One review of McShane-type inequalities

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***Abstract*—In this paper Diaz-Metcalf inequality is refined upon the conversion of McShane-type inequality. Extension of generalized Hadamard inequality on functions of two variable is reviewed. The inverse of Hölder's inequality is proven using the property of two-variable function. One estimation of Jensen's functional is rewritten.**

Keywords: Concave functions, Linear mean, McShane inequality, Diaz-Metcalf inequality, Hadamard inequality, Inverse Hölder inequality

I. Introduction

Throughout this paper Ω will denote a nonempty set and L will denote a linear class of real-valued functions $f:\Omega\to\mathbb{R}$ and a linear mean $A:L\to\mathbb{R}$ will be considered as linear, positive and normalized functional (see [1, p. 47] and [5]). Mathematical expectation $E[X]$ is a linear mean for random variable X as a function on a probability space (Ω,Σ,P).

McShane's generalization of the Jensen's inequality is presented in [1, p. 49].

Theorem 1 (McShane): Let φ be a continuous convex function on a closed convex set K in $\mathbb{R}^n$ and A be a linear mean on L. Let g_i be a function in $L, i=1,...,n$, such that $\mathbf{g}(x)=(g_1(x),...,g_n(x))$ is in K for all $x\in\Omega$ and the components of $\varphi(\mathbf{g})$ are in the class L. Then $A(\mathbf{g})=(A(g_1),\ldots,A(g_2))$ is in K and $\varphi(A(\mathbf{g}))\le A(\varphi(\mathbf{g}))$.

Following the tag of the relationship between Jensen's functionals of the shape
$J_n(\varphi,\mathbf{x},\mathbf{p}):=\sum_{i=1}^n p_i\varphi(x_i)-\varphi(\sum_{i=1}^n p_ix_i)$, given in [3] for different weights $p_i,q_i>0$, $\sum_i p_i=\sum_i q_i=1$ and for x_i as vectors from the vector space X, we have obtained relations for Hölder and Minkowski type inequalities.

II. Conversions on a rectangular

In this section we consider Theorem for a concave function φ defined on $K=D\subset\mathbb{R}^2$ to obtain a conversion of inequality $A(\varphi(\mathbf{g}))\le\varphi(A(\mathbf{g}))$ for two variables.

For the statement of the next Theorem that has been proved in [5] we now consider continuous functions
$M_{ij},m_{ij}:\mathbb{R}^2\to\mathbb{R}$ defined by

$$\begin{aligned} M_{ij}(t,s) &= \frac{(\lambda_i+\lambda_j)t+(\mu_i+\mu_j)s+\nu_i+\nu_j}{2} \\ &+\frac{|(\lambda_i-\lambda_j)t+(\mu_i-\mu_j)s+\nu_i-\nu_j|}{2}, \\ m_{ij}(t,s) &= \frac{(\lambda_i+\lambda_j)t+(\mu_i+\mu_j)s+\nu_i+\nu_j}{2} \\ &-\frac{|(\lambda_i-\lambda_j)t+(\mu_i-\mu_j)s+\nu_i-\nu_j|}{2}, \end{aligned} \quad (1)$$

with the coefficients:

$$\begin{aligned} \lambda_{1,4} &= \frac{\varphi(A,b)-\varphi(a,b)}{A-a}; \quad \mu_{1,3}=\frac{\varphi(a,B)-\varphi(a,b)}{B-b}; \\ \lambda_{2,3} &= \frac{\varphi(A,B)-\varphi(a,B)}{A-a}; \quad \mu_{2,4}=\frac{\varphi(A,B)-\varphi(A,b)}{B-b}; \\ \nu_1 &= \varphi(a,b)-\lambda_1a-\mu_1b; \quad \nu_2=\varphi(A,B)-\lambda_2A-\mu_2B \\ \nu_3 &= \varphi(a,B)-\lambda_3a-\mu_3B; \quad \nu_4=\varphi(A,b)-\lambda_4A-\mu_4b. \end{aligned} \quad (2)$$

Theorem 2.1: Let $A:L\to\mathbb{R}$ be a linear mean and $g_1,g_2\in L$ are functions with $g_1(t)\in[a,A], g_2(t)\in[b,B]$ for all $t\in\Omega$. Functions M_{12},M_{34},m_{12} and m_{34} are defined by (1).

Suppose that $\varphi:D\to\mathbb{R}$ is a continuous and concave function.

(i) If $\Delta\varphi\ge0$, then

$$M_{12}(A(g_1),A(g_2))\le A(m_{34}(g_1,g_2))\le A(\varphi(g_1,g_2)).$$

(ii) If $\Delta\varphi\le0$, then

$$M_{34}(A(g_1),A(g_2))\le A(m_{12}(g_1,g_2))\le A(\varphi(g_1,g_2)).$$

The well-known inequality of Hadamard is given in [9, p.11] and [8] An extension of the weighted Hadamard's inequality proved by Fejér is given in [1, p.138] and [8].

As application of Theorem 2.1 for a linear mean defined as weighted integral over the rectangle D, we obtain in [5] a refinement of Feyér's inequalities calculated by $O(|\Delta\varphi|)$.

Theorem 2: Let $w:D=[a,A]\times[b,B]\to\mathbb{R}$ be a nonnegative integrable function such that $w(s,t)=u(s)v(t)$, where $u:[a,A]\to\mathbb{R}$ is an integrable function, $\int_a^A u(s)ds=1$, $u(s)=u(a+A-s)$, for all $s\in[a,A]$ and $v:[b,B]\to\mathbb{R}$ is an integrable function, $\int_b^B v(t)dt=1$, $v(t)=v(b+B-t)$, for all $t\in[b,B]$. If $\varphi:D\to\mathbb{R}$ is a continuous concave function, then

$$\begin{aligned} &\max\{\frac{\varphi(a,b)+\varphi(A,B)}{2},\frac{\varphi(A,b)+\varphi(a,B)}{2}\}-O(|\Delta\varphi|) \\ &\le\int_D w(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}\le\varphi(\frac{a+A}{2},\frac{b+B}{2}), \end{aligned}$$

where $O(|\Delta\varphi|)=$

$$\begin{aligned} &= \frac{|\Delta\varphi|}{2}\Bigg[\frac{1}{A-a}\int_a^A su(s)(\int_{b+\frac{B-b}{A-a}(s-a)}^{B-\frac{B-b}{A-a}(s-a)}v(t)dt)ds \\ &+\frac{1}{B-b}\int_b^B tv(t)(\int_{a+\frac{A-a}{B-b}(t-b)}^{A-\frac{A-a}{B-b}(t-b)}u(s)ds)dt\Bigg]. \end{aligned}$$

Similar enlargements on two-variables functions and their refinements are given in [8] and [5], for the inequalities given by Lupaş and Petrović for functions of one variable.

Well-known inequality of Schwarz-Cauchy- Buniakowsky for mathematical expectation of random variables ξ, η defined on a probability space (Ω, Σ, P) is given with $(E[\xi\eta])^2 \leq E[\xi^2]E[\eta^2]$. If $X = \xi^2$ and $Y = \eta^2$, then it turns into $E[\sqrt{XY}] \leq \sqrt{E[X]E[Y]}$.

For $P(m_1 \leq \xi \leq M_1) = P(m_2 \leq \eta \leq M_2) = 1$, $m_1, m_2 > 0$, Diaz and Metcalf have proved a conversion in [7]:

$$m_2M_2E[\xi^2] + m_1M_1E[\eta^2] \leq (m_1m_2 + M_1M_2)E[\xi\eta].$$

Csiszár and Móri in [2] obtained $\lambda E[\xi^2] + \mu E[\eta^2] + \nu \leq E[\xi\eta]$ with coefficients $\lambda_{(1)}$, $\mu_{(1)}$, $\nu_{(1)}$ and respectively $\lambda_{(2)}$, $\mu_{(2)}$, $\nu_{(2)}$, calculated from (2). Theorem 2.1 refined the result of Csiszár and Móri.

Corollary 2.1: Suppose (Ω, Σ, P) is a probability space and $g_1 = \xi^2$ and $g_2 = \eta^2$ to be random variables with $P(m_1 \leq \xi \leq M_1) = 1$ and $P(m_2 \leq \eta \leq M_2) = 1$, for $m_1, m_2 > 0$. Taking $\varphi(x, y) = \sqrt{xy}$ and taking mathematical expectation E as linear mean, we obtain $\lambda E[\xi^2] + \mu E[\eta^2] + \nu \leq E[\xi\eta]$ calculating (2).

If $(M_2^2 - m_2^2)E[\xi^2] - (M_1^2 - m_1^2)E[\eta^2] \leq m_1^2M_2^2 - M_1^2m_2^2$, then

$\lambda_{(3)} = \dfrac{M_2}{m_1 + M_1}$, $\mu_{(3)} = \dfrac{m_1}{m_2 + M_2}$ and

$\nu_{(3)} = (M_1m_2 - m_1M_2)\lambda_{(3)}\mu_{(3)}$.

In opposite,

$(M_2^2 - m_2^2)E[\xi^2] - (M_1^2 - m_1^2)E[\eta^2] \geq m_1^2M_2^2 - M_1^2m_2^2$

gives the coefficients

$\lambda_{(4)} = \dfrac{m_2}{m_1 + M_1}$, $\mu_{(4)} = \dfrac{M_1}{m_2 + M_2}$ and

$\nu_{(4)} = (m_1M_2 - M_1m_2)\lambda_{(4)}\mu_{(4)}$.

Simple algebra ensures that

$\lambda_{(i)}E[\xi^2] + \mu_{(i)}E[\eta^2] + \nu_{(i)} \leq \lambda_{(j)}E[\xi^2] + \mu_{(j)}E[\eta^2] + \nu_{(j)}$, $i = 1, 2;\ j = 3, 4$.

III. Advanced conversions on rectangular

Conversions of the McShane inequality are obtained applying the functions of two variables according to the idea for conversions of Jensen's inequality given in [1, p.101].

Theorem 2.1, proved in [4], has inspired the following result.

Theorem 3.1: Let $\varphi, f : D \to \mathbb{R}$ such that φ is continuous and concave and assume that for $g_1, g_2 \in L$, compositions $\varphi(g_1, g_2), f(g_1, g_2) \in L$. A is a linear mean on L. After Theorem 1, $A(\varphi(g_1, g_2)) \leq \varphi(A(g_1), A(g_2))$.

Suppose $\mathcal{F} : U \times V \subset \mathbb{R}^2 \to \mathbb{R}$ increases in the first variable and $\varphi(D) \subset U$, $f(D) \subset V$.

(i) If $\Delta\varphi \geq 0$, then

$$\min_{(t,s)\in D} \mathcal{F}\left(M_{12}(t, s), f(t, s)\right) \leq \mathcal{F}\left(A(\varphi(g_1, g_2)), g(A(g_1), A(g_2))\right).$$

(ii) In the case $\Delta\varphi \leq 0$, we have

$$\min_{(t,s)\in D} \mathcal{F}\left(M_{34}(t, s), g(t, s)\right) \leq \mathcal{F}\left(A(\varphi(g_1, g_2)), g(A(g_1), A(g_2))\right).$$

Proof: *(i)* Condition $\Delta\varphi \geq 0$, after Theorem 2.1 entails

$$M_{12}(A(g_1), A(g_2)) \leq E[\varphi(g_1, g_2)].$$

From increasing $\mathcal{F}(\cdot, v)$ it follows that

$$\mathcal{F}\left(A(\varphi(g_1, g_2)), g(A(g_1), A(g_2))\right) \geq \mathcal{F}\left(M_{12}(A(g_1), A(g_2)), g(A(g_1), A(g_2))\right).$$

Now $(A(g_1), A(g_2)) \in D$ ensures

$$\mathcal{F}\left(M_{12}(A(g_1), A(g_2)), g(A(g_1), A(g_2))\right) \geq \min_{(t,s)\in D} \mathcal{F}\left(M_{12}(t, s), g(A(g_1), A(g_2))\right)$$

and the Theorem is proved. ∎

The next statements follow from Theorem 3.1 for specially defined function $\mathcal{F}$.

Corollary 3.1: Assume $g_1, g_2 \in L$ such that for $\varphi : D \to \mathbb{R}$ we have $\varphi(g_1, g_2) \in L$ and suppose φ is a continuous concave function.

(i) If $\Delta\varphi \geq 0$ then

$$\varphi(A(g_1), A(g_2)) + \min_{(t,s)\in D} (M_{12}(s, t) - \varphi(t, s)) \leq A(\varphi(g_1, g_2)).$$

(ii) If $\Delta\varphi \leq 0$, then

$$\varphi(A(g_1), A(g_2)) + \min_{(t,s)\in D} (M_{34}(s, t) - \varphi(t, s)) \leq A(\varphi(g_1, g_2)).$$

(iii) If $\Delta\varphi \geq 0$ and $\varphi(D) > 0$, then

$$\min_{(t,s)\in D} \frac{M_{12}(t, s)}{\varphi(t, s)} \cdot \varphi(A(g_1), A(g_2)) \leq A(\varphi(g_1, g_2)).$$

(iv) In opposite, if $\Delta\varphi \leq 0$ together with $\varphi(D) > 0$, then

$$\min_{(t,s)\in D} \frac{M_{34}(t, s)}{\varphi(t, s)} \cdot \varphi(A(g_1), A(g_2)) \leq A(\varphi(g_1, g_2)).$$

Proof: To prove *(i)* and *(ii)* use $\mathcal{F}(x, y) = x - y$. For *(iii)* and *(iv)* take $\mathcal{F}(x, y) = \frac{x}{y}$. Then apply Theorem 3.1. ∎

The next Lemma is a consequence of the fact that $\alpha x + \beta y \leq \max\{x, y\}$ for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

Lemma 3.1: Let $g_1, g_2 \in L$ such that for continuous concave function $\varphi : D \to \mathbb{R}$, $\varphi(g_1, g_2)$ belongs to L. If $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$, then

(i) For $\Delta\varphi \geq 0$ we have:

$$(\alpha\lambda_1 + \beta\lambda_2)A(g_1) + (\alpha\mu_1 + \beta\mu_2)A(g_2) + \alpha\nu_1 + \beta\nu_2 \leq A(\varphi(g_1, g_2)).$$

(ii) In the case $\Delta\varphi \leq 0$, there is

$$(\alpha\lambda_3 + \beta\lambda_4)A(g_1) + (\alpha\mu_3 + \beta\mu_4)A(g_2) + \alpha\nu_3 + \beta\nu_4 \leq A(\varphi(g_1, g_2)).$$

For the next results we take assumption that ν_1 and ν_2 from (2) are of the opposite sign.

Proposition 3.1: Let $\varphi : D \to \mathbb{R}$ be a continuous concave function, let $g_1, g_2 \in L$ such that $\varphi(g_1, g_2) \in L$ and A as a linear mean on L.

(i) Condition $\Delta\varphi \geq 0$ together with presumption $\nu_1 \cdot \nu_2 < 0$ ensures

$$U_{12}A(g_1) + V_{12}A(g_2) \leq A(\varphi(g_1, g_2)),$$

whereby:

$$U_{12} = \frac{\nu_2\lambda_1 - \nu_1\lambda_2}{\nu_2 - \nu_1},\ V_{12} = \frac{\nu_2\mu_1 - \nu_1\mu_2}{\nu_2 - \nu_1}.$$

(ii) Condition $\Delta\varphi \leq 0$ with $\nu_3 \cdot \nu_4 < 0$ gives

$$U_{34}Ag_1 + V_{34}Ag_2 \leq A(\varphi(g_1, g_2)),$$

whereby

$$U_{34} = \frac{\nu_4\lambda_3 - \nu_3\lambda_4}{\nu_4 - \nu_3},\ V_{34} = \frac{\nu_4\mu_3 - \nu_3\mu_4}{\nu_4 - \nu_3}.$$

Proof: For *(i)* it is enough to solve the system $\begin{cases} \alpha + \beta &= 1 \\ \alpha\nu_1 + \beta\nu_2 &= 0. \end{cases}$ and apply Lemma 3.1 ■

Specially defined function $\mathcal{F}$ appears in the next corollaries:

Corollary 3.2: Let $\varphi : D \to \mathbb{R}$ be continuous concave positive function. Let $g_1, g_2 \in L$ such that $\varphi(g_1, g_2) \in L$ and A is a linear mean on L.

(i) Case $\Delta\varphi \geq 0$ under the condition $\nu_1 \cdot \nu_2 < 0$ gives

$$\min_{(t,s)\in D} \frac{U_{12}t + V_{12}s}{\varphi(t, s)} \cdot \varphi(A(g_1), A(g_2)) \leq A(\varphi(g_1, g_2)).$$

(ii) Case $\Delta\varphi \leq 0$ under the condition $\nu_3 \cdot \nu_4 < 0$ gives

$$\min_{(t,s)\in D} \frac{U_{34}t + V_{34}s}{\varphi(t, s)} \cdot \varphi(A(g_1), A(g_2)) \leq A(\varphi(g_1, g_2)).$$

IV. An Example

Using results given in the previous section, as an example, a conversion of Hölder inequality is proved.

Theorem 4.1 (General Gheorghiu inequality): Let $g_1(\Omega) \subset [a, A]$ and $g_2(\Omega) \subset [b, B]$ for positive real numbers a, b and take positive real numbers p, q such that $\frac{1}{p} + \frac{1}{q} = 1$ holds. Under these presumptions the following is valid:

$$\frac{p^{\frac{1}{p}} q^{\frac{1}{q}} (abAB)^{\frac{1}{pq}} \left((AB)^{\frac{1}{p}} - (ab)^{\frac{1}{p}}\right)^{\frac{1}{p}} \left((AB)^{\frac{1}{q}} - (ab)^{\frac{1}{q}}\right)^{\frac{1}{q}}}{AB - ab} \cdot (A(g_1))^{\frac{1}{p}} (A(g_2))^{\frac{1}{q}} \leq A\left(g_1^{\frac{1}{p}} g_2^{\frac{1}{q}}\right).$$

Proof: Function $\varphi(x, y) = x^{\frac{1}{p}} y^{\frac{1}{q}}$ is continuous, concave and $\varphi(x, y) > 0$ for all $(x, y) \in D = [a, A] \times [b, B]$. Furthermore, $\left(A^{\frac{1}{p}} - a^{\frac{1}{p}}\right)\left(B^{\frac{1}{q}} - b^{\frac{1}{q}}\right) > 0$.

Presumption $\nu_1 \cdot \nu_2 \leq 0$ from Theorem 2.1 is a consequence of Lagrange mean-value theorem for differentiable function, ensuring

$$\frac{1}{p}a^{\frac{1}{p}-1} \geq \frac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A - a} \geq \frac{1}{p}A^{\frac{1}{p}-1} \tag{3}$$

and

$$\frac{1}{q}b^{\frac{1}{q}-1} \geq \frac{B^{\frac{1}{q}} - b^{\frac{1}{q}}}{B - b} \geq \frac{1}{p}A^{\frac{1}{p}-1}.$$

The inequalities in (3) give:

$$\nu_2 \leq A^{\frac{1}{p}} B^{\frac{1}{q}} - \frac{A^{\frac{1}{p}} B^{\frac{1}{q}}}{p} - \frac{A^{\frac{1}{p}} B^{\frac{1}{q}}}{q} = A^{\frac{1}{p}} B^{\frac{1}{q}} \left(1 - \frac{1}{p} - \frac{1}{q}\right) = 0.$$

and

$$\nu_1 \geq a^{\frac{1}{p}} b^{\frac{1}{q}} - \frac{a^{\frac{1}{p}} b^{\frac{1}{q}}}{p} - \frac{a^{\frac{1}{p}} b^{\frac{1}{q}}}{q} = a^{\frac{1}{p}} b^{\frac{1}{q}} \left(1 - \frac{1}{p} - \frac{1}{q}\right) = 0.$$

It remains to minimize the function $\frac{U_{12}t + V_{12}s}{\varphi(t, s)}$:

$$\min_{(t,s)\in D} \frac{U_{12}t + V_{12}s}{t^{\frac{1}{p}} s^{\frac{1}{q}}} = \min_{(t,s)\in D} \left(U \cdot \left(\frac{t}{s}\right)^{\frac{1}{q}} + V \cdot \left(\frac{s}{t}\right)^{\frac{1}{p}}\right).$$

Differential calculus renders the minimum in $\left(\frac{t}{s}\right)_{min} = \frac{V \cdot q}{U \cdot p}$, the points on the straight-line $\frac{t}{s} = \frac{Vq}{Up}$ inside rectangular D.

The minimum value is $U^{\frac{1}{p}} V^{\frac{1}{q}} p^{\frac{1}{p}} q^{\frac{1}{q}}$. Substituting U_{12} and V_{12} from Remark 3.1 gives as follows:

$$U_{12} = \frac{B^{\frac{1}{q}} b^{\frac{1}{q}} \left((AB)^{\frac{1}{p}} - (ab)^{\frac{1}{p}}\right)}{AB - ab}$$
$$V_{12} = \frac{A^{\frac{1}{p}} a^{\frac{1}{p}} \left((AB)^{\frac{1}{q}} - (ab)^{\frac{1}{q}}\right)}{AB - ab}$$

and proof is finished. ■

Direct consequence of Theorem 4.1 is presented in [6] and [8] as Gheorghiu inequality for specially boarded values of random variables.

V. Estimations of Jensen's functionals

In [4] we have considered Jensen's functional:

$J(\varphi, \mathbf{f}, \gamma; A) := A(\gamma\varphi(\mathbf{f})) - A(\gamma)\varphi\left(\frac{A(\gamma\mathbf{f})}{A(\gamma)}\right)$, where φ is continuous, convex function on a convex set $K \subseteq \mathbb{R}^n$. For a linear mean A and $f_1, \ldots, f_n \in L$ in [1, p. 48] is defined: $A(\mathbf{f}) = A(f_1, \ldots, f_n) = (A(f_1), \ldots, A(f_n))$. $\gamma \in L$ is a non-negative weight function. Real constants m and M are such that for non-negative $p, q \in L$ and for all $t \in \Omega$ the next inequalities hold

$$\begin{aligned} &p(t) - mq(t) \geq 0,\ Mq(t) - p(t) \geq 0, \\ &A(p) - mA(q) > 0,\ MA(q) - A(p) > 0. \end{aligned} \tag{4}$$

Next we rewrite the main Theorem from [4] as Theorem 5.1.

Theorem 5.1: Besides the mentioned above, assume that for all functions $f_i \in L, i = 1, ..., n$, mapping $\mathbf{f}(x) = (f_1(x), ..., f_n(x))$ is in K for all $x \in \Omega$. If the components of $p\mathbf{f}, q\mathbf{f}, \varphi(\mathbf{f}), q\varphi(\mathbf{f}), p\varphi(\mathbf{f})$ are in the class L, then $\varphi\left(\frac{A(q\mathbf{f})}{A(q)}\right)$ and $\varphi\left(\frac{A(p\mathbf{f})}{A(p)}\right)$ are well defined if $A(p) \neq 0$, and $A(q) \neq 0$. And the next inequalities hold:

$$M\ J(\varphi, \mathbf{f}, q; A) \geq J(\varphi, \mathbf{f}, p; A) \geq m\ J(\varphi, \mathbf{f}, q; A).$$

The inequalities are reversed if the function φ is concave.

In [4] we consider generalized means $M_\chi(\varphi(\mathbf{f}), w; A)$ for a function $\mathbf{f} = (f_1, f_2, ..., f_n) : \Omega \to \mathbb{R}^n$, function φ of n variables, with respect to the isotonic positive linear functional A and a continuous and strictly monotonic function $\chi : I \to \mathbb{R}$.

$$M_\chi(\varphi(\mathbf{f}), w; A) = \chi^{-1}\left(\frac{A(w\chi(\varphi(\mathbf{f})))}{A(w)}\right), \quad \chi(\varphi(\mathbf{f}(x))) \in L.$$

The next Theorem is also proved in [4].

Theorem 5.2: Let $A : L \to \mathbb{R}$ be a linear mean. Let $\chi, \psi_i : I \to \mathbb{R}$, $i = 1, ..., n$ be continuous and strictly monotonic functions, and let φ be a function of n variables. Moreover, let m and M be real constants such that the (4) hold for $p, q \in L$. If we suppose that the function $H(s_1, s_2, ..., s_n) = \chi \circ \varphi(\psi_1^{-1}(s_1), ..., \psi_n^{-1}(s_n))$ is convex then for every $\mathbf{g} = (g_1, g_2, ..., g_n) : \Omega \to \mathbb{R}^n$, such that the functions $\psi_i(g_i), p\psi_i(g_i), q\psi_i(g_i), \chi(\varphi(\mathbf{g}))$ are in L, we have that $H\left(\frac{A(p\psi_1(g_1))}{A(p)}, \ldots, \frac{A(p\psi_n(g_n))}{A(p)}\right)$ and $H\left(\frac{A(q\psi_1(g_1))}{A(q)}, \ldots, \frac{A(q\psi_n(g_n))}{A(q)}\right)$ are well defined.

And the next inequalities hold:

$$\begin{aligned} & MA(q) \cdot [\chi(M_\chi(\varphi(\mathbf{g}), q; A)) \\ & -\chi(\varphi(M_{\psi_1}(g_1, q; A), .., M_{\psi_n}(g_n, q; A)))] \\ \geq\ & A(p) \cdot [\chi(M_\chi(\varphi(\mathbf{g}), p; A)) \\ & -\chi(\varphi(M_{\psi_1}(g_1, p; A), .., M_{\psi_n}(g_n, p; A)))] \\ \geq\ & mA(q) \cdot [\chi(M_\chi(\varphi(\mathbf{g}), q; A)) \\ & -\chi(\varphi(M_{\psi_1}(g_1, q; A), .., M_{\psi_n}(g_n, q; A)))] \end{aligned}$$

The inequalities are reversed if the function H is concave.

In the following two corollaries of Theorem 5.2 we give extensions of the multiplicative type inequality and the additive type inequality investigating in [10] and [4].

Corollary 5.1: Assume that $\varphi(x, y, z) = x + y + z$. Let $M, m, p, q, \chi, g_i, \psi_i$ be as in Theorem 5.2 for $n = 3$ and $H(s_1, s_2, s_3) = \chi(\psi_1^{-1}(s_1) + \psi_2^{-1}(s_2) + \psi_3^{-1}(s_3))$. Moreover, let

$$F_1 = \frac{\psi_1^{'}}{\psi_1^{''}}, \quad F_2 = \frac{\psi_2^{'}}{\psi_2^{''}}, \quad F_3 = \frac{\psi_3^{'}}{\psi_3^{''}} \quad \text{and} \quad G = \frac{\chi'}{\chi''}.$$

If $\psi_1^{'}, \psi_2^{'}, \psi_3^{'}, \chi'$ are positive and $\psi_1^{''}, \psi_2^{''}, \psi_3^{''}, \chi''$ are negative, then $H(s_1, s_2, s_3)$ is convex and

$$\begin{aligned} & MA(q) \cdot [\chi(M_\chi(g_1 + g_2 + g_3, q; A)) \\ & -\chi(M_{\psi_1}(g_1, q; A) + M_{\psi_2}(g_2, q; A) + M_{\psi_3}(g_3, q; A))] \\ \geq\ & A(p) \cdot [\chi(M_\chi(g_1 + g_2 + g_3, p; A)) \\ & -\chi(M_{\psi_1}(g_1, p; A) + M_{\psi_2}(g_2, p; A) + M_{\psi_3}(g_3, p; A))] \\ \geq\ & mA(q) \cdot [\chi(M_\chi(g_1 + g_2 + g_3, q; A)) \\ & -\chi(M_{\psi_1}(g_1, q; A) + M_{\psi_2}(g_2, q; A) + M_{\psi_3}(g_3, q; A))] \end{aligned}$$

hold iff $G(g_1 + g_2 + g_3) \leq F_1(g_1) + F_2(g_2) + F_3(g_3)$.

If all of $\psi_1^{'}, \psi_2^{'}, \psi_3^{'}, \chi'$, $\psi_1^{''}, \psi_2^{''}, \psi_3^{''}, \chi''$ are positive, then $H(s_1, s_2, s_3)$ is concave and the inequalities are reversed iff $G(g_1 + g_2 + g_3) \geq F_1(g_1) + F_2(g_2) + F_3(g_3)$.

Corollary 5.2: Assume the function $\varphi(x, y) = x \cdot y \cdot z$. Let $M, m, p, q, \chi, g_i, \psi_i$ be as in Theorem 5.2 for $n = 3$ and $H(s_1, s_2, s_3) = \chi(\psi_1^{-1}(s_1) \cdot \psi_2^{-1}(s_2) \cdot \psi_3^{-1}(s_3))$. Moreover, let

$$B_1(x) = \frac{\psi_1'(x)}{\psi_1'(x) + x\psi_1''(x)}, \; B_2(x) = \frac{\psi_2'(x)}{\psi'(x) + x\psi_2''(x)},$$

$$B_3(x) = \frac{\psi_3'(x)}{\psi'(x) + x\psi_3''(x)} \text{ and } C(x) = \frac{\chi'(x)}{\chi'(x) + x\chi''(x)}.$$

If g_1, g_2, g_3, χ' are positive and $B_1(g_1), B_2(g_2), B_3(g_3)$ and $C(g_1 g_2 g_3)$ are negative, then the function $H(s_1, s_2, s_3)$ is convex and

$$\begin{aligned} & MA(q) \cdot [\chi(M_\chi(g_1 \cdot g_2 \cdot g_3, q; A)) \\ & -\chi(M_{\psi_1}(g_1, q; A) \cdot M_{\psi_2}(g_2, q; A) \cdot M_{\psi_3}(g_3, q; A))] \\ \geq\ & A(p) \cdot [\chi(M_\chi(g_1 \cdot g_2 \cdot g_3, p; A)) \\ & -\chi(M_{\psi_1}(g_1, p; A) \cdot M_{\psi_2}(g_2, p; A) \cdot M_{\psi_3}(g_3, p; A))] \\ \geq\ & mA(q) \cdot [\chi(M_\chi(g_1 \cdot g_2 \cdot g_3, q; A)) \\ & -\chi(M_{\psi_1}(g_1, q; A) \cdot M_{\psi_2}(g_2, q; A) \cdot M_{\psi_3}(g_3, q; A))] \end{aligned}$$

hold iff $C(g_1 \cdot g_2 \cdot g_3) \leq B_1(g_1) + B_2(g_2) + B_3(g_3)$.

If $g_1, g_2, g_3, \chi', B_1(g_1), B_2(g_2), B_3(g_3), C(g_1 g_2 g_3)$ are positive then the function $H(s_1, s_2, s_3)$ is concave and the inequalities are reversed iff $C(g_1 \cdot g_2 \cdot g_3) \geq B_1(g_1) + B_2(g_2) + B_3(g_3)$.

Applications of two-variables cases are presented in [4] for some elementary functions. Further studies can be taken in the direction of expanding the function φ on more than variables.

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