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A Structuralist Account of Logic

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The lynch-pin of the structuralist account of logic endorsed by Koslow is the definition of logical and modal operators with respect to implication relations, i.e. relative to implication structures. Logical operators are depicted independently of any possible semantic or syntactic limitations. It turns out that it is possible to define conjunction as well as other logical operators much more generally than it has usually been, and items on which the logical operators may be applied need not be syntactic objects and need not have truth values.
In this paper I analyse Koslow's structuralist theory and point out certain objectionable aspects to as well as reasons why such a theory does not fulfil the (possibly unjustified) expectation of getting defined a universal logical structure.

Key words: logic, Koslow's theory, structure, implication relation, logical operators

Introduction

What does a structuralist account of logic amount to? And what does *structuralism* mean in the domain of logic? One of the most appealing contemporary formulations of the answer to the questions just asked is Koslow's structuralist account of logic[1].
In this paper I firstly present the basic tenets of his theory; secondly, I discuss it and give reasons for rejecting some aspects of it and thirdly, I defend the view that such a structuralist account of logic, even though most appealing, does not satisfactorily answer the questions posed above.
Let us begin with a brief look at how structuralism applies to the paradigmatic cases of astronomy and mathematics and see how, if at all, logic has something in common with these two fields.
In the history of astronomy apparently opposing theories have been endorsed about the structure of the Solar System, e.g. the heliocentric and geocentric system and their respective theses on having either the Sun or the Earth as the centre of the Solar

[1] Koslow (1992), (2007).

System. That these theories were mutually exclusive, so could not possibly both be right, seemed to be more than plain common sense. But, surprisingly and counter-intuitively, these theories turned out to share not just the same mathematical apparatus but also to be translatable into equivalent mathematical theories. Analogously with the analyses of the points in a plane in which the coordinates of a point change depending on what point we choose as the origin of the coordinate system, the mathematical translation of astronomical theories depends on what we choose as the "origin" of the system: the Earth or the Sun. Mathematically we can decide to do either - and in this sense there is no "right answer".[2] The theories share the same (mathematical) structure.

Do we encounter theories sharing the same structure in other domains of interest as well? To a certain extent, yes. In mathematics, the science of structures *par excellence*, there are many interesting philosophical consequences (both ontological and epistemological) of the fact that mathematics *is* basically about structures: the natural number structure, the group structure, the vector space structure, etc.; and the ontological reduction of mathematical objects to structures aims to solve same basic ontological (as well as epistemological) problems concerning different versions of realism in the philosophy of mathematics.[3]

Nevertheless, the problems and approach in mathematics are slightly different and much more complex than the case in astronomy. Structuralism in mathematics is twofold: in one sense different branches and theories are about different structures, in another the possibility of the reduction of different theories to set theory makes us think of one common set-theoretic structure.

What about logic? Is it possible that logics share a universal structure? Is there any analogy with the case of astronomy or mathematics, and would it be possible to reduce different logical theories to a common structure? As is very well known, in logic there are different theories that hold opposing views even on some very basic topics. Examples are legion. We might mention the case of intuitionist logic rejecting some basic rules of classical logic such as the *tertium non datur* or double negation; or that of relevance logic introducing certain constraints absolutely absent in classical logic.

[2] In the same way in which we choose in mathematics the origin of a coordinate system depending on what result we aim to get and what objects we aim to depict or analyse.
[3] See, e.g. Shapiro (1997), Resnik (1997), Hellman (2001).

Does it mean that the proposal of a common logical structure and, consequently, of a universal logic, is destined to fail? In this paper I will try to answer this question through the analysis of one of the most prominent structuralist theories of logic: Koslow's structuralist account of logic.

Koslow's structuralist theory of logic

The basic idea in Koslow's structuralist theory of logic amounts to the introduction of the notion of *implication structure*, and the development of a theory of the operators by depicting how such operators act on implication structures.

What is an implication structure? It is any order pair: ((S, $\Rightarrow$); where S is a non-empty set, and "$\Rightarrow$" is an implication relation.

An implication relation is any relation that satisfies the following conditions:

(1) *Reflexivity*: $A \Rightarrow A$, for every A in S

(2) *Projection*: $A_1, A_2, \ldots, A_n \Rightarrow A_k$, for every k=1, …, n, and for all A_i in S (i = 1, …,n)

(3) *Simplification*: If $A_1, A_1, A_2, \ldots, A_n \Rightarrow B$, then $A_1, A_2, \ldots, A_n \Rightarrow B$, for each A_i (i = 1, …,n) and B in S

(4) *Permutation*: If $A_1, A_2, \ldots, A_n \Rightarrow B$, then $A_{f(1)}, A_{f(2)}, \ldots, A_{f(n)} \Rightarrow B$, for any permutation f of {1, …,n}

(5) *Dilution* (or *Thinning*): If $A_1, A_2, \ldots, A_n \Rightarrow B$, then $A_1, A_2, \ldots, A_n, C \Rightarrow B$, for every A_i (i = 1, …,n), B and C in S

(6) *Cut*: If $A_1, A_2, \ldots, A_n \Rightarrow B$, and $B, B_1, B_2, \ldots, B_m \Rightarrow C$, then $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m \Rightarrow C$, for every A_i, B_j, B and C (i, j = 1, …,n)

It can easily be noted that certain conditions follow from others[4]; Koslow nevertheless keeps such a formulation for the sake of greater articulateness.

Apart from standard examples of implication relations such as the notion of (semantic) consequence and the (syntactic) notion of deducibility for a set of sentences of first-order logic, there are other examples as well, e.g. set inclusion, given a non-empty set of sets.

Implication structures are, in fact, with respect to their generality, comparable to equivalence relations and, analogously to the latter, such structures can be more or less mathematically, logically, scientifically or philosophically interesting and fruitful.

[4] *Reflexivity* follows from *Projection*; *Dilution* follows from *Projection* and *Cut*.

One of the tenets of Koslow's theory is also the characterization of logical operators relative to an implication structure. How do the logical operators interact with respect to implication? Any explanation of the logical operators, as well as modal operators, is based *exclusively* on the notion of an *implication structure* or, more precisely, such operators are functions defined on implication structures.

As Koslow (1994, 4) points out:

> The answer we shall give to the question of what it is like for an item to belong to a certain logical category appeals solely to the role that the element plays with respect to the implication relation of the structure under consideration. Because the elements of an implication structure need not be syntactical objects having a special sign design, and they need not have some special semantic value, an explanation of what can count as hypothetical, disjunctions, negations, and quantifies items (existential or universal) can proceed in a way that is free of such restrictions.

So the basic idea of Koslow's theory amounts to having logical operators depicted independently of any possible semantic or syntactic limitations. Logical operators (conjunction, disjunction, negation, quantifications, etc.) are determinate functions defined upon implication structures.

Let us take the example of the *conjunction* operator. Given an implication structure $(S, \Rightarrow)$, the conjunction operator is a function C that assigns to any elements A and B of S, a subset $C_{\Rightarrow}(A, B)$ of S containing all those members (if any)[5] that satisfy the following conditions:

(C1) $C_{\Rightarrow}(A, B) \Rightarrow A$ and $C_{\Rightarrow}(A, B) \Rightarrow B$[6]

(C2) $C_{\Rightarrow}(A, B)$ is the weakest member of the implication structure that satisfies (C1);

i.e. if T is any member of the implication structure such that $T \Rightarrow A$, $T \Rightarrow B$,

then $T \Rightarrow C_{\Rightarrow}(A, B)$.

Such a concept of conjunction is based on – even though it is much more general than – the one determined by the standard Gentzen-style Introduction and Elimination rules[7] that concern the conjunction operator ('$\wedge$'). The results are analogous for other logical operators (in first-order logic) in the sense that standard Introduction and Elimination rules are replaced by more general conditions.

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[5] There are structures in which conjunctions exist, and others in which they do not exist. See Koslow (1992, 108-109).

[6] The notation $C_{\Rightarrow}(A, B)$ is used to denote both a subset of S and its elements (that are all equivalent with respect to the implication relation).

[7] Introduction rule: $A, B \Rightarrow A \wedge B$; Elimination rules: $A \wedge B \Rightarrow A$, and $A \wedge B \Rightarrow B$.

So, even though Koslow uses the same rules (1)-(6) for an implication relation that Gentzen does, he poses different, more general conditions for the logical operators. And he does so because the Introduction and Elimination rules do not allow the generality that Koslow's definitions of conjunction and other logical operators do. As Koslow indicates, pointing to the parity between the account of the implication structures and logical operators:

> On our theory, parity is restored, and the account of the operators is as abstract as the theory of implication upon which it rests.[8]

Apart from the missing generality, Gentzen's rules imply a certain ontological commitment to the existence of the logical operators while Koslow refrains from giving the operators an existential character. Since any logical operator is defined in dependence on the role it plays with respect to an implication structure, it can appear differently given a different implication relation of a structure.

The richness of Koslow's theory reveals itself especially in the finding that logical operators can be identified in non-standard implication structures too. Non-standard examples include logical operators defined on structures of sets, names and interrogatives.[9]

Let us take the example of set theory, the implication structure being (S, $\Rightarrow$) where S is a non-empty set of sets and the implication relation is set inclusion '$\subseteq$'. In this case the conjunction of two sets A and B, is some set $C_{\Rightarrow}(A, B)$[10] such that:

(C1) $C_{\Rightarrow}(A, B) \subseteq A$ and $C_{\Rightarrow}(A, B) \subseteq B$

(C2) $C_{\Rightarrow}(A, B)$ is the weakest member of the implication structure that satisfies (C1);

i.e. if T is any element of S such that $T \subseteq A$, $T \subseteq B$, then $T \subseteq C_{\Rightarrow}(A, B)$.

Given the definition, $C_{\Rightarrow}(A, B)$ corresponds to $A \cap B$. Hence, it turns out that the intersection of two sets is not just analogous to conjunction – it *is* the conjunction[11].

And it also turns out that it is possible to define conjunction much more generally than it has usually been, by doing it on objects that are not truth bearers. So, items on

[8] See Koslow (1992, 14-17).

[9] See Koslow (1992, 209-222).

[10] $C_{\Rightarrow}(A, B)$ does not necessarily exist.

[11] The disjunction of two sets turns out to be the their union, the negation of any set A its complement S-A, etc. For more details see Koslow (1992, 77-127).

which the conjunction operator may be applied need not be syntactic objects and need not have truth values[12].

Given that operators are defined as special functions *relative to* implication relations, such relativization to implication structures make the problems of stability and distinctness naturally arise. The first problem concerns the following issue: if $C_{\Rightarrow}(A, B)$ is a conjunction in one implication structure, will it be a conjunction in a different implication structure in which the same elements are included? And the answer is positive: given an implication structure, any conjunction of two elements will still be the conjunction of those elements in every conservative extension of the structure[13].

The problem of distinctness is due to the general character of any implication structure and it amounts to distinguishing the operators from each other. Koslow defines two operators as distinct (given an implication structure) if and only if for some items in the structure the two yield at least two nonequivalent members of the structure.

The structuralist account: Why not?

Even though the structuralist account of logic that Koslow develops appears to be logically and philosophically most appealing, there are certain objectionable aspects.

It has already been mentioned that set inclusion (given an non-empty set of sets) is an example of implication relation. Let us have a closer look at this example. Set inclusion should and do satisfy the conditions (1)-(6) for any implication relation. And the conjunction of two sets turns out to be their intersection. So, given e.g. *Projection*[14], the question arises as to what the left-hand side of the condition - A_1, A_2, …, A_n - amounts to. The answer is that such left-hand side is precisely the intersection of sets: $A_1 \cap A_2 \cap \ldots \cap A_n$. It turns out then, that we ought to know what the intersection of sets is prior to having defined the implication relation on sets even though formally the definition of conjunction, i.e. intersection should follow the one of implication structure. Hence, such definition of implication relation turns out to be circular.

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[12] Koslow's way of defining the logical operators has also the interesting advantage of getting possible *the conjunction of different theories* (either in logic or physical sciences), since the standard requirement of the conjunction being defined only to sentences does not longer exist.

[13] For more details see Koslow (1992, 389).

[14] *Projection*: $A_1, \ldots, A_n \Rightarrow A_k$, for each $k=1,\ldots n$

One of the most interesting parts of the structuralist programme concerns some aspects of erotetic logic, the logic of questions and answers. Koslow (1992, 223-225) defines an implication relation for interrogatives in the following way:

Let Q be a collection of interrogatives (every question is denoted by a capital letter followed by a question mark), and S a set of sentences inclusive of the sentential direct answers to the questions in Q. We denote their union with S^*: $S^* = S \cup Q$; while '$\Rightarrow$' is an implication relation on sentences of S.

What needs to be defined is an implication relation '$\Rightarrow_q$' on the set S^*, that involves just the questions of Q, or any combination of questions in Q and statements in S.

Let $M_1?$, $M_2?$, …, $M_n?$ and $R?$ be questions in Q, and let F_1, F_2, …, F_m and G be statements of S (the set of M's or the set of F's may be empty but not both), and let A_i be a direct answer to the question $M_i?$ (i=1,…,n); then

(1.) $F_1, F_2, \ldots, F_m, M_1?, M_2?, \ldots, M_n? \Rightarrow_q R?$ if and only if there is some direct answer B to the question $R?$ such that

$F_1, F_2, \ldots, F_m, A_1, A_2, \ldots, A_n \Rightarrow B$

(2.) $F_1, F_2, \ldots, F_m, M_1?, M_2?, \ldots, M_n? \Rightarrow_q G$ if and only if

$F_1, F_2, \ldots, F_m, A_1, A_2, \ldots, A_n \Rightarrow G$

There are certain unclarities in such a definition. Let us take the example in which the set S is a collection of sentences in classical propositional logic where the implication relation is the standard semantic notion of logical consequence. Let Q be the collection of interrogatives that, among others**,** includes e.g. $M_1?$: *"How many satellites does the planet Earth have?"* and $R?$: *"Does the number '3' solve the equation 'x-4=0'?"* and let us say, for the sake of simplicity, that the set of *F's* is empty[15]. The set S is infinite, there are infinitely many possible direct answers to the question $M_1?$, given that the direct answer need not be the correct one; as Koslow (1992, 220) points out:

> We shall use the term "interrogative" to include any question that has a *direct answer*. The most important feature of the direct answers to a question is that they are statements that, whether they are true or false, tell the questioner exactly what he wants to know[16] – neither more nor less.

[15] Since the set of *M's* is non-empty, the set of *F's* may be empty (see the definition above).

[16] The specified feature of direct answers is interesting since it controversially includes false answers in what the questioner ***exactly*** wants to know - unlike Belnap and Steel, according to whom 'the direct answer…is what counts as completely, but just completely, answering the question' (Belnap and Steel,

According to the definition of an implication relation '$\Rightarrow_q$', it follows that:

$M_1? \Rightarrow_q R?$ if and only if there is some direct answer B to the question $R?$ such that $A_1 \Rightarrow B$.

Whether A_1 is a logical consequence of B or not, depends on what answer A_1 (to the question $M_1?$) we decide to choose[17]. Given the possibility of choosing a wrong answer (i.e., 'The planet Earth has *n* satellites', where *n* is any natural number different from 1), and given the possibility to do the same for any question $M_i?$ it turns out that for any questions $M_i?$, $R?$ with a direct answer, we get: $M_1? \Rightarrow_q R?$. What the application of such a definition is, and what its fruitfulness amounts to, remains unclear.

The case in which a question implies a statement (the second condition in the definition) is slightly different. Let us take the question to be the same as before - $M_1?$: "*How many satellites does the planet Earth have?*" and the statement G to be any false statement, e.g. the false answer to the previous question $R?$: "*Yes, the number '3' solves the equation 'x-4=0'*". In this case, whether $M_1? \Rightarrow_q G$ or not depends on whether $A_1 \Rightarrow G$, and the latter depends on what answer A_1 (to the question $M_1?$) we choose. If the answer we choose is a false one, then $M_1? \Rightarrow_q G$, otherwise $M_1? \not\Rightarrow_q G$. More generally, the same problem appears whenever the statement G is a false one. In this case, given a collection of interrogatives $M_i?$ (i=1,…n), their respective direct answers A_i, and a set of true statements F_i (i=1,…n), there is nothing in Koslow's definition that allows us to uniquely determine whether $F_1, F_2, \ldots, F_m, M_1?, M_2?, \ldots, M_n? \Rightarrow_q G$ or not, or to rule out the possibility of discussing it in the first place. As soon as we choose at least one false answer, it follows that

$A_i, F_1, F_2, \ldots, F_m, M_1?, M_2?, \ldots, M_n? \Rightarrow_q G$;

while by choosing all the correct answers we get:

$F_1, F_2, \ldots, F_m, M_1?, M_2?, \ldots, M_n? \not\Rightarrow_q G$.

1976, 13). It is not clear in what contexts, if any, the questioner would exactly want to know a *wrong* answer to a question posed.

[17] A_1 is, according to the definition, *any* answer to the question $M_1?$.

Apart from the details that have just been referred to concerning Koslow's structuralist account of logic, there are certain more general concerns that should be brought to the surface.

Does Koslow's theory prove that there is a unique logical structure? Well, the fact that Koslow shows how certain logical notions are universally present in many extra-logical theories does not prove *per se* that there is a universal logical structure and that different logics exemplify such a structure in the same way.

Koslow's programme might seem to be analogous to what the standard mathematical practice is, in the sense of defining a determinate structure that can be exemplified by completely different systems. Let us take the example of vector space. Utterly different objects – e.g. either geometric vectors or real numbers[18] - count as vectors[19], in the same way in which e.g. either the standard conjunction in classical propositional logic (that have the sign '$\wedge$') or the intersection of sets, both count as conjunctions $C_{\Rightarrow}(A, B)$. How far does the analogy go?

The theory of vector spaces determines not just what a vector space (over a field) is, but it also allows the projection of many other properties from the structure to single templates (or systems), e.g. a base for every finitely dimensional vector space.

It is not the case with Koslow's theory of implication structure. Let us observe one example. According to the definition,[20] the conjunction operator C on an implication structure $(S, \Rightarrow)$, is a *function* which assigns to any two elements A and B of S, a conjunction of them, i.e. a subset $C_{\Rightarrow}(A, B)$ of S:

$$C:\ (S, \Rightarrow) \ \rightarrow\ (S, \Rightarrow)$$

$$C:\ A,\ B\ \mapsto\ C_{\Rightarrow}(A, B)$$

$C_{\Rightarrow}(A, B)$ is the subset of all those elements of S (if they exist) that satisfy the conditions ($C1$) and ($C2$) we have already mentioned (see above). Let us have a look at the example in which we take $C_{\Rightarrow}(A, B)$ to be, e.g., the standard logical conjunction operator in classical propositional logic in which the implication structure is the set of formulas of the language together with the 'standard' implication. In this case the conjunction is not defined, as it is usually the case, through its truth tables nor through the Elimination and Introduction rules. Given Koslow's definition, we ought to be

[18] Real numbers are interpreted formally as members of the real number vector space over the field of real numbers.
[19] Here 'vector' is used in the sense of an arbitrary element of a vector space.
[20] Koslow (2007, 170) and Koslow (1992, 6).

able to get such results for the conjunction operator out of Koslow's definition, because there is simply no other way in which we could do it. But, the conjunction is defined independently of any syntactic or semantic features, and it is unclear how this definition is to be combined with the syntactic rules for formula formation and (semantic) truth tables. Once the implication structure is defined and it turns out that the semantic concept of logical consequence in classical propositional logic exemplifies the structure as well as the conjunction '$\wedge$' fulfils the conjunction $C_{\Rightarrow}(A, B)$ requirements, none of the semantic properties of the conjunction '$\wedge$' follow from the structure. How can we define the truth table for it? How are such truth tables related to the characterisation of the operator within the theory?

Koslow very clearly endorses the view that 'the tasks of a logical theory of statements can be carried out without appeal to either syntax or semantics'[21]; nevertheless, in order to develop the logical theories we are interested in both syntax and semantics are necessary. Otherwise our classical logical theories are exemplified by Koslow's structures just fragmentally. And different logics turn out to have just partial fragments reducible to or exemplified by the same structure.

Certainly, while the result of reducing logical operators belonging to different domains to the same structure is remarkable, it nevertheless leaves us in dismay in respect of the expectation of getting a universal logical structure defined.

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[21] Koslow (1992, 219).