A NON-LINEAR PLANAR BEAM FINITE ELEMENT BASED ON FIXED-POLE APPROACH

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1 Introduction and motivation

The motivation to introduce the fixed-pole approach (Borri and Bottasso, 1998) in non-linear beam structures analysis lies in its ability to conserve mechanical constants in numerical time integration. Therefore the main area of application of such an approach is non-linear dynamic analysis of beam structures. In this work, we will present a non-linear planar beam finite element based on Reissner's theory (Reissner, 1972) and the fixed pole approach and its application in static analysis. The numerical results will be shown and compared with the ones obtained using the non-linear planar beam finite element by Simo and Vu-Quoc (Simo and Vu-Quoc, 1986*a*).

2 The fixed-pole approach

In this section, a brief explanation of the fixed-pole approach will be given. All of the expressions will be written for a general spatial beam and will later be reduced to a planar case. The principle of virtual work states:

$$V_i + V_m - V_e = 0\,, \tag{1}$$

where V_i, V_m and V_e are the virtual works of internal, inertial and external forces, respectively.

2.1 Fixed-pole virtual work of internal forces

The theory given in (Borri and Bottasso, 1994) introduces a new set of fixed-pole stress resultants:

$$\begin{Bmatrix} \overline{\boldsymbol{n}} \\ \overline{\boldsymbol{m}} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{n} \\ \boldsymbol{m} + \boldsymbol{r} \times \boldsymbol{n} \end{Bmatrix}\,, \tag{2}$$

where $\boldsymbol{n}$ and $\boldsymbol{m}$ are the spatial cross-sectional stress and stress-couple resultants and $\boldsymbol{r}$ is a position vector of a point along the beam reference axis with respect to the origin of a spatial co-ordinate system.
The corresponding fixed-pole strain measures follow from the strain-energy balance (Jelenić, 2011) as

$$\begin{aligned} \overline{\boldsymbol{\gamma}} &= \boldsymbol{\gamma} + \boldsymbol{r} \times \boldsymbol{\kappa} \\ \overline{\boldsymbol{\kappa}} &= \boldsymbol{\kappa}\,, \end{aligned} \tag{3}$$

where the spatial strain measures $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ are defined as $\boldsymbol{\gamma} = \boldsymbol{r}' - \boldsymbol{e}_1$ and $\hat{\boldsymbol{\kappa}} = \boldsymbol{\Lambda}'\boldsymbol{\Lambda}^T$ (Simo, 1985) with $(\cdot)'$ as the first derivative with respect to x, $\widehat{(\cdot)}$ as the cross-product operator, $\boldsymbol{e}_1$ as unit vector normal to the cross-section in the deformed state, and $\boldsymbol{\Lambda}$ as the rotation matrix of a triad rigidly attached to the cross-section.
The virtual work of internal forces reads

$$V_i = \int_0^L \langle \delta\overline{\boldsymbol{\gamma}}^T \quad \delta\overline{\boldsymbol{\kappa}}^T \rangle \begin{Bmatrix} \overline{\boldsymbol{n}} \\ \overline{\boldsymbol{m}} \end{Bmatrix} dx \tag{4}$$

and after introducing the fixed-pole virtual displacements and rotations:

$$\begin{aligned} \delta\overline{\boldsymbol{r}} &= \delta\boldsymbol{r} + \boldsymbol{r} \times \delta\boldsymbol{\theta} \\ \delta\overline{\boldsymbol{\theta}} &= \delta\boldsymbol{\theta}\,, \end{aligned} \tag{5}$$

it turns into

$$V_i = \int_0^L \left(\frac{d}{dx}\right) \langle \delta\overline{\boldsymbol{r}}^T \quad \delta\overline{\boldsymbol{\theta}}^T \rangle \begin{Bmatrix} \overline{\boldsymbol{n}} \\ \overline{\boldsymbol{m}} \end{Bmatrix} dx\,. \tag{6}$$

2.2 Fixed-pole virtual work of inertial forces

We introduce $\boldsymbol{k} = A_\rho \dot{\boldsymbol{r}}$ and $\boldsymbol{\pi} = \boldsymbol{\Lambda}\boldsymbol{J_\rho}\boldsymbol{W}$ as the specific translational and angular momenta with respect to the beam centroid axis and $\boldsymbol{J_\rho}$ as the inertia tensor and $\boldsymbol{W}$ as the material angular velocity for which

$\hat{\boldsymbol{W}} = \boldsymbol{\Lambda}^T \dot{\boldsymbol{\Lambda}}$. We introduce the fixed-pole specific momenta:

$$\begin{aligned} \overline{\boldsymbol{k}} &= \boldsymbol{k} \\ \overline{\boldsymbol{\pi}} &= \boldsymbol{\pi} + \boldsymbol{r} \times \boldsymbol{k} \end{aligned} \tag{7}$$

and write the virtual work of inertial forces as

$$V_m = \int_0^L \langle \delta\overline{\boldsymbol{r}}^T \quad \delta\overline{\boldsymbol{\theta}}^T \rangle \begin{Bmatrix} \dot{\overline{\boldsymbol{k}}} \\ \dot{\overline{\boldsymbol{\pi}}} \end{Bmatrix} dx \,. \tag{8}$$

2.3 Fixed-pole virtual work of external forces

Upon introducing the distributed fixed-pole loads and moments:

$$\begin{aligned} \overline{\boldsymbol{n}}_e &= \boldsymbol{n}_e \\ \overline{\boldsymbol{m}}_e &= \boldsymbol{m}_e + \boldsymbol{r} \times \boldsymbol{n}_e \end{aligned} \tag{9}$$

and nodal forces and moments:

$$\begin{aligned} \overline{\boldsymbol{F}}_0 &= \boldsymbol{F}_0 \\ \overline{\boldsymbol{F}}_L &= \boldsymbol{F}_L \end{aligned} \tag{10}$$

$$\begin{aligned} \overline{\boldsymbol{M}}_0 &= \boldsymbol{M}_0 + \boldsymbol{r}_0 \times \boldsymbol{F}_0 \\ \overline{\boldsymbol{M}}_L &= \boldsymbol{M}_L + \boldsymbol{r}_L \times \boldsymbol{F}_L \,, \end{aligned} \tag{11}$$

with $\boldsymbol{n}_e$ as the external distributed load and $\boldsymbol{m}_e$ as the external distributed moment, $\boldsymbol{F}_0$ and $\boldsymbol{F}_L$ as the external point load and $\boldsymbol{M}_0$ and $\boldsymbol{M}_L$ as the external concentrated moment at outer nodes and $\boldsymbol{r}_0$ and $\boldsymbol{r}_L$ as the position vectors of the outer nodes.
The fixed-pole virtual work of external forces is written as

$$\begin{aligned} V_e = \int_0^L \langle \delta\overline{\boldsymbol{r}}^T \quad \delta\overline{\boldsymbol{\theta}}^T \rangle \begin{Bmatrix} \overline{\boldsymbol{n}}_e \\ \overline{\boldsymbol{m}}_e \end{Bmatrix} dx + \\ + \langle \delta\overline{\boldsymbol{r}}_0^T \quad \delta\overline{\boldsymbol{\theta}}_0^T \rangle \begin{Bmatrix} \overline{\boldsymbol{F}}_0 \\ \overline{\boldsymbol{M}}_0 \end{Bmatrix} + \\ + \langle \delta\overline{\boldsymbol{r}}_L^T \quad \delta\overline{\boldsymbol{\theta}}_L^T \rangle \begin{Bmatrix} \overline{\boldsymbol{F}}_L \\ \overline{\boldsymbol{M}}_L \end{Bmatrix} . \end{aligned} \tag{12}$$

2.4 Interpolation of the fixed-pole virtual quantities

We interpolate the fixed-pole virtual quantities using Lagrangian polynomials

$$\begin{aligned} \delta\overline{\boldsymbol{r}} \doteq \delta\overline{\boldsymbol{r}}^h &= \sum_{i=1}^N I^i(x) \delta\overline{\boldsymbol{r}}_i \\ \delta\overline{\boldsymbol{\theta}} \doteq \delta\overline{\boldsymbol{\theta}}^h &= \sum_{i=1}^N I^i(x) \delta\overline{\boldsymbol{\theta}}_i \end{aligned} \tag{13}$$

where N is the number of interpolation points (nodes) and I^i is the i-th Lagrangian interpolation function of order $N-1$.
The following results in an approximate weak form

$$G^h \equiv \sum_{i=1}^N \langle \delta\overline{\boldsymbol{r}}_i^T \quad \delta\overline{\boldsymbol{\theta}}_i^T \rangle \overline{\boldsymbol{g}}^i = 0 \,, \tag{14}$$

from which, because of the arbitrariness of $\delta\overline{\boldsymbol{r}}$ and $\delta\overline{\boldsymbol{\theta}}$, follows

$$\overline{\boldsymbol{g}}^i \equiv \overline{\boldsymbol{q}}_i^i + \overline{\boldsymbol{q}}_m^i - \overline{\boldsymbol{q}}_e^i = \boldsymbol{0} \,, \tag{15}$$

with $\overline{\boldsymbol{g}}^i$ as the fixed-pole vector of nodal dynamic residuals and the following vectors of internal, inertial and external forces (Jelenić, 2011):

$$\overline{\boldsymbol{q}}_i^i = \int_0^L I^{i\prime} \begin{Bmatrix} \overline{\boldsymbol{n}} \\ \overline{\boldsymbol{m}} \end{Bmatrix} dx \tag{16}$$

$$\overline{\boldsymbol{q}}_m^i = \int_0^L I^i \begin{Bmatrix} \dot{\overline{\boldsymbol{k}}} \\ \dot{\overline{\boldsymbol{\pi}}} \end{Bmatrix} dx \tag{17}$$

$$\overline{\boldsymbol{q}}_e^i = \int_0^L I^i \begin{Bmatrix} \overline{\boldsymbol{n}}_e \\ \overline{\boldsymbol{m}}_e \end{Bmatrix} dx + \delta_1^i \begin{Bmatrix} \overline{\boldsymbol{F}}_0 \\ \overline{\boldsymbol{M}}_0 \end{Bmatrix} + \delta_N^i \begin{Bmatrix} \overline{\boldsymbol{F}}_L \\ \overline{\boldsymbol{M}}_L \end{Bmatrix} , \tag{18}$$

where δ_1^i and δ_N^i are the Kronecker delta symbols, equal to unity when the indices are equal, and zero otherwise.

3 The modified fixed-pole approach

As shown in (14), the fixed-pole nodal dynamic residual $\overline{\boldsymbol{g}}^i$ is virtual-work conjugate to the *fixed-pole* nodal virtual displacements and rotations which are non-standard and prevent the elements from being combined with standard elements in a finite-element mesh. This problem, however, is easily overcome by noting that from (5) the nodal fixed-pole virtual quantities are related to the standard nodal virtual quantities via:

$$\begin{Bmatrix} \delta\overline{\boldsymbol{r}}_i \\ \delta\overline{\boldsymbol{\theta}}_i \end{Bmatrix} = \begin{Bmatrix} \delta\boldsymbol{r}_i + \boldsymbol{r}_i \times \delta\boldsymbol{\theta}_i \\ \delta\boldsymbol{\theta}_i \end{Bmatrix} = \begin{bmatrix} \boldsymbol{I} & \hat{\boldsymbol{r}}_i \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \delta\boldsymbol{r}_i \\ \delta\boldsymbol{\theta}_i \end{Bmatrix} , \tag{19}$$

so that the fixed-pole weak form (14) becomes

$$\begin{aligned} G^h &\equiv \sum_{i=1}^N \langle \delta\boldsymbol{r}_i^T \quad \delta\boldsymbol{\theta}_i^T \rangle \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\hat{\boldsymbol{r}}_i & \boldsymbol{I} \end{bmatrix} \overline{\boldsymbol{g}}^i = \\ &= \sum_{i=1}^N \langle \delta\boldsymbol{r}_i^T \quad \delta\boldsymbol{\theta}_i^T \rangle \tilde{\boldsymbol{g}}^i = 0 \,, \end{aligned} \tag{20}$$

with

$$\tilde{\boldsymbol{g}}^i = \tilde{\boldsymbol{q}}_i^i + \tilde{\boldsymbol{q}}_m^i - \tilde{\boldsymbol{q}}_e^i \tag{21}$$

and

$$\tilde{\boldsymbol{q}}_i^i = \int_0^L I^{i'} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \widehat{\boldsymbol{r} - \boldsymbol{r}_i} & \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{n} \\ \boldsymbol{m} \end{Bmatrix} dx \tag{22}$$

$$\tilde{\boldsymbol{q}}_m^i = \int_0^L I^{i} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \widehat{\boldsymbol{r} - \boldsymbol{r}_i} & \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{k}} \\ \dot{\boldsymbol{\pi}} \end{Bmatrix} dx \tag{23}$$

$$\tilde{\boldsymbol{q}}_e^i = \int_0^L I^{i} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \widehat{\boldsymbol{r} - \boldsymbol{r}_i} & \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{n}_e \\ \boldsymbol{m}_e \end{Bmatrix} dx + \\ + \delta_1^i \begin{Bmatrix} \boldsymbol{F}_0 \\ \boldsymbol{M}_0 \end{Bmatrix} + \delta_N^i \begin{Bmatrix} \boldsymbol{F}_L \\ \boldsymbol{M}_L \end{Bmatrix}, \tag{24}$$

where $i = 1, 2 \ldots N$. After substituting (22), (23) and (24) in $\tilde{\boldsymbol{g}}^i$ and then back to (14), it can be shown that, using a non-linear interpolation of the type:

$$\delta \boldsymbol{r} = \sum_{i=1}^{N} I^i [\delta \boldsymbol{r}_i + (\boldsymbol{r}_i - \boldsymbol{r}) \times \delta \boldsymbol{\theta}_i] \\ \delta \boldsymbol{\theta} = \sum_{i=1}^{N} I^i \delta \boldsymbol{\theta}_i \tag{25}$$

is equivalent to the standard (Lagrangian) interpolation of the fixed-pole virtual quantities (13).

4 Static analysis of a non-linear beam element

In a static analysis the inertial term (23) vanishes and we are left with

$$\tilde{\boldsymbol{g}}^i = \tilde{\boldsymbol{q}}_i^i - \tilde{\boldsymbol{q}}_e^i \,. \tag{26}$$

4.1 Kinematic equations

Kinematic equations in material co-ordinate system follow from the Reissner-Simo theory (Simo, 1985):

$$\boldsymbol{\Gamma} = \boldsymbol{\Lambda}^T \boldsymbol{\gamma} = \boldsymbol{\Lambda}^T \boldsymbol{r}' - \boldsymbol{E}_1 \\ \hat{\boldsymbol{K}} = \boldsymbol{\Lambda}^T \hat{\boldsymbol{\kappa}} \boldsymbol{\Lambda} = \boldsymbol{\Lambda}^T \boldsymbol{\Lambda}' \tag{27}$$

4.2 Constitutive equations

The constitutive law is linear, and the cross-sectional stress and stress-couple resultants in the material co-ordinate system are defined as follows:

$$\begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{C}_N & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C}_M \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{K} \end{Bmatrix} \tag{28}$$

with

$$\boldsymbol{C}_N = \begin{bmatrix} EA & 0 & 0 \\ 0 & GA_{s1} & 0 \\ 0 & 0 & GA_{s2} \end{bmatrix} \tag{29}$$

$$\boldsymbol{C}_M = \begin{bmatrix} GI_t & 0 & 0 \\ 0 & EI_1 & 0 \\ 0 & 0 & EI_2 \end{bmatrix}, \tag{30}$$

where E is the Young's modulus, G the shear modulus, A the cross-sectional area, A_{s1}, A_{s2} cross-sectional shear areas, I_t the cross-sectional torsional second moment of area and I_1, I_2 as the cross-sectional second moments of area of the beam.

4.3 Newton-Raphson solution procedure

The non-linear equation (26) is solved using the Newton-Raphson solution procedure. After expanding the equation into a Taylor series around a known configuration and omitting higher-order terms we have:

$$\tilde{\boldsymbol{g}}^{\boldsymbol{i}} + \Delta \tilde{\boldsymbol{g}}^{\boldsymbol{i}} = \boldsymbol{0}\,, \tag{31}$$

where

$$\Delta \tilde{\boldsymbol{g}}^{\boldsymbol{i}} = \Delta \tilde{\boldsymbol{q}}_i^i - \Delta \tilde{\boldsymbol{q}}_e^i \,. \tag{32}$$

Linearizing the modified nodal internal force vector (22) we have:

$$\Delta \tilde{\boldsymbol{q}}_i^i = \int_0^L I^{i'} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \hat{\boldsymbol{n}} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \Delta \boldsymbol{r}_i - \Delta \boldsymbol{r} \\ \Delta \boldsymbol{\theta}_i - \Delta \boldsymbol{\theta} \end{Bmatrix} dx + \int_0^L I^{i'} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}_i & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & -\hat{\boldsymbol{n}} \\ \boldsymbol{0} & -\hat{\boldsymbol{m}} \end{bmatrix} \begin{Bmatrix} \Delta \boldsymbol{r} \\ \Delta \boldsymbol{\theta} \end{Bmatrix} dx + \\ + \int_0^L I^{i'} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}_i & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Lambda} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Lambda} \end{bmatrix} \begin{bmatrix} \boldsymbol{C}_N & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C}_M \end{bmatrix} \begin{bmatrix} \boldsymbol{\Lambda}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Lambda}^T \end{bmatrix} \begin{Bmatrix} \Delta \boldsymbol{r}' + \boldsymbol{r}' \times \Delta \boldsymbol{\theta} \\ \Delta \boldsymbol{\theta}' \end{Bmatrix} dx\,, \tag{33}$$

where account has been taken of the fact that $\Delta\boldsymbol{\Lambda} = \widehat{\Delta\boldsymbol{\theta}}\boldsymbol{\Lambda}$ and $\Delta\boldsymbol{\Gamma} = \boldsymbol{\Lambda}^T(\Delta\boldsymbol{r}' + \boldsymbol{r}' \times \Delta\boldsymbol{\theta})$ as well as $\Delta\hat{\boldsymbol{K}} = \boldsymbol{\Lambda}^T \left(\widehat{\Delta\boldsymbol{\theta}}^T\boldsymbol{\Lambda}' + \widehat{\Delta\boldsymbol{\theta}'}\boldsymbol{\Lambda} + \widehat{\Delta\boldsymbol{\theta}}\boldsymbol{\Lambda}'\right) = \boldsymbol{\Lambda}^T\widehat{\Delta\boldsymbol{\theta}'}\boldsymbol{\Lambda} \Longleftrightarrow \Delta\boldsymbol{K} = \boldsymbol{\Lambda}^T\widehat{\Delta\boldsymbol{\theta}'}$ (Simo, 1985).

Linearizing the modified nodal external force vector (24) we have:

$$\Delta\tilde{\boldsymbol{q}}_e^i = \int_0^L I^i \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\hat{\boldsymbol{n}}_e & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{r}_i - \Delta\boldsymbol{r} \\ \Delta\boldsymbol{\theta}_i - \Delta\boldsymbol{\theta} \end{Bmatrix} dx \quad (34)$$

An interesting phenomenon occurs here which is a direct consequence of the use of the fixed-pole virtual quantities and their conversion back to standard virtual quantities: even though the external loading is configuration-independent, the modified nodal external loading vector (24) is not. As a result of this, the stiffness matrix will have an additional part as a result of the linearization of $\tilde{\boldsymbol{q}}_{\boldsymbol{e}}^{\boldsymbol{i}}$.
From here on, we have to decide how to interpolate the unknown functions and their iterative changes.

4.4 Interpolation options

4.4.1 Interpolation option 1

Within this option $\Delta\boldsymbol{r}$ and $\Delta\boldsymbol{\theta}$ are interpolated in the same way as $\delta\boldsymbol{r}$ and $\delta\theta$ (25):

$$\begin{aligned} \Delta\boldsymbol{r} &= \sum_{j=1}^{N} I^j[\Delta\boldsymbol{r}_j + (\boldsymbol{r}_j - \boldsymbol{r}) \times \Delta\boldsymbol{\theta}_j] \\ \Delta\boldsymbol{\theta} &= \sum_{j=1}^{N} I^j\Delta\boldsymbol{\theta}_j \, , \end{aligned} \quad (35)$$

while the unknown displacement functions are interpolated using the Lagrangian polynomials:

$$\boldsymbol{r} = \sum_{k=1}^{N} I^k\boldsymbol{r}_{\boldsymbol{k}} \, . \quad (36)$$

It should be noted that this interpolation is in contradiction with the interpolation for $\Delta\boldsymbol{r}$ in (35), which obviously does not follow as a linearisation of (36). For a 2D case this should be contrasted to $\Delta\theta = \sum_{k=1}^{N} I^k\Delta\theta_k \Longleftrightarrow \theta = \sum_{k=1}^{N} I^k\theta_k$.
The above contradiction necessarily results in the loss of quadratic convergence of the Newton-Raphson solution process, and has been introduced only for the sake of evaluation of $\boldsymbol{r}$ in (35) from its nodal values.

After introducing (35) into (33) and (34) we have:

$$\Delta\tilde{\boldsymbol{q}}_i^i = \boldsymbol{K}_{int}^{ij} \begin{Bmatrix} \Delta\boldsymbol{r}_j \\ \Delta\boldsymbol{\theta}_j \end{Bmatrix} \quad (37)$$

$$\Delta\tilde{\boldsymbol{q}}_e^i = \boldsymbol{K}_{ext}^{ij} \begin{Bmatrix} \Delta\boldsymbol{r}_j \\ \Delta\boldsymbol{\theta}_j \end{Bmatrix} , \quad (38)$$

where:

$$\begin{aligned} \boldsymbol{K}_{int}^{ij} = &\int_0^L I^{i\prime} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \hat{\boldsymbol{n}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} (\delta_{ij} - I^j)\boldsymbol{I} & -I^j(\hat{\boldsymbol{r}}_j - \hat{\boldsymbol{r}}) \\ \mathbf{0} & (\delta_{ij} - I^j)\boldsymbol{I} \end{bmatrix} dx + \\ &+ \int_0^L I^{i\prime} I^j \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ \hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}_i & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\hat{\boldsymbol{n}} \\ \mathbf{0} & -\hat{\boldsymbol{m}} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \hat{\boldsymbol{r}}_j - \hat{\boldsymbol{r}} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix} dx + \\ &+ \int_0^L I^{i\prime} I^{j\prime} \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ \hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}_i & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Lambda} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda} \end{bmatrix} \begin{bmatrix} \boldsymbol{C}_N & \mathbf{0} \\ \mathbf{0} & \boldsymbol{C}_M \end{bmatrix} \\ &\begin{bmatrix} \boldsymbol{\Lambda}^T & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \hat{\boldsymbol{r}}_j - \hat{\boldsymbol{r}} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix} dx \end{aligned} \quad (39)$$

and

$$\boldsymbol{K}_{ext}^{ij} = \int_0^L I^i \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\hat{\boldsymbol{n}}_e & \mathbf{0} \end{bmatrix} \begin{bmatrix} (I^j - \delta_{ij})\boldsymbol{I} & I^j(\hat{\boldsymbol{r}}_j - \hat{\boldsymbol{r}}) \\ \mathbf{0} & (I^j - \delta_{ij})\boldsymbol{I} \end{bmatrix} dx \quad (40)$$

and finally:

$$\boldsymbol{K}^{ij} = \boldsymbol{K}_{int}^{ij} - \boldsymbol{K}_{ext}^{ij} \, . \quad (41)$$

4.4.2 Interpolation option 2

Within this option $\boldsymbol{r}$ is interpolated using Lagrangian polynomials (36) and so is also the rotation $\boldsymbol{\theta}$ (the latter is only legitimate in the presently analysed 2D case). Interpolation of the displacement increments follows consistently by linearizing $\boldsymbol{r}$ and $\boldsymbol{\theta}$:

$$\Delta\boldsymbol{r} = \sum_{j=1}^{N} I^j\Delta\boldsymbol{r}_j \qquad \Delta\theta = \sum_{j=1}^{N} I^j\Delta\theta_j \, . \quad (42)$$

After introducing (42) into (33) and (34) we have:

$$\Delta\tilde{\boldsymbol{q}}_i^i = \boldsymbol{K}_{int}^{ij} \begin{Bmatrix} \Delta\boldsymbol{r}_j \\ \Delta\boldsymbol{\theta}_j \end{Bmatrix} \quad (43)$$

$$\Delta\tilde{\boldsymbol{q}}_e^i = \boldsymbol{K}_{ext}^{ij} \begin{Bmatrix} \Delta\boldsymbol{r}_j \\ \Delta\boldsymbol{\theta}_j \end{Bmatrix} , \quad (44)$$

where:

$$\boldsymbol{K}_{int}^{ij} = \int_0^L I^{i'} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \hat{\boldsymbol{n}} & \boldsymbol{0} \end{bmatrix} (\delta_{ij} - I^j)\, dx +$$
$$+ \int_0^L I^{i'} I^j \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}_i & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & -\hat{\boldsymbol{n}} \\ \boldsymbol{0} & -\hat{\boldsymbol{m}} \end{bmatrix} dx +$$
$$+ \int_0^L I^{i'} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}_i & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Lambda} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Lambda} \end{bmatrix} \begin{bmatrix} \boldsymbol{C}_N & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C}_M \end{bmatrix}$$
$$\begin{bmatrix} \boldsymbol{\Lambda}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Lambda}^T \end{bmatrix} \begin{bmatrix} I^{j'} \boldsymbol{I} & I^j \hat{\boldsymbol{r}}' \\ \boldsymbol{0} & I^{j'} \boldsymbol{I} \end{bmatrix} dx \qquad (45)$$

and

$$\boldsymbol{K}_{ext}^{ij} = \int_0^L I^i (I^j - \delta_{ij}) \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ -\hat{\boldsymbol{n}}_e & \boldsymbol{0} \end{bmatrix} dx \qquad (46)$$

and finally:

$$\boldsymbol{K}^{ij} = \boldsymbol{K}_{int}^{ij} - \boldsymbol{K}_{ext}^{ij} \,. \qquad (47)$$

5 Numerical example

In this section we present a numerical example of a planar cantilever beam with a transversally acting point load at the free end. The exact analytical solutions are given in (Mattiasson, 1981), assuming that the axial and shear deformations are ignored. In order to simulate this effect, the shear modulus has been set to $G = 1000E$. The following constants have been used: $L = 1$, $E = 20000$, $A = 0.02$, $As = A/1.2$, $I = 0.00025$. In the linear limit, the shear contribution to the total deformation amounts to less than 0.005% thus giving an indication how many decimal places should be considered when comparing the results with the analytical solution.

By reducing the presented modified fixed-pole theory to a planar case and using reduced Gaussian integration for solving the integrals in order to avoid shear locking, algorithms have been coded for the interpolation options 1 and 2 and it was found that they give exactly the same solutions as the element by Simo and Vu-Quoc (Simo and Vu-Quoc, 1986*a*). The results were obtained using a three-noded element and loaded with a point force with a ratio $\frac{FL^2}{EI}$ ranging from 0.2 to 10.0 and are shown in Table 1:

	max $\frac{FL^2}{EI}$
Simo and Vu Quoc	10.0
Interpolation op. 1	0.8
Interpolation op. 2	6.5

Table 1. Maximal values of $\frac{FL^2}{EI}$ for which a solution can be obtained

After using load control procedure, both interpolation options give the same results as Simo and Vu Quoc element.

It is also interesting to compare the number of iterations needed to obtain a solution for a given force shown in Table 2:

$\frac{FL^2}{EI}$	0.2	1.0	2.0	5.0	6.0	10.0
Simo and Vu Quoc	18	9	12	29	21	35
Interpolation op. 1	13	-	-	-	-	-
Interpolation op. 2	12	10	12	37	21	-

Table 2. Iterations needed to obtain solutions for a given $\frac{FL^2}{EI}$ ratio

6 Concluding remarks

A formulation for a static non-linear beam element using the modified fixed-pole approach was presented and three interpolation options presented and coded. The example shows that the presented element has the same accuracy as the Simo and Vu Quoc element but is less robust (which is easily overcome by using the load control procedure). As mentioned in the introduction, the most interesting application of these finite elements is in non-linear dynamic analysis of spatial beams. The theoretical formulations for a spatial non-linear beam element under dynamic loading have been derived and the numerical examples will be shown in future works.

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