Time and graphical properties of processes indexed by time-like graphs - Tvrtko Tadić, UW, Seattle

Introduction

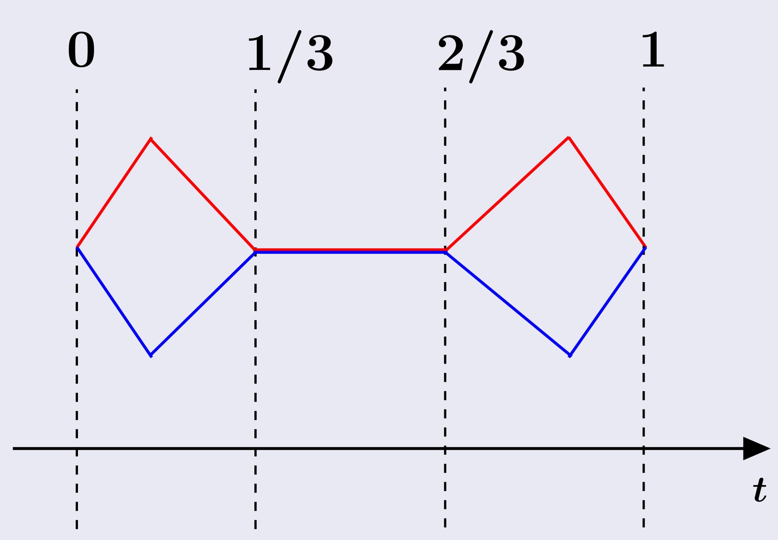
- stochastic process $(X(t) : t \in T)$ — a collection of random variables indexed by set T

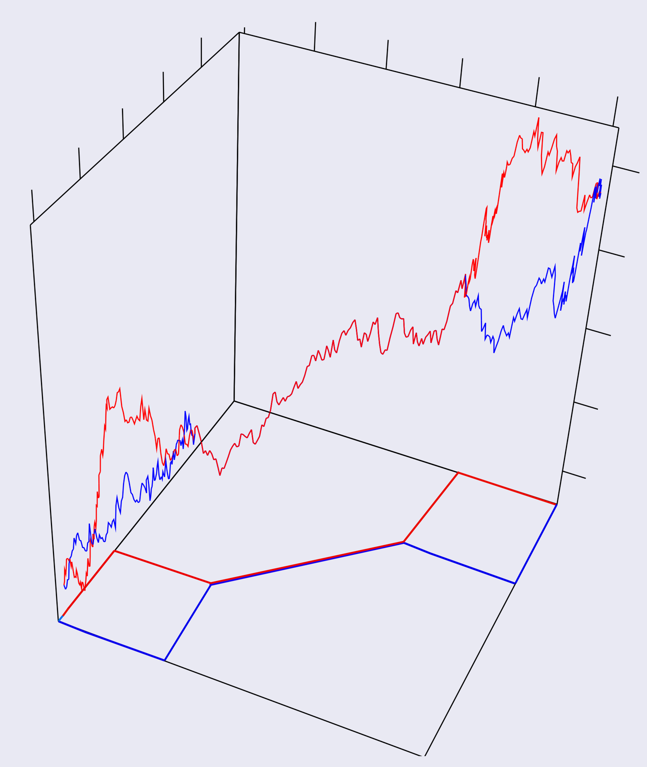
What can T be?

- discrete or continuous time ($T \subset \mathbb{R}$)
- vertices of a graph (graphical models)
- continuous graph-type structure?

What are we talking about?

- stochastic processes indexed by a specific graph with a time structure
- processes on a representation of a graph (the process defined in each point of the representation)



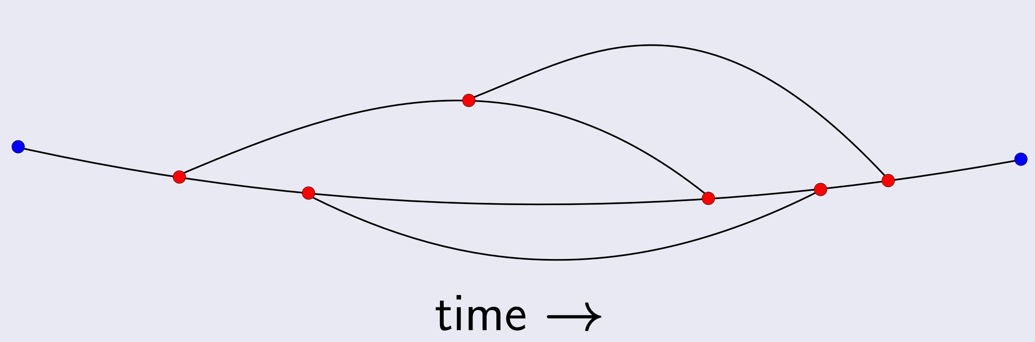


- two processes are together at 0, 1 and in the time interval $[1/3, 2/3]$

Original model

Time-like graphs and processes on them introduced in

[1] Burdzy, K., Pal S., *Markov processes on time-like graphs*, Ann. Probab. **39** (2011)



Key graph features:

- beginning and end of degree 1 (●)
- other vertices of degree 3 (●)
- defined time-like graphs with infinite number of vertices

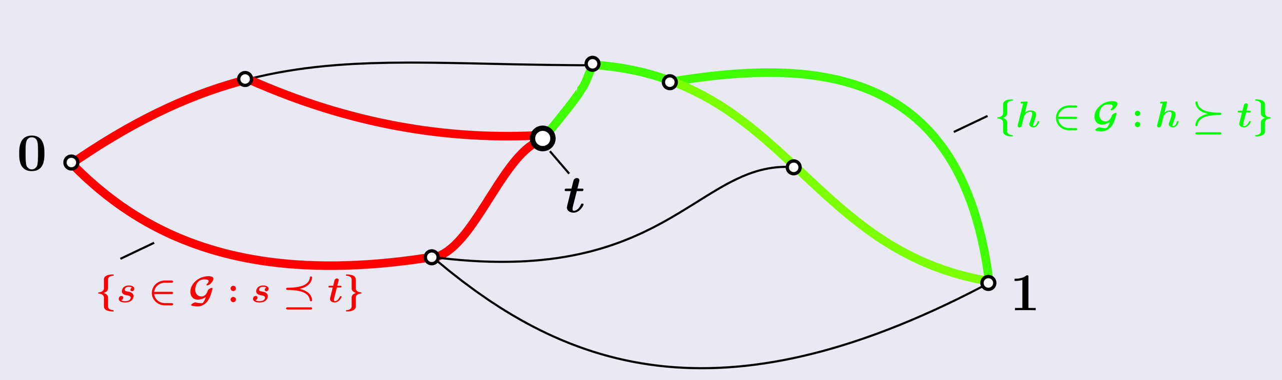
Features of processes:

- defined on a NCC subfamily
- distributions of processes along different time-paths the same
- (some) time and graph induced properties

Time-Markovian property (Burdzy-Pal '11, T' 12)

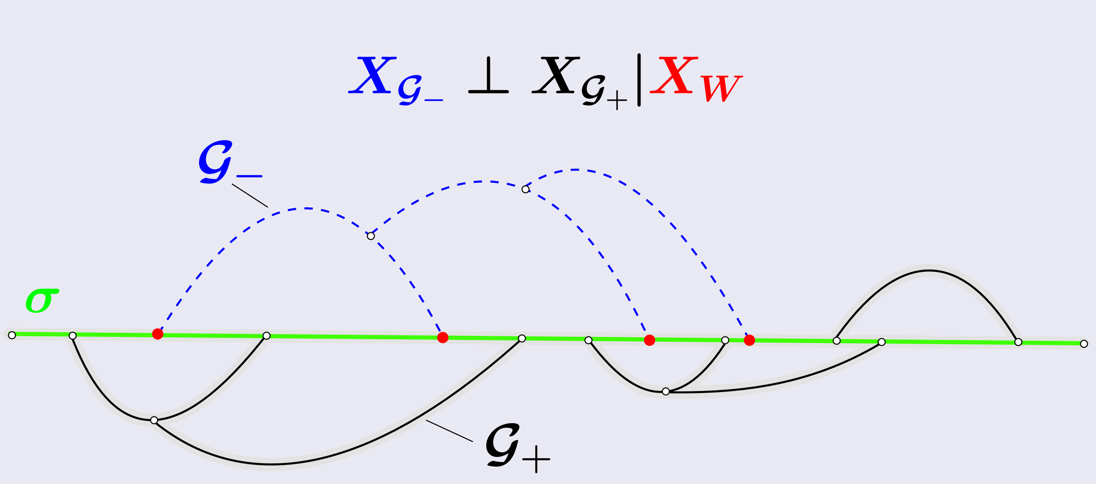
If distributions along time-paths are Markov, then:

$$(X(s) : s \preceq t) \perp (X(h) : h \succeq t) \,|\, X(t)$$



Spine-Markovian property (T' 12)

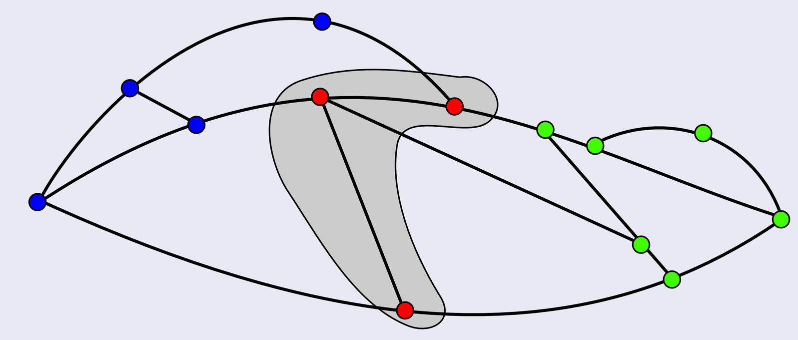
$$X_{\mathcal{G}_-} \perp X_{\mathcal{G}_+} | X_W$$



- path σ from beginning to end
- $\mathcal{G}$ after removing σ decomposes into several components
- $\mathcal{G}_-$ component connected to σ through roots W (●)
- $\mathcal{G}_+$ the rest of the graph (not $\mathcal{G}_-$)

Graphical models

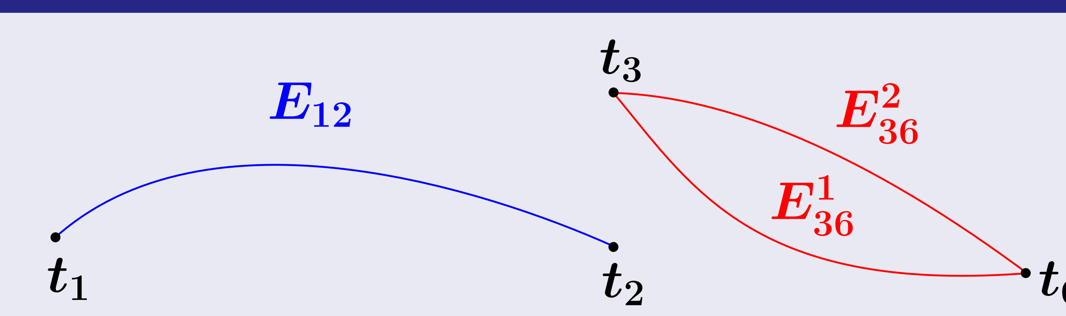
- graphical model $(X_v : v \in V)$
- $G = (V, E)$ a (un)directed graph
- conditional independencies encoded in the structure of the graph G



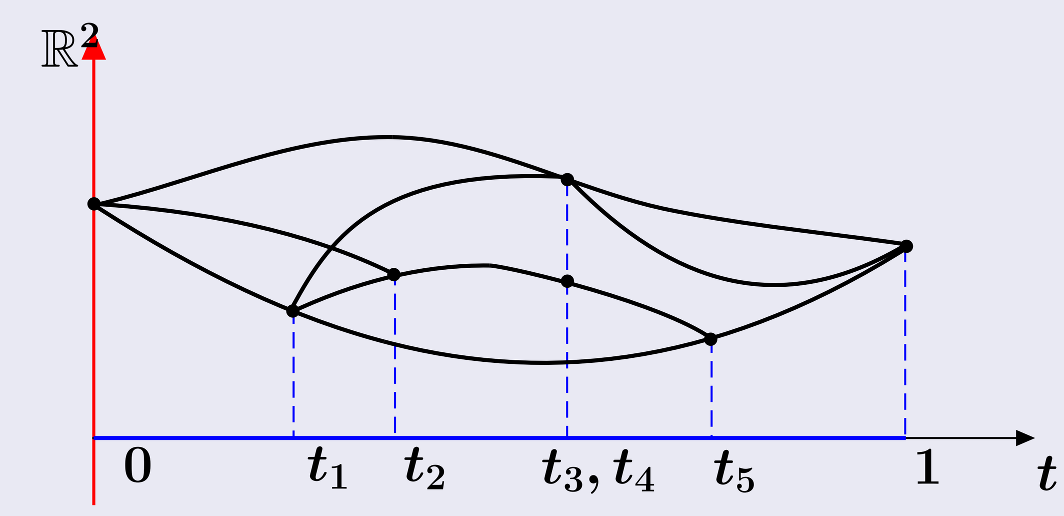
$$X_A \perp X_C | X_B$$

Time-like graphs (TLG's)

- each vertex k has an attribute — time t_k;
- E_{jk} is an edge between t_j and t_k where $t_j < t_k$;



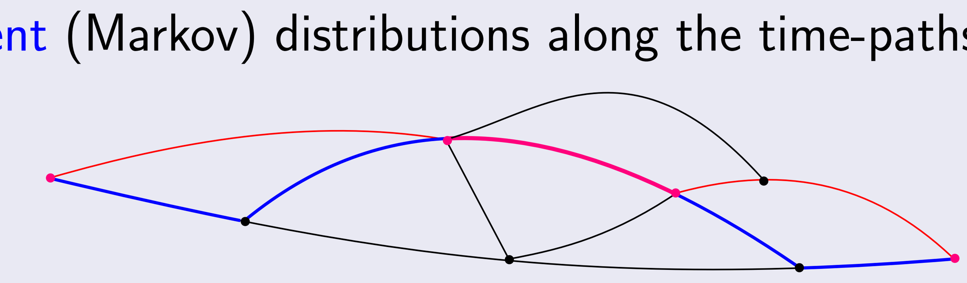
TLG's are represented in $\mathbb{R}^3$ where one dimension is time (t).



Construction of the process (T' 12)

To construct a process X on $\mathcal{G}$ we need:

- $\mathcal{G}$ to have a special structure
- a family of consistent (Markov) distributions along the time-paths;



If these conditions hold …

1. processes behave as expected;
2. there are some Markov-type properties induced by the structure of $\mathcal{G}$

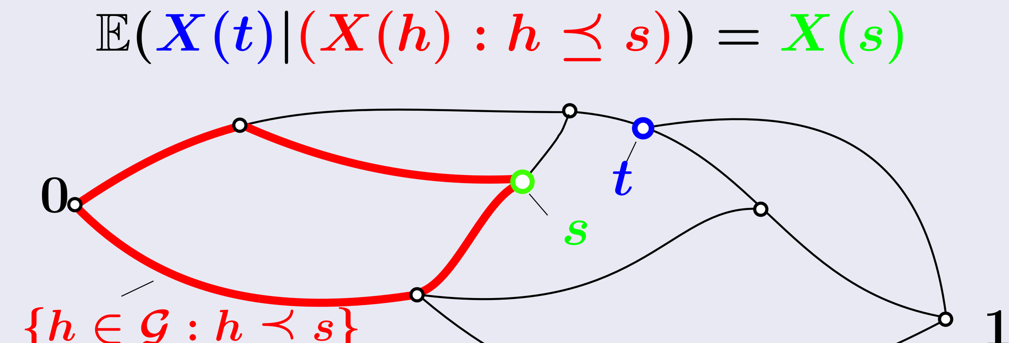
Theorem (T '12)

These two properties guarantee that the distribution of $(X(t) : t \in \mathcal{G})$ is independent of the construction.

Martingales

For $s \preceq t$

$$\mathbb{E}(X(t) | (X(h) : h \preceq s)) = X(s)$$



Theorem (T' 13)

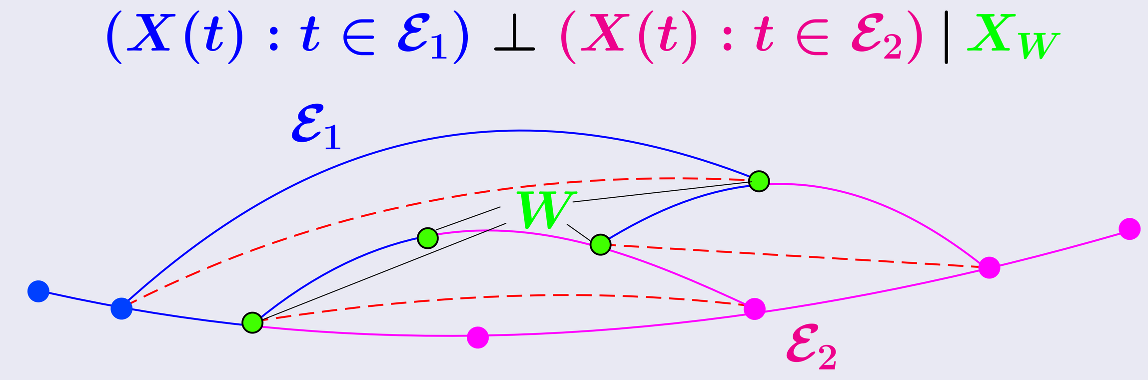
Let $\mathcal{G}$ be a TLG*. Let $X(t)$ be a RCLL martingale with respect to the right continuous filtration $(\mathcal{F}_t)_{t \in \mathcal{G}}$. For stopping times $T_1 \preceq T_2$, if $\mathbb{E}(|X(T_2)|) < \infty$ then

$$\mathbb{E}(X(T_2) | \mathcal{F}_{T_1}) = X(T_1).$$

Moralized graph-Markovian property (T '14)

Adding new edges to the graph we can read the conditional independencies:

$$(X(t) : t \in \mathcal{E}_1) \perp (X(t) : t \in \mathcal{E}_2) \,|\, X_W$$



- version of the global Markov property
- projection to undirected graphical models