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A geometrically exact spatial beam finite element based on the fixed-pole approach

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Abstract. *A spatial geometrically exact finite element based on the fixed-pole approach of an arbitrary order is presented. In contrast to the original formulation by Bottasso and Borri, this formulation uses standard kinematic unknowns and is therefore combinable with standard finite elements. Preliminary results show that when updating displacements and rotations at integration points, this formulation may yield more accurate results.*

Keywords: 3D beams; non-linear analysis; configuration-dependent interpolation, fixed-pole approach.

1 INTRODUCTION

For a load-free beam of length L in motion the power balance reads

$$\int_0^L \left[(\mathbf{v}' + \widehat{\mathbf{r}'}\mathbf{w}) \cdot \mathbf{n} + \mathbf{w}' \cdot \mathbf{m}\right] dx + \int_0^L \left(\mathbf{v} \cdot \dot{\mathbf{k}} + \mathbf{w} \cdot \dot{\boldsymbol{\pi}}\right) dx = 0\,, \tag{1}$$

where $\mathbf{n}$ and $\mathbf{m}$ are vectors of spatial stress and stress-couple resultants, $\mathbf{k} = A\rho\mathbf{v}$ and $\boldsymbol{\pi} = \mathbf{R}\mathbf{J}_\rho\mathbf{R}^{\mathrm{t}}\mathbf{w}$ are the vectors of specific momentum and angular momentum with respect to the beam reference axis at a cross-section, $\mathbf{r}$ and $\mathbf{R}$ are the position vector of the reference line and the orientation tensor of the principal axes of the cross-section with respect to their position in the reference state, a dot and a dash indicate differentiation with respect to time t and the beam-length parameter x, a superimposed hat indicates a cross-product operator, $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{w}$ for which $\widehat{\mathbf{w}} = \mathbf{R}^t\dot{\mathbf{R}}$ are the velocity and the angular velocity vectors, A and ρ are the cross-sectional area and density of the material, and $\mathbf{J}_\rho$ the tensor of cross-sectional moments of inertia.

Originally [1], only the velocity fields in the power-balance equation have been interpolated using Lagrangian polynomials $I^i(x)$ via $\mathbf{v}(x) = \sum_{i=1}^N I^i(x)\mathbf{v}_i$ and $\mathbf{w}(x) = \sum_{i=1}^N I^i(x)\mathbf{w}_i$. For arbitrary nodal velocities, this has resulted in the nodal balance $\mathbf{g}^i \equiv \mathbf{q}_i^i + \mathbf{q}_m^i = 0$ at any node $i = 1, \ldots, N$ with the nodal internal and inertial force vectors $\mathbf{q}_i^i$ and $\mathbf{q}_m^i$ as

$$\mathbf{q}_i^i = \int_0^L \begin{bmatrix} I^{i\prime}\mathbf{I} & \mathbf{0} \\ -I^i\widehat{\mathbf{r}'} & I^{i\prime}\mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{n} \\ \mathbf{m} \end{Bmatrix} dx \qquad \text{and} \qquad \mathbf{q}_m^i = \int_0^L I^i \begin{Bmatrix} \dot{\mathbf{k}} \\ \dot{\boldsymbol{\pi}} \end{Bmatrix} dx\,, \tag{2}$$

The system of non-linear equations $\mathbf{g}^i \equiv \mathbf{q}_i^i + \mathbf{q}_m^i = 0$ (for $i = 1, \ldots, N$) may now be solved for the kinematic unknowns $\mathbf{r}(x)$ and $\mathbf{R}(x)$ using the Newton–Raphson solution procedure in which the linear part of the changes in these unknowns may be interpolated in a manner which is kinematically consistent with the applied interpolation for $\mathbf{v}(x)$ and $\mathbf{w}(x)$. This approach turns out to be incompatible with algorithmic preservation of strain invariance with respect to a rigid-body motion, or simultaneous conservation of energy and the momentum vectors during a free motion of an unloaded beam.

2 THE FIXED-POLE THEORY

In [2, 3] Borri and Bottasso thoroughly investigated the idea of replacing the stress-couple resultant $\mathbf{m}$ and the specific angular momentum $\boldsymbol{\pi}$, which are defined with respect to the beam reference axis at a cross-section, with another stress-couple resultant $\bar{\mathbf{m}}$ and specific angular momentum $\bar{\boldsymbol{\pi}}$, which are to be defined with respect to a unique point for the whole structure - the fixed pole - naturally taken to be the origin of the spatial frame, i.e. $\bar{\mathbf{m}} = \mathbf{r}\times\mathbf{n}+\mathbf{m}$ and $\bar{\boldsymbol{\pi}} = \mathbf{r}\times\mathbf{k}+\boldsymbol{\pi}$. Substituting this into (1) results in $\int_0^L (\bar{\mathbf{v}}'\cdot\mathbf{n}+\mathbf{w}'\cdot\bar{\mathbf{m}})\,dx + \int_0^L \left(\bar{\mathbf{v}}\cdot\dot{\mathbf{k}}+\mathbf{w}\cdot\dot{\bar{\boldsymbol{\pi}}}\right)dx = 0$, where the 'new' velocity $\bar{\mathbf{v}} = \mathbf{v}+\mathbf{r}\times\mathbf{w}$ is nothing but the relative velocity vector of the reference line at a cross-section as seen by the observer rigidly attached to the frame rotating with the cross-section. Choosing to interpolate $\bar{\mathbf{v}}(x)$ as well as $\mathbf{w}(x)$, an alternative nodal balance $\bar{\mathbf{g}}^i \equiv \bar{\mathbf{q}}_i^i + \bar{\mathbf{q}}_m^i = \mathbf{0}$ is obtained with the corresponding nodal internal and inertial force vectors $\bar{\mathbf{q}}_i^i$ and $\bar{\mathbf{q}}_m^i$ as

$$\bar{\mathbf{q}}_i^i = \int_0^L I^{i\prime}\begin{Bmatrix}\bar{\mathbf{n}}\\ \bar{\mathbf{m}}\end{Bmatrix}dx \qquad \text{and} \qquad \bar{\mathbf{q}}_m^i = \int_0^L I^{i}\begin{Bmatrix}\dot{\mathbf{k}}\\ \dot{\bar{\boldsymbol{\pi}}}\end{Bmatrix}dx \tag{3}$$

Different implementations of this general concept result in the algorithms which naturally inherit the strain-invariance of the underlying formulation with respect to a rigid-body motion [2] or are capable of simultaneous conservation of energy and both momentum vectors [3].
It should be noted, however, that in the lower halves of these vectors we now have the nodal internal and inertial moment vectors with respect to the fixed-pole, rather than the reference line at the cross-section. As a result, at least in the classical Bubnov–Galerkin approach, the linear part of the changes in kinematic unknowns is to be interpolated consistently with the applied interpolation for $\bar{\mathbf{v}}(x)$ and $\mathbf{w}(x)$. The translational kinematic unknown thus ceases to be the standard position vector $\mathbf{r}(x)$, which makes the algorithm untypical and not easy to merge with existing finite-element codes.
The benefits of the fixed-pole approach may be easily combined with the standard choice of the unknowns in the earlier 'moving-frame' approach by simply expressing the relative velocity vectors at the nodal points in terms of the (absolute) velocity vectors and the angular velocities via $\bar{\mathbf{v}}_i = \mathbf{v}_i + \mathbf{r}_i\times\mathbf{w}_i$ for each $i = 1,\ldots,N$. The power balance equation thus turns into

$$\sum_{i=1}^N \int_0^L I^{i\prime}\left[(\mathbf{v}_i+\mathbf{r}_i\times\mathbf{w}_i)\cdot\mathbf{n}+\mathbf{w}_i\cdot\bar{\mathbf{m}}\right]dx + \sum_{i=1}^N \int_0^L I^{i}\left[(\mathbf{v}_i+\mathbf{r}_i\times\mathbf{w}_i)\cdot\dot{\mathbf{k}}+\mathbf{w}_i\cdot\dot{\bar{\boldsymbol{\pi}}}\right]dx = 0\,,$$

which eventually leads to the nodal balance $\tilde{\mathbf{g}}^i \equiv \tilde{\mathbf{q}}_i^i + \tilde{\mathbf{q}}_m^i = \mathbf{0}$ with the corresponding nodal internal and inertial force vectors $\tilde{\mathbf{q}}_i^i$ and $\tilde{\mathbf{q}}_m^i$ as

$$\tilde{\mathbf{q}}_i^i = \int_0^L I^{i\prime}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \widehat{\mathbf{r}-\mathbf{r}_i} & \mathbf{I}\end{bmatrix}\begin{Bmatrix}\mathbf{n}\\ \mathbf{m}\end{Bmatrix}dx \qquad \text{and} \qquad \tilde{\mathbf{q}}_m^i = \int_0^L I^{i}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \widehat{\mathbf{r}-\mathbf{r}_i} & \mathbf{I}\end{bmatrix}\begin{Bmatrix}\dot{\mathbf{k}}\\ \dot{\boldsymbol{\pi}}\end{Bmatrix}dx\,. \tag{4}$$

This is equivalent to using non-linear velocity interpolation $\mathbf{v}(x) = \sum_{i=1}^N I^i(x)\left[\mathbf{v}_i + (\mathbf{r}_i - \mathbf{r}(x))\times\mathbf{w}_i\right]$ in the original power balance (1).
In static analysis, $\tilde{\mathbf{q}}_m^i = \mathbf{0}$ and the nodal vector of external forces reads

$$\tilde{\mathbf{q}}_e^i = \int_0^L I^{i}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \widehat{\mathbf{r}-\mathbf{r}_i} & \mathbf{I}\end{bmatrix}\begin{Bmatrix}\mathbf{n}_e\\ \mathbf{m}_e\end{Bmatrix}dx + \delta_1^i\begin{Bmatrix}\mathbf{F}_0\\ \mathbf{M}_0\end{Bmatrix} + \delta_N^i\begin{Bmatrix}\mathbf{F}_L\\ \mathbf{M}_L\end{Bmatrix}\,, \tag{5}$$

where $\mathbf{n}_e$, $\mathbf{m}_e$ are vectors of applied distributed forces and torques and $\mathbf{F}_0$, $\mathbf{F}_L$,$\mathbf{M}_0$, $\mathbf{M}_L$ are applied concentrated forces at ends of the beam.

3 SOLUTION PROCEDURE

The non-linear equation $\tilde{\mathbf{g}}^i \equiv \tilde{\mathbf{q}}_i^i - \tilde{\mathbf{q}}_e^i = \mathbf{0}$ is solved using the Newton-Raphson solution procedure. When linearizing either of these equations, account has to be taken of the fact that $\Delta\mathbf{R} = \Delta\widehat{\boldsymbol{\vartheta}}\mathbf{R}$ [4] where $\Delta\boldsymbol{\vartheta}$ is the spin vector. In order to complete the solution procedure the unknown functions $\mathbf{r}$ and/or their iterative changes $\Delta\mathbf{r}$ must be interpolated in some way. Three different solution procedures are considered which arise as a consequence of the fact that the velocity vector $\mathbf{v}(x)$ has been interpolated in a non-linear manner, dependent on the actual position vector $\mathbf{r}$ not only at the nodal points, but also at x (i.e. the integration point).

3.1 Interpolation option 1

The incremental displacements $\Delta\mathbf{r}$ and spin vectors $\Delta\boldsymbol{\vartheta}$ are interpolated in a same manner as $\mathbf{v}$ and $\mathbf{w}$ respectively.

$$\Delta\mathbf{r} = \sum_{j=1}^{N} I^j [\Delta\mathbf{r}_j + (\mathbf{r}_j - \mathbf{r}) \times \Delta\boldsymbol{\vartheta}_j] \qquad \text{and} \qquad \Delta\boldsymbol{\vartheta} = \sum_{j=1}^{N} I^j \Delta\boldsymbol{\vartheta}_j \,, \tag{6}$$

while the unknown displacement functions are interpolated using the Lagrangian polynomials:

$$\mathbf{r} = \sum_{k=1}^{N} I^k \mathbf{r}_k \,. \tag{7}$$

The interpolation (7) is in contradiction with the interpolation for $\Delta\mathbf{r}$ in (6) which necessarily results in the loss of quadratic convergence of the Newton-Raphson solution process.

3.2 Interpolation option 2

Within this option $\mathbf{r}$ is interpolated using the Lagrangian polynomials (7). Interpolation of the displacement increments follows consistently by linearizing $\mathbf{r}$ and $\Delta\boldsymbol{\vartheta}$ is interpolated using the Lagrangian polynomials:

$$\Delta\mathbf{r} = \sum_{j=1}^{N} I^j \Delta\mathbf{r}_j \,, \qquad \Delta\boldsymbol{\vartheta} = \sum_{j=1}^{N} I^j \Delta\boldsymbol{\vartheta}_j \,. \tag{8}$$

This option makes the tangent stiffness matrix strongly non-symmetric because different interpolations have been used for test and trial functions.

3.3 Interpolation option 3

Within this option, $\mathbf{r}$ is not interpolated, but updated as follows:

$$\mathbf{r}_{new}(x) = \mathbf{r}_{old}(x) + \Delta\mathbf{r}(x) \,, \tag{9}$$

where $\mathbf{r}_{old}(x)$ is the last known value for $\mathbf{r}(x)$, not necessarily associated with an equilibrium state and, $\Delta\mathbf{r}(x)$ is given in (6). This is a consistent choice which yields a tangent stiffness matrix. In this case the values of $\mathbf{r}$ at both integration and nodal points must be saved.

4 NUMERICAL EXAMPLE

We consider a hinged right-angle frame given in [5], with cross-sectional moment of inertia and area $I = 2$, $A = 6$, respectively, Young's modulus $Y = 7.2 \times 10^6$, Poisson's ratio $\nu = 0.3$, and length of each leg $l = 120$. The frame is divided into ten quadratic elements, five along each leg. The horizontal leg of the frame is loaded with a point force $P = 15000$ as shown in Figure 1.

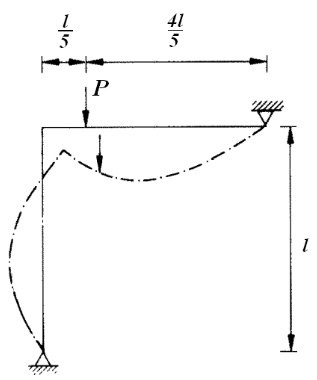

Figure 1: Lee's frame [5]

The reference solution for this example is obtained using 40 quadratic elements by Simo and Vu-Quoc [4], 20 along each leg with reduced Gaussian integration employed for evaluation of the integrals. The displacements of the loaded node are shown in Table 1 and the deformed shape of the structure in Figure 2.

	u	v
Reference solution	**8.028174**	**-25.892510**
10 linear Simo Vu-Quoc	6.460728	-22.486339
10 linear FP 1	6.460728	-22.486339
10 linear FP 2	6.460728	-22.486339
10 linear FP 3	**8.593919**	**-26.028418**

Table 1: Displacements of the loaded node

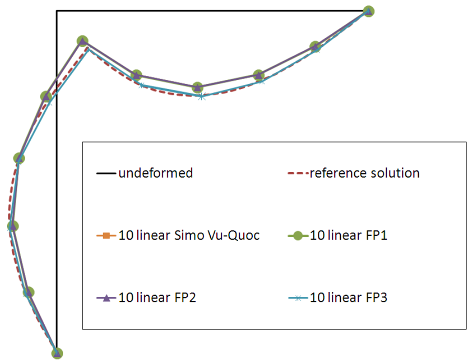

Figure 2: Lee's frame displacements

5 CONCLUSIONS

The proposed elements are spatial, geometrically non-linear, use standard kinematic unknowns and are generalized to an arbitrary order. The preliminary results indicate that the the third of the interpolation options considered may yield higher accuracy than the standard Simo and Vu-Quoc [5] element, while the first two interpolation options give the same results. As shown by Bottasso and Borri [2, 3] the fixed-pole approach is particularly effective in non-linear dynamics. The modification of this approach along the lines given here will be shown in future works.

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