DYNAMIC ANALYSIS OF BEAM-LIKE STRUCTURES SUBJECT TO MOVING LOADS

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INTRODUCTION

The vehicle-induced vibration of bridges has been subject of interest for more than one and a half centuries. Due to tremendous growth of traffic, most of bridges are nowadays heavily loaded. The increase in traffic intensity and speed requires a more complex analysis of structures than in the past.

The simplest case of a moving-load (dynamic) analysis is the case of a simply supported beam over which a concentrated load is moving. The problem is represented with a 4^{th} order partial differential equation (PDE), which is nowadays usually solved numerically. While we use finite elements for discretization in space, the discretization in time is more conveniently handled by finite differences. Finite difference method converts the procedure of solving a differential equation into a procedure of solving a system of linear equations. The results are the displacement, velocity and acceleration for every finite element node at every time step.

DIRECT NUMERICAL INTEGRATION

Problem of a massless load moving on a beam is described by the well-known partial differential equation [1]

$$EI\,\frac{\partial^4 u(x.t)}{\partial x^4} + m\cdot\frac{\partial^2 u(x,t)}{\partial t^2} + c\cdot\frac{\partial u(x,t)}{\partial t} - P(x,t) = 0 \qquad (1)$$

where *u(x,t)* represents the displacement of the beam, *x* represents the travelling direction of the moving load, and *t* represents time. Also, *EI* is flexural rigidity of the beam, *E* is Young's modulus, *I* is second moment of area of the beam, *m* is mass of the beam per unit length, *c* is a viscous damping coefficient and *P(x,t)* is the applied external force.

The easiest way to solve the partial differential equation is by numerical integration in contrast to modal analysis, which applies only to linear analysis, the direct numerical integration can be used for both linear and nonlinear problems.

Discrete form of equation (1) in matrix notation is

$$\boldsymbol{M}\ddot{\boldsymbol{D}} + \boldsymbol{C}\dot{\boldsymbol{D}} + \boldsymbol{S}\boldsymbol{D} = \boldsymbol{A}(t) \qquad (2)$$

where $\boldsymbol{M}$ is the mass matrix, $\boldsymbol{C}$ is the damping matrix, $\boldsymbol{S}$ is the stiffness matrix, $\boldsymbol{D}$, $\dot{\boldsymbol{D}}$ and $\ddot{\boldsymbol{D}}$ are the displacement, velocity and acceleration vectors, and $\boldsymbol{A}$ is the external force.

Direct linear extrapolation procedures may be devised for both the average and linear acceleration methods. We choose the average acceleration method which is unconditionally stable although the linear acceleration method is somewhat more accurate. Writing equation (2) for two consecutive time steps and subtracting the results we obtain the following incremental form:

$$\boldsymbol{M}\Delta\ddot{\boldsymbol{D}}_{\boldsymbol{j}} + \boldsymbol{C}\Delta\dot{\boldsymbol{D}}_j + \boldsymbol{S}\Delta\boldsymbol{D}_j = \Delta\boldsymbol{A}_j \qquad (3)$$

where j is *j-th* time step, $\Delta\ddot{\boldsymbol{D}}_{\boldsymbol{j}}$, $\Delta\dot{\boldsymbol{D}}_j$, $\Delta\boldsymbol{D}$, are the incremental acceleration, velocity and displacement vectors and $\Delta\boldsymbol{A}_j$ is the incremental external force.

In this equation, we have unknown incremental accelerations, velocities and displacements. Introducing the assumption that the acceleration is constant within a time interval, their value can be deduced and substituted into eq. (3) to obtain:

$$\boldsymbol{M}\left[\frac{4}{(\Delta t_j)^2}\Delta\boldsymbol{D}_j - \overline{\boldsymbol{Q}}\right] + \boldsymbol{C}\left[\frac{2}{(\Delta t_j)^2}\Delta\boldsymbol{D}_j - \overline{\boldsymbol{R}}_{\boldsymbol{j}}\right] + \boldsymbol{S}\Delta\boldsymbol{D}_j = \Delta\boldsymbol{A}_j \qquad (4)$$

where

$$\overline{\boldsymbol{Q}}_{\boldsymbol{j}} = \frac{4}{\Delta t_j}\dot{\boldsymbol{D}}_{\boldsymbol{j}} + 2\ddot{\boldsymbol{D}}_{\boldsymbol{j}} \qquad (5)$$

$$\overline{\boldsymbol{R}}_{\boldsymbol{j}} = 2\dot{\boldsymbol{D}}_{\boldsymbol{j}}. \qquad (6)$$

We rewrite this equation in the form

$$\overline{\boldsymbol{S}}\Delta\boldsymbol{D}_j = \Delta\overline{\boldsymbol{A}}_j \qquad (7)$$

where

$$\overline{\boldsymbol{S}} = \boldsymbol{S} + \frac{4}{(\Delta t_j)^2}\boldsymbol{M} + \frac{2}{\Delta t_j}\boldsymbol{C} \quad \text{and} \quad \Delta\overline{\boldsymbol{A}}_{\boldsymbol{j}} = \Delta\boldsymbol{A}_{\boldsymbol{j}} + \boldsymbol{M}\overline{\boldsymbol{Q}}_{\boldsymbol{j}} + \boldsymbol{C}\overline{\boldsymbol{R}}_{\boldsymbol{j}}. \qquad (8)$$

Thus, the pseudostatic equation (7) is to be solved for the incremental displacements at each time step. The incremental velocities and accelerations may then be found using equations (9) and (10):

$$\Delta\dot{\boldsymbol{D}}_j = \frac{2}{(\Delta t_j)^2}\Delta\boldsymbol{D}_j - \overline{\boldsymbol{R}}_{\boldsymbol{j}} \qquad (9)$$

$$\Delta\ddot{\boldsymbol{D}}_j = \frac{4}{(\Delta t_j)^2}\Delta\boldsymbol{D}_j - \overline{\boldsymbol{Q}}_{\boldsymbol{j}} \qquad (10)$$

Finally, the total values of $\Delta \boldsymbol{D}_{j+1}, \Delta \dot{\boldsymbol{D}}_{j+1}$ and $\Delta \ddot{\boldsymbol{D}}_{j}$ are:

$$\Delta \boldsymbol{D}_{j+1} = \boldsymbol{D}_j + \Delta \boldsymbol{D}_j \tag{11}$$

$$\Delta \dot{\boldsymbol{D}}_{j+1} = \dot{\boldsymbol{D}}_j + \Delta \dot{\boldsymbol{D}}_j \tag{12}$$

$$\Delta \ddot{\boldsymbol{D}}_{j+1} = \ddot{\boldsymbol{D}}_j + \Delta \ddot{\boldsymbol{D}}_j . \tag{13}$$

COMPARISON BETWEEN NUMERICAL AND ANALYTICAL SOLUTION

Numerical solution is coded in MathCad 2001 using the direct linear extrapolation method with assumption of average acceleration within a time step. Numerical analysis has been preformed of undamped or damped simply supported beams. The supports have been modelled as pinned and movable or as undamped or damped springs.

SIMPLY SUPPORTED BEAM WITHOUT DAMPING

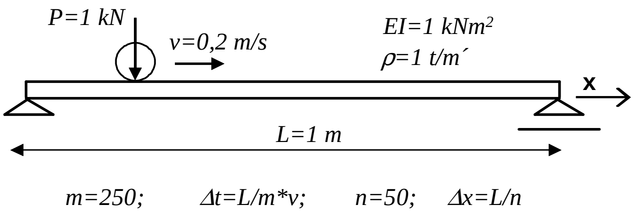


Fig. 1
Beam model

The numerical solution for the total beam displacement in time is shown in Fig. 2.

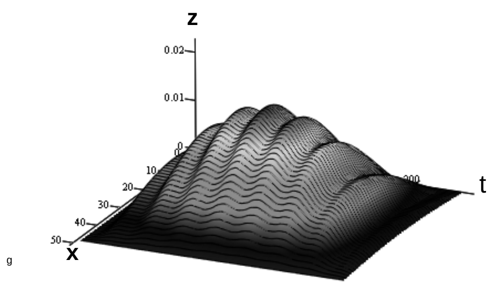


Fig. 2
Beam displacement (*z*-displacement, *x*-position on the beam, *t*-time)

3D picture in Fig. 2 shows the beam displacement in every finite element node (n = 0-50) in every time step (m = 0-250).
The analytical solution [5] is represented by

$$z(x,t)=\frac{2\cdot P}{\rho\cdot L\cdot\omega(1)^2}\left[\sum_{k=1}^{50}\left[\frac{1}{k^2(k^2-\alpha^2)}\cdot\left(\sin\left(k\cdot\pi\cdot v\cdot\frac{t}{L}\right)-\frac{\alpha}{k}\cdot\sin(\omega(k)\cdot t)\right)\cdot\sin\left(k\cdot\pi\cdot\frac{x}{L}\right)\right]\right] \quad (14)$$

where

$$\omega(k)=\frac{\sqrt{\frac{EI}{\rho}}\cdot k^2\cdot\pi^2}{L^2};\ \ \alpha=\pi\cdot\frac{v}{L\cdot\omega(1)} \quad (15)$$

Comparison of the results between the numerical and the analytical solution is shown in 2D plot (Fig. 3). As it can be seen, there is excellent agreement between the two solutions.

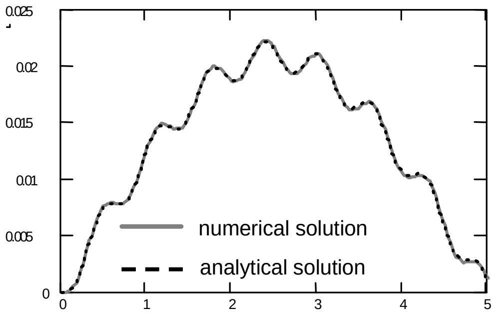
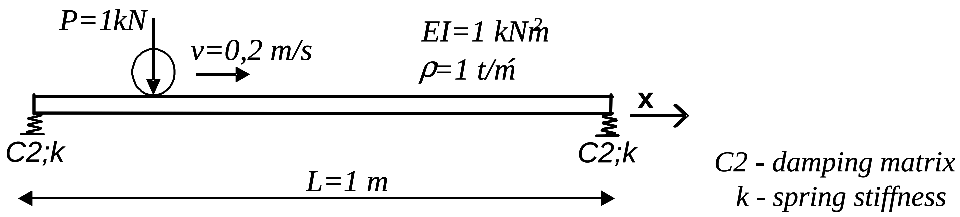


Fig 3.
Mid-point displacement in time

SIMPLY SUPPORTED BEAM WITH SPRINGS AT SUPPORTS

The bridge superstructure is nowadays usually placed on the neoprene bearings. The neoprene is a flexible structure, therefore it is not correct to neglect its vibration, and impact to the whole structure. The neoprene bearings are modelled as damped linear elastic springs. The problem is designed in Fig. 4.



*m=250; Δt=L/m*v; n=50; Δx=L/n* $C2_{0,0}=0{,}2$ $C2_{n,n}=0{,}2$ $C2_{i,i}=0$

Fig. 4
Beam model

If the springs are very stiff, there is little between the model with springs and the model with fixed supports. (Fig. 5)

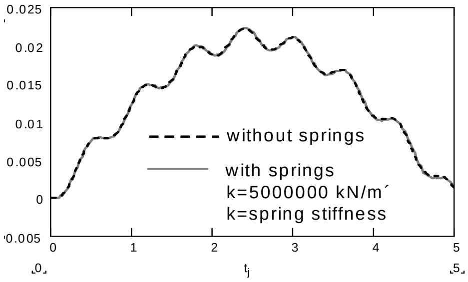


Fig 5.

The impact of a moderate spring stiffness to mid-point displacement can be seen from the diagrams in Fig. 6.

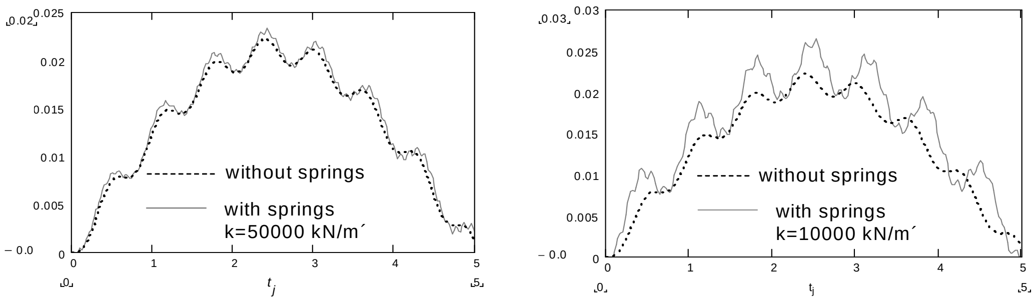


Fig. 6

SIMPLY SUPPORTED BEAM WITH STRUCTURAL DAMPING AND SPRINGS AT THE SUPPORTS

Structural damping is incorporated in numerical procedure and it is represented by matrix *C1*.

$$C1 = \alpha m \cdot M + \alpha k \cdot S\,; \qquad (\alpha m=0{,}1;\ \ \alpha k=0{,}002) \quad (16)$$

The vibration of the damped structure is reduced in comparison to the undamped model. It can be seen especially from the right part of the Fig. 7 (when moving load is out of the structure).

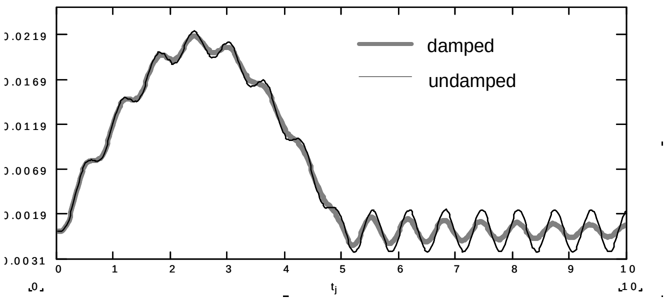


Fig. 7

The influence of the spring $k_{\frac{n}{2},j}$ at the mid-point displacement can be easily calculated from

$$k_{\frac{n}{2},j} = \frac{zz_{0,j} + zz_{n,j}}{2} \quad (17)$$

where $zz_{0,j}$ and $zz_{n,j}$ are the displacement of the springs. Thus,

$$p_{\frac{n}{2},j} = zz_{\frac{n}{2},j} - k_{\frac{n}{2},j} \quad (18)$$

where $zz_{\frac{n}{2},j}$ is the total mid-point deflection and $p_{\frac{n}{2},j}$ is the mid-point deflection due to the moving load without deflection due to the springs. (Fig. 8).

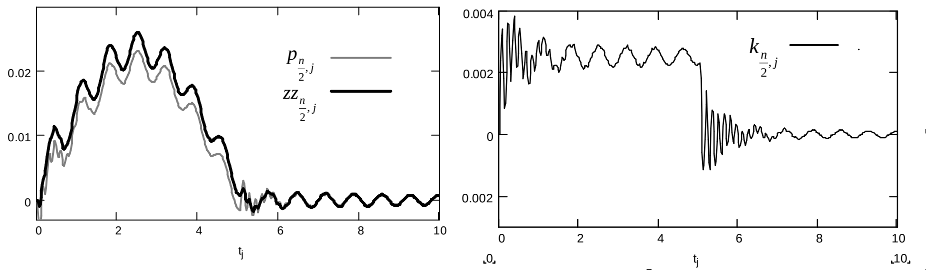


Fig. 8

CONCLUSION

As can be seen from the examples, the agreement between numerical approach and analytical solution is excellent. The procedure based on average acceleration is robust. Further benefit of the numerical formulation is that various boundary conditions and damping can all be easily taken into analysis.

Based on the above procedure, an existing 2D finite element computer program “DARK” has been extended to accommodate the moving-load analysis. “DARK” calculates the eigenfrequencies and the mode shapes as well as the displacements, velocities and accelerations due to moving load at any construction point at every time step. These results enable a straight forward calculation of strains and stresses.

References

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