi'\psi'}\frac{1}{D_4} \\ \frac{z\chi'}{\varphi'\psi'}\frac{1}{D_4} & \frac{xz\chi'}{y\psi'^2}\left(\frac{1}{D_4}-\frac{1}{D_2}\right) \end{vmatrix} \geq 0,$$

equivalent with

$$\frac{1}{D_1}\frac{1}{D_2}-\frac{1}{D_1}\frac{1}{D_4}-\frac{1}{D_2}\frac{1}{D_4} \geq 0. \tag{2.13}$$

Third condition in the terms of (2.10) is

$$\begin{vmatrix} \frac{yz\chi'}{x\varphi'^2}\left(\frac{1}{D_4}-\frac{1}{D_1}\right) & \frac{z\chi'}{\varphi'\psi'}\frac{1}{D_4} & \frac{y\chi'}{\varphi'\rho'}\frac{1}{D_4} \\ \frac{z\chi'}{\varphi'\psi'}\frac{1}{D_4} & \frac{xz\chi'}{y\psi'^2}\left(\frac{1}{D_4}-\frac{1}{D_2}\right) & \frac{x\chi'}{\psi'\rho'}\frac{1}{D_4} \\ \frac{y\chi'}{\varphi'\rho'}\frac{1}{D_4} & \frac{x\chi'}{\psi'\rho'}\frac{1}{D_4} & \frac{xy\chi'}{z\rho'^2}\left(\frac{1}{D_4}-\frac{1}{D_3}\right) \end{vmatrix} \leq 0.$$

Extractions and elementary transformations simplify the determinant obtaining the condition

$$\chi'^3 \cdot \begin{vmatrix} \frac{1}{D_4}-\frac{1}{D_1} & \frac{1}{D_4} & \frac{1}{D_4} \\ \frac{1}{D_1} & -\frac{1}{D_2} & 0 \\ \frac{1}{D_1} & 0 & -\frac{1}{D_3} \end{vmatrix} \leq 0,$$

equivalent with

$$\chi'^3\left(\frac{1}{D_4}\frac{1}{D_2}\frac{1}{D_3}+\frac{1}{D_4}\frac{1}{D_1}\frac{1}{D_3}+\frac{1}{D_4}\frac{1}{D_2}\frac{1}{D_1}-\frac{1}{D_1}\frac{1}{D_2}\frac{1}{D_3}\right) \leq 0. \tag{2.14}$$

Discussions is analogue with discussion obtained in the proof of Theorem 2.1, so it is left to reader. □

3. Applications

Ordinary well-known Minkowski and Hölder inequality from [1] now is going to be generalized and enlarged.

3.1. Generalized Minkowski inequality

Well-known inequality now is generalized and enlarged for the case of three variables and for different potential means. For the inequality

$$\left(\sum_{i=1}^{n} a_i x_i^{\mu}\right)^{\frac{1}{\mu}} + \left(\sum_{i=1}^{n} a_i y_i^{\nu}\right)^{\frac{1}{\nu}} + \left(\sum_{i=1}^{n} a_i z_i^{\eta}\right)^{\frac{1}{\eta}} \geq \left(\sum_{i=1}^{n} a_i (x_i + y_i + z_i)^{\lambda}\right)^{\frac{1}{\lambda}}, \quad (3.1)$$

four auxiliary functions are appearing according to the (2.3)

$$F_1(x) = \frac{x}{\mu - 1},\ F_2(y) = \frac{y}{\nu - 1},\ F_3(z) = \frac{z}{\eta - 1}, \text{ and } F_4(x + y + z) = \frac{x + y + z}{\lambda - 1}.$$

The inequality (3.1) holds if $\lambda < 1$ and $\mu, \nu, \eta > 1$. If all $\mu, \nu, \eta, \lambda > 1$, the (3.1) holds if

$$\frac{x + y + z}{\lambda - 1} \geq \frac{x}{\mu - 1} + \frac{y}{\nu - 1} + \frac{z}{\eta - 1}. \quad (3.2)$$

The inequality (3.2) holds, if one of the following is fulfilled:

- when $\mu, \nu, \eta > \lambda > 1$ for every positive x, y, z;
- when $\nu, \eta > \lambda > \mu > 1$, under the condition by the components

$$0 < x \leq \frac{\mu - 1}{\lambda - \mu}\left(\frac{\nu - \lambda}{\nu - 1} y + \frac{\eta - \lambda}{\eta - 1} z\right);$$

- when $\mu > \lambda > \nu, \eta > 1$, under the condition by the components

$$x \geq \frac{\mu - 1}{\mu - \lambda}\left(\frac{\lambda - \nu}{\nu - 1} y + \frac{\lambda - \eta}{\eta - 1} z\right) > 0.$$

The inequality in (3.1) is opposite if $\lambda > 1$ and $\mu, \nu, \eta < 1$. If all $\mu, \nu, \eta, \lambda < 1$, the opposite inequality in (3.1) holds, if

$$\frac{x+y+z}{\lambda-1} \le \frac{x}{\mu-1} + \frac{y}{\nu-1} + \frac{z}{\eta-1}. \tag{3.3}$$

The inequality (3.3) holds, if one of the following is fulfilled:

- when $\mu, \nu, \eta < \lambda$ for every positive x, y, z;
- when $\nu, \eta < \lambda < \mu < 1$, under the condition by the components

$$0 < x \le \frac{1-\mu}{\mu-\lambda}\left(\frac{\lambda-\nu}{1-\nu} y + \frac{\lambda-\eta}{1-\eta} z\right);$$

- when $\mu < \lambda < \nu, \eta < 1$, under the condition by the components

$$x \ge \frac{1-\mu}{\lambda-\mu}\left(\frac{\nu-\lambda}{1-\nu} y + \frac{\eta-\lambda}{1-\eta} z\right) > 0.$$

3.2. Generalized Hölder inequality

An enlargement of generalization given in [9] is presented as the inequality

$$\left(\sum_{i=1}^{n} a_i x_i^{\mu}\right)^{\frac{1}{\mu}} \cdot \left(\sum_{i=1}^{n} a_i y_i^{\nu}\right)^{\frac{1}{\nu}} \cdot \left(\sum_{i=1}^{n} a_i z_i^{\eta}\right)^{\frac{1}{\eta}} \ge \left(\sum_{i=1}^{n} a_i (x_i y_i z_i)\right)^{\frac{1}{\lambda}}. \tag{3.4}$$

The suitable auxiliary functions are now constants with given exponets as their values $D_1(x) = \frac{1}{\mu}$, $D_2(y) = \frac{1}{\nu}$, $D_3(z) = \frac{1}{\eta}$, and $D_4(xyz) = \frac{1}{\lambda}$.

The inequality (3.4) holds if $\lambda < 0$ and $\mu, \nu, \eta > 0$. If $\mu, \nu, \eta, \lambda > 0$, then the (3.4) holds if

$$\frac{1}{\lambda} \ge \frac{1}{\mu} + \frac{1}{\nu} + \frac{1}{\eta}.$$

The inequality in (3.4) is opposite when $\lambda > 0$ and $\mu, \nu, \eta < 0$. If $\mu, \nu, \eta, \lambda < 0$ and

$$\frac{1}{\lambda} \le \frac{1}{\mu} + \frac{1}{\nu} + \frac{1}{\eta},$$

the inequality in (3.4) is opposite too.

4. Comparison the Generalized Quasi-Arithmetic Means

For a non-empty set Ω, the set of real-valued functions, $\mathcal{L} = \{x, x : \Omega \to \mathbb{R}\}$, constitutes a linear space. A linear functional $A : \mathcal{L} \to \mathbb{R}$ is entitled as positive if $A(x)$ is positive when $x \in \mathcal{L}$ is getting positive values. A positive linear functional is entitled as linear mean if $A(e) = 1$ whenever $e : \Omega \to \{1\}$.

For a real-valued function φ and a positive (weight) function $a \in \mathcal{L}$, the generalized quasi-arithmetic mean

$$M_{\varphi}(x, a; A) := \varphi^{-1}\left(\frac{A(a\varphi(x))}{A(a)}\right), \tag{4.1}$$

is well defined if $\varphi(x) \in \mathcal{L}$ and if φ is continuous and strictly monotone on the real-number interval J_x with $x(\Omega) \subseteq J_x$. Obligatory $A(a) \neq 0$.

Note that definition (1.1) is a special case of (4.1) for $\Omega = \{1, 2, \dots, n\}$,

$x(\omega) = x_{\omega} \in J_x$, $J_x \subseteq \mathbb{R}$, $\varphi : J_x \to \mathbb{R}$, $a(\omega) = a_{\omega} \geq 0$, with $\sum_{\omega=1}^{n} a_{\omega} = 1$ and $A(a) = \sum_{\omega \in \Omega} a(\omega)$.

Lemma 4.1. *Suppose that* φ, ψ, ρ *and* χ *are continuous, strictly monotone real functions defined on the real intervals* J_x, J_y, J_z, *and* J_u. *From* (1.2), *the inequality*

$$\chi(f(M_{\varphi}(x; a), M_{\psi}(y; a), M_{\rho}(z; a))) - \chi(M_{\chi}(f(x, y, z); a)) \geq 0, \tag{4.2}$$

holds if the function H defined by (1.3) *is concave and* χ *increases or if H is convex and* χ *decreases. The values are well defined for* $y, z \in \mathcal{L}$ *if* $y(\Omega) \subseteq J_y$, $z(\Omega) \subseteq J_z$ *and if* $f : J_x \times J_y \times J_z \to J_u$. *The inequality is opposite if H is convex and* χ *increases or if the H is concave and H decreases.*

Proof. Analogue to the proof of Lemma 1.1. □

On the left side in (4.2) is a difference that can be observed in dependence of non-negative function $a \in \mathcal{L}$. In [4], the authors considered relationship between two differences that arise by non-negative (weight) functions $a, b \in \mathcal{L}$. The next proposition is a corollary of the general Theorem 3.2 in [4]. The proof is given in spite of easier understanding.

Proposition 4.1. *Take the values and functions from Lemma* 4.1 *and assume that A is a linear, positive functional on* $\mathcal{L}$. *Suppose that for the two given non-negative functions* $a, b \in \mathcal{L}$ *there exist constants* m, M *such that* $Mb(\omega) - a(\omega) \geq 0$ *and* $a(\omega) - mb(\omega) \geq 0$ *hold for every* $w \in \Omega$. *If the function H defined by* (1.3) *is concave, then*

$$MA(b) \cdot [\chi(f(M_{\varphi}(x, b; A), M_{\psi}(y, b; A), M_{\rho}(z, b; A))) - \chi(M_{\chi}(f(x, y, z), b; A))]$$

$$\geq A(a) \cdot [\chi(f(M_{\varphi}(x, a; A), M_{\psi}(y, a; A), M_{\rho}(z, a; A))) - \chi(M_{\chi}(f(x, y, z), a; A))]$$

$$\geq mA(b) \cdot [\chi(f(M_{\varphi}(x, b; A), M_{\psi}(y, b; A), M_{\rho}(z, b; A))) - \chi(M_{\chi}(f(x, y, z), b; A))]. \tag{4.3}$$

The inequalities are reversed whether if the function H is convex.

Proof. For any non-negative $a \in \mathcal{L}$, a positive linear functional $x \rightarrowtail \frac{A(ax)}{A(a)}$ is a linear mean on $\mathcal{L}$ if $A(a) \neq 0$. Jessen's and McShane's generalizations of the Jensen's inequality (see [1, p. 48-49]), for a concave function H, gives $A(bH(s, t, r)) \leq A(b)H\left(\frac{A(bs)}{A(b)}, \frac{A(bt)}{A(b)}, \frac{A(br)}{A(b)}\right)$. By shortcut $T := (s, t, r)$ and $A(bT) = (A(bs), A(bt), A(br))$, we use concavity of H. Note that only m and M are constants here and assume all denominators differ from zero.

$$MA(bH(T)) - A(aH(T)) + A(a)H\left(\frac{A(aT)}{A(a)}\right)$$

$$\leq A(Mb-a)H\left(\frac{A(MbT-aT)}{A(Mb-a)}\right) + A(a)H\left(\frac{A(aT)}{A(a)}\right) \tag{4.4}$$

$$\leq MA(b)H\left(\frac{MA(bT)-A(aT)}{MA(b)} + \frac{A(aT)}{MA(b)}\right) = MA(b)H\left(\frac{A(bT)}{A(b)}\right). \tag{4.5}$$

Further steps are based on definitions (1.3) and (4.1), because

$$A(aH(T)) - A(a)H\left(\frac{A(aT)}{A(a)}\right)$$

$$= A(a)\left[\frac{A(a\chi f(x,\ y,\ z))}{A(a)} - \chi f\left(\varphi^{-1}\left(\frac{A(as)}{A(a)}\right),\ \psi^{-1}\left(\frac{A(at)}{A(a)}\right),\ \rho^{-1}\left(\frac{A(ar)}{A(a)}\right)\right)\right]$$

$$= A(a)\left[\chi\chi^{-1}\left(\frac{A(a\chi f(x,y,z))}{A(a)}\right) - \chi f\left(\varphi^{-1}\left(\frac{A(a\varphi(x))}{A(a)}\right), \psi^{-1}\left(\frac{A(a\psi(y))}{A(a)}\right), \rho^{-1}\left(\frac{A(a\rho(x))}{A(a)}\right)\right)\right].$$

Extend $A(bH(T)) - A(b)H\left(\frac{A(bT)}{A(b)}\right)$ in similarly manner, multiply (4.4) and (4.5) by (-1) and (4.3) will be obtained. □

In the next two corollaries, the extensions of the multiplicative type inequality and the additive type inequality investigating in [9] and [4] are given.

Corollary 4.1. *Substitute* $f(x,\ y,\ z) = x + y + z$ *in Proposition* 4.1 *and assume that for* $\varphi, \psi, \rho,$ *and* $\chi,$ *we can define* $F_i,\ i = 1, 2, 3, 4$ *by* (2.3). *Under presumptions of Proposition* 4.1, *the inequalities*

$$MA(b)\cdot[\chi(M_\varphi(x,\ b;\ A) + M_\psi(y,\ b;\ A) + M_\rho(z,\ b;\ A)) - \chi(M_\chi(x+y+z,\ b;\ A))]$$

$$\geq A(a)\cdot[\chi(M_\varphi(x,\ a;\ A) + M_\psi(y,\ a;\ A) + M_\rho(z,\ a;\ A)) - \chi(M_\chi(x+y+z,\ a;\ A))]$$

$$\geq mA(b)\cdot[\chi(M_\psi(x, b;\ A) + M_\varphi(y,\ b;\ A) + M_\rho(z, b;\ A)) - \chi(M_\chi(x+y+z,\ b;\ A))]$$

hold if $\chi',\ F_1,\ F_2,\ F_3$ *are positive and* χ'' *is negative or if all of* $\chi',\ \chi'',$ $F_1,\ F_2,\ F_3$ *are positive and* $F_1(x) + F_2(y) + F_3(z) \leq F_4(y+y+z)$. *If* χ'

is negative, then inequalities (4.6) *hold when* χ'', F_1, F_2, F_3 *are negative or when* χ'' *is positive,* F_1, F_2, F_3 *are negative and* $F_1(x)+F_2(y)+F_3(z) \geq F_4(y+y+z)$.

The inequalities (4.6) *are opposite if* χ', χ'' *are positive and* F_1, F_2, F_3 *are negative or if* χ' *is positive, all of* χ'', F_1, F_2, F_3 *are negative and* $F_1(x)+F_2(y)+F_3(z) \geq F_4(x+y+z)$. *If* χ' *is negative, then inequalities in* (4.6) *are opposite when* χ'', F_1, F_2, F_3 *are positive or when* χ'' *is negative,* F_1, F_2, F_3 *are positive and* $F_1(x)+F_2(y)+F_3(z) \leq F_4(x+y+z)$.

Proof. For inequalities (4.6), check the conditions (2.6), (2.7), and (2.8). For the opposite inequalities, take all conditions to be greater than zero. □

Corollary 4.2. *Substitute* $f(x, y, z) = x \cdot y \cdot z$ *in Proposition* 4.1 *and assume that functions* x, y, z *are positive. Presume that for* φ, υ', ρ *and* χ *the functions* D_i, $i = 1, 2, 3, 4$ *are definable by* (2.10). *Under the conditions of Proposition* 4.1, *the inequalities*

$$MA(b) \cdot [\chi(M_\varphi(x, b; A) \cdot M_\psi(y, b; A) \cdot M_\rho(z, b; A)) - \chi(M_\chi(x \cdot y \cdot z, b; A))]$$

$$\geq A(a) \cdot [\chi(M_\varphi(x, a; A) \cdot M_\psi(y, a; A) \cdot M_\rho(z, a; A)) - \chi(M_\chi(x \cdot y \cdot z, a; A))]$$

$$\geq mA(b) \cdot [\chi(M_\varphi(x, b; A) \cdot M_\psi(y, b; A) \cdot M_\rho(z, b; A)) - \chi(M_\chi(x \cdot y \cdot z, b; A))]$$

hold if χ', D_1, D_2, D_3 *are positive and* D_4 *is negative or if all of* χ', D_1, D_2, D_3, D_4 *are positive and* $D_1(x)+D_2(y)+D_3(z) \leq D_4(xyz)$. *If* χ' *is negative, then inequalities* (4.6) *hold when* D_1, D_2, D_3 *are negative and* D_4 *is positive or when all* D_1, D_2, D_3, D_4 *are negative and* $D_1(x) + D_2(y)+D_3(z) \geq D_4(xyz)$.

The inequalities (4.6) *are opposite if* χ', D_4 *are positive and* D_1, D_2, D_3 *are negative or if* χ' *is positive, all of* D_1, D_2, D_3, D_4 *are negative and* $D_1(x)+D_2(y)+D_3(z) \geq D_4(xyz)$. *If* χ' *is negative, then* *inequalities in* (4.6) *are opposite when* D_1, D_2, D_3 *are positive and* D_4 *is negative or when all* D_1, D_2, D_3, D_4 *are positive and* $D_1(x) + D_2(y) + D_3(z) \leq D_4(xyz)$.

Proof. Analogue to the proof of Corollary 4.1, but the conditions (2.12), (2.13), and (2.14) are examined now. □

5. Examples and Discussions

The results of Corollaries 4.1 and 4.2 are applied on concrete quasi-arithmetic means. In the next example, the potential means $M^{[\mu]}(x, a; A) = \left(\frac{A(ax^{\mu})}{A(a)}\right)^{\frac{1}{\mu}}$ are considered in multiplicative case.

Example 5.1. Assume that a, b, M, m, A are as in Proposition 4.1. For positive real functions $x, y, z \in \mathcal{L}$, the inequalities

$$M\left\{A(b) \cdot \left[\left(\frac{A(bx^{\mu})}{A(b)}\right)^{\frac{1}{\mu}} \cdot \left(\frac{A(by^{\nu})}{A(b)}\right)^{\frac{1}{\nu}} \cdot \left(\frac{A(bz^{\eta})}{A(b)}\right)^{\frac{1}{\eta}}\right]^{\lambda} - A(b \cdot (xyz)^{\lambda})\right\}$$

$$\geq A(a) \cdot \left[\left(\frac{A(ax^{\mu})}{A(a)}\right)^{\frac{1}{\mu}} \cdot \left(\frac{A(ay^{\nu})}{A(a)}\right)^{\frac{1}{\nu}} \cdot \left(\frac{A(az^{\eta})}{A(a)}\right)^{\frac{1}{\eta}}\right]^{\lambda} - A(a \cdot (xyz)^{\lambda}) \qquad (5.1)$$

$$\geq m\left\{A(b) \cdot \left[\left(\frac{A(bx^{\mu})}{A(b)}\right)^{\frac{1}{\mu}} \cdot \left(\frac{A(by^{\nu})}{A(b)}\right)^{\frac{1}{\nu}} \cdot \left(\frac{A(bz^{\eta})}{A(b)}\right)^{\frac{1}{\eta}}\right]^{\lambda} - A(b \cdot (xyz)^{\lambda})\right\} \geq 0,$$

hold in the following cases:

(i) If $\mu, \nu, \eta, \lambda > 0$ and $\frac{1}{\lambda} \geq \frac{1}{\mu} + \frac{1}{\nu} + \frac{1}{\eta}$.

(ii) When $\mu, \nu, \eta, \lambda < 0$ and $\frac{1}{\lambda} \leq \frac{1}{\mu} + \frac{1}{\nu} + \frac{1}{\eta}$.

The inequalities (5.1) are opposite in the following cases:

(iii) If $\lambda < 0 < \mu, \nu, \eta$.

(iv) When $\mu, \nu, \eta < 0 < \lambda$.

Proof. The conditions are following directly from Corollary 4.2, after considering that for the function $H(s, t, r) = \left(s^{\frac{1}{\mu}} t^{\frac{1}{\nu}} r^{\frac{1}{\eta}}\right)^{\lambda}$, we have $D_1 = \frac{1}{\mu}$, $D_2 = \frac{1}{\nu}$, $D_3 = \frac{1}{\eta}$, $D_4 = \frac{1}{\lambda}$, and $\chi'(xyz) = \lambda(xyz)^{\lambda-1}$. Checking the conditions (2.12), (2.13), and (2.14), we obtain that H is concave in (i), (ii) or (iii). Changing conditions (2.12) and (2.14) into greater than zero, we obtain that H is convex when (iii) or (iv) appear. □

Similar example is constructed for additive case in the next example.

Example 5.2. Assume that a, b, M, m, A are as in Proposition 4.1. For positive real functions $x, y, z \in \mathcal{L}$, the inequalities

$$M\left\{A(b)\cdot\left[\left(\frac{A(bx^{\mu})}{A(b)}\right)^{\frac{1}{\mu}} + \left(\frac{A(by^{\nu})}{A(b)}\right)^{\frac{1}{\nu}} + \left(\frac{A(bz^{\eta})}{A(b)}\right)^{\frac{1}{\eta}}\right]^{\lambda} - A(b\cdot(x+y+z)^{\lambda})\right\}$$

$$\geq A(a)\cdot\left[\left(\frac{A(ax^{\mu})}{A(a)}\right)^{\frac{1}{\mu}} + \left(\frac{A(ay^{\nu})}{A(a)}\right)^{\frac{1}{\nu}} + \left(\frac{A(az^{\eta})}{A(a)}\right)^{\frac{1}{\eta}}\right]^{\lambda} - A(a\cdot(x+y+z)^{\lambda})$$

$$\geq m\left\{A(b)\cdot\left[\left(\frac{A(bx^{\mu})}{A(b)}\right)^{\frac{1}{\mu}} + \left(\frac{A(by^{\nu})}{A(b)}\right)^{\frac{1}{\nu}} + \left(\frac{A(bz^{\eta})}{A(b)}\right)^{\frac{1}{\eta}}\right]^{\lambda} - A(b\cdot(x+y+z)^{\lambda})\right\} \geq 0, \tag{5.2}$$

hold in the following cases:

(i) If $\mu, \nu, \eta > 1 > \lambda > 0$.

(ii) When $\mu, \nu, \eta, \lambda > 1$ and $\frac{x+y+z}{\lambda-1} \geq \frac{x}{\mu-1} + \frac{y}{\nu-1} + \frac{z}{\eta-1}$, which is valid when (3.2) is valid.

(iii) When $\mu, \nu, \eta < 1, \lambda < 0$ and $\frac{x+y+z}{\lambda-1} \leq \frac{x}{\mu-1} + \frac{y}{\nu-1} + \frac{z}{\eta-1}$, which is valid when (3.3) is valid.

The inequalities (5.2) are opposite in the following cases:

(iv) If $\lambda > 1 > \mu, \nu, \eta$.

(v) When $\lambda < 0$ and $1 < \mu, \nu, \eta$.

(vi) When $\mu, \nu, \eta < 1, 0 < \lambda < 1$ and $\frac{x+y+z}{\lambda-1} \leq \frac{x}{\mu-1} + \frac{y}{\nu-1} + \frac{z}{\eta-1}$, which is valid when (3.3) is valid.

Proof. The conditions are following directly from Corollary 4.2, after considering that for the function $H(s.\, t,\, r) = \left(s^{\frac{1}{\mu}} + t^{\frac{1}{\nu}} + r^{\frac{1}{\eta}}\right)^{\lambda}$, we have $F_1 = \frac{x}{\mu-1}$, $F_2 = \frac{y}{\nu-1}$, $F_3 = \frac{z}{\eta-1}$, $F_4 = \frac{x+y+z}{\lambda-1}$, and $\chi'(x+y+z) = \lambda(x+y+z)^{\lambda-1}$. Checking the conditions (2.6), (2.7), and (2.8), we obtain that H is concave in (i), (ii) or (iii). Changing conditions (2.6) and (2.8) into greater than zero, we obtain that H is convex when (iv), (v) or (vi) appears. □

Example 5.3. Let a, b, A, M, m be defined as in Corollary 4.1. Let assume that $x, y, z \in L$ such that the next inequalities

$$M\left\{A(b) \cdot \cos\left[Arc\cos\frac{A(b \cdot \cos x)}{A(b)} + Arc\cos\frac{A(b \cdot \cos y)}{A(b)} + Arc\cos\frac{A(b \cdot \cos z)}{A(b)}\right] - A(b\cos(x+y+z))\right\}$$

$$\leq A(a)\cdot\cos\left[Arc\cos\frac{A(a\cdot\cos x)}{A(a)}+Arc\cos\frac{A(a\cdot\cos y)}{A(a)}+Arc\cos\frac{A(a\cdot\cos z)}{A(a)}\right]$$

$$-A(a\cos(x+y+z))$$

$$\leq m\left\{A(b)\cdot\cos\left[Arc\cos\frac{A(b\cdot\cos x)}{A(b)}+Arc\cos\frac{A(b\cdot\cos y)}{A(b)}\right.\right.$$

$$\left.\left.+Arc\cos\frac{A(b\cdot\cos z)}{A(b)}\right]-A(b\cos(x+y+z))\right\}$$

holds if for all $\omega\in\Omega$, $0<x(\omega),\ y(\omega),\ z(\omega)<\frac{\pi}{6}$. If $0>x(\omega),\ y(\omega),\ z(\omega)>-\frac{\pi}{6}$ for all $\omega\in\Omega$, then the inequalities are reversed.

Proof. For the functions $\chi(x)=\varphi(x)=\psi(x)=\rho(x)=\cos x$, we can apply Corollary 4.1 in the case that $\chi'\chi''$ is negative, F_1, F_2, F_3 are positive and $F_1(x)+F_2(y)+F_3(z)<F_4(x+y+z)$. Namely, $\chi'(x+y+z)=-\sin(x+y+z)<0$ and $\chi''(x+y+z)=-\cos(x+y+z)<0$, because $0<x+y+z<\frac{\pi}{2}$. Furthermore, $F_1(x)=F_2(x)=F_3(x)=\tan x>0$ and $\tan x+\tan y+\tan z\leq\tan(x+y+z)$ for $0<x,\ y,\ z<\frac{\pi}{6}$. So the inequalities are true. The proof of the opposite is analogue and it is left to a reader. □

In the next remarks, we notice that well-known inequalities are the border cases of the examples given above. The terms a, b, A, M, m are defined as in Corollary 4.1.

Remark 5.1. Assume that $f(x, y, z)=x+y+z$ and $\chi(x)=\varphi(x)=\psi(x)=\rho(x)=x^{\mu}$ in (1.2). Then $H(s.\ t,\ r)=(s^{\frac{1}{\mu}}+t^{\frac{1}{\mu}}+r^{\frac{1}{\mu}})^{\mu}$ in (1.3).

If $0<\mu<1$, then Example 5.2 implies the Minkowski type inequalities for positive functions $x, y, z\in L$:

$$MA(b)\cdot[\{M^{[\mu]}(x+y+z,\ b;A)\}^{\mu}-(M^{[\mu]}(x,\ b;\ A)+M^{[\mu]}(y,\ b;A)+M^{[\mu]}(z,\ b;A)^{\mu}]$$

$$\geq A(a)\cdot[\{M^{[\mu]}(x+y+z,a;A)\}^{\mu}-(M^{[\mu]}(x,\ a;\ A)+M^{[\mu]}(y,a;A)+M^{[\mu]}(z,\ a;A)^{\mu}]$$

$$\geq mA(b)\cdot[\{M^{[\mu]}(x+y+z,b;A)\}^{\mu}-(M^{[\mu]}(x,\ b;\ A)+M^{[\mu]}(y,b;A)+M^{[\mu]}(z,\ b;A)^{\mu}].$$

If $\mu > 1$ or if $\mu < 0$, then the inequalities are reversed.

Remark 5.2. Assume that $f(x,\ y,\ z) = xyz$, $\chi(x) = x$ and $\varphi(x) = x^{\mu}$, $\psi(y) = y^{\upsilon}$, $\rho(z) = z^{\eta}$. Then $H(s,t,r) = (s^{\frac{1}{\mu}}\cdot t^{\frac{1}{\upsilon}}\cdot r^{\frac{1}{\eta}})$ in (1.3). If $\mu, \nu, \eta > 0$ and $\frac{1}{\mu}+\frac{1}{\upsilon}+\frac{1}{\eta} = 1$, then Example 5.1 implies Hölder type of inequalities for positive functions $x,\ y,\ z \in L$:

$$MA(b)\cdot[M^{[1]}(xyz,\ b;\ A)-M^{[\mu]}(x,\ b;\ A)\cdot M^{[\nu]}(y,\ b;\ A)\cdot M^{[\nu]}(z,\ b;\ A)]$$

$$\leq A(a)\cdot[M^{[1]}(xyz,\ a;\ A)-M^{[\mu]}(x,\ a;\ A)\cdot M^{[\nu]}(y,\ a;\ A)\cdot M^{[\nu]}(z,\ a;\ A)]$$

$$\leq mA(b)\cdot[M^{[1]}(xyz,\ b;\ A)-M^{[\mu]}(x,\ b;\ A)\cdot M^{[\nu]}(y,\ b;\ A)\cdot M^{[\nu]}(z,\ b;\ A)].$$

If $\mu,\ \upsilon,\ \eta < 0$, then the inequalities are reversed.

Theorems 2.1, 2.2, Proposition 4.1, and Example 5.1 are the refinements of Theorems 4.1 and 4.31 and Corollaries 4.33 and 4.34, respectively, from [1].

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