A LINES-POINTS RELATION FORMULA FOR SYMMETRIC BLOCK DESIGN

Vladimir Ćepulić
University of Zagreb,
10000 Zagreb, CROATIA

Pajo Slamić
University of Rijeka
51000 Rijeka, CROATIA

We proved the following result

Theorem. *If there are t lines of a $(v, k, \lambda)-$symmetric block design, all containing the same s points, $0 \leq s < \lambda$, then the union of their point sets should contain at least*

$$m_s(t) = tk + \binom{t-1}{2} s - \binom{t}{2} \lambda$$

points, for $1 \leq t \leq \mu_s = \lfloor \frac{k-s}{\lambda-s} + 1 \rfloor$, and $m_s(t) = m_s(\mu_s)$, for $t > \mu_s$.

This result has useful applications in constructing symmetric block designs, especially for building their orbital structures.

At the beginning we recall some basic definitions.

Definition 1. A finite *incidence structure* $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ consists of two finite sets, a point set $\mathcal{P}$ and a line set $\mathcal{B}$, and of the incidence relation $I \subseteq \mathcal{P} \times \mathcal{B}$. We say that P is on x (or x is going through P), if $(P, x) \in I$.

For $P \in \mathcal{P}, x \in \mathcal{B}$, denote with

$$\langle P \rangle = \{y \in \mathcal{B} \mid (P, y) \in I\},$$

the set of lines through P, and with

$$\langle x \rangle = \{Q \in \mathcal{P} \mid (Q, x) \in I\},$$

the set of points on x. The cardinal numbers of these sets we denote by $|P|$ and $|x|$, respectively.

Definition 2. A $(v, k, \lambda)-$*symmetric block design,* $v, k, \lambda \in \mathbb{N}$, $k > \lambda$ is an incident structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ such that:

(i) $|\mathcal{P}| = |\mathcal{B}| = v = k(k-1)\lambda/\lambda + 1$,

(ii) $|x| = |P| = k$,

(iii) $|\langle x\rangle \cap \langle y\rangle| = |\langle P\rangle \cap \langle Q\rangle| = \lambda$,

for all $x, y \in \mathcal{B}, P, Q \in \mathcal{P}$,

with $x \neq y$, $P \neq Q$.

In the following we shall use the term *design* for symmetric block design.

Definition 3. Let $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, I_1)$ and $\mathcal{D}_2 = (\mathcal{P}_2, \mathcal{B}_2, I_2)$ be two incidence structures. An *isomorphism of* $\mathcal{D}_1$ *onto* $\mathcal{D}_2$ is a bijection $\alpha : \mathcal{P}_1 \cup \mathcal{B}_1 \Rightarrow \mathcal{P}_2 \cup \mathcal{B}_2$ such that:

(1.) $\mathcal{P}_1\alpha = \mathcal{P}_2$

(2.) $\mathcal{B}_1\alpha = \mathcal{B}_2$ and

(3.) $(P, x) \in I_1 \Leftrightarrow (P\alpha, x\alpha) \in I_2$.

If $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic, we write $\mathcal{D}_1 \cong \mathcal{D}_2$.

If α is an isomorphism of $\mathcal{D}$ onto $\mathcal{D}$, we say that α is an *automorphism.* The full group of automorphism of $\mathcal{D}$ we denote by $Aut(\mathcal{D})$.

Definition 4. For $x \in \mathcal{B}, P \in \mathcal{P}$ and a group $G \leq Aut(\mathcal{D})$, we denote by $xG = \{xg|g \in G\}$, $PG = \{Pg|g \in G\}$ the $G-$*orbits* of x and P, respectively. There are as many point orbits as line orbits. Denoting this number by t, we have the partitions:

$$\mathcal{B} = \bigsqcup_{i=1}^{t} \mathcal{B}_i, \mathcal{P} = \bigsqcup_{r=1}^{t} \mathcal{P}_r$$

Obviously,$|\mathcal{B}_i|, |\mathcal{P}_r|$ divide $|G|$.

PROOF OF THE THEOREM:

According to the above definitions a set of different lines of a $(v, k, \lambda)-$design $\mathcal{D}$ can have at most λ points in common, and, similarly, for a set of different points there exist at most λ lines containing all of them.

Let $\mathcal{T} = \{x_1, x_2, ..., x_t\}$ be a set of t lines in $\mathcal{D}$, $t \geq 2$, all of them containing the same point set $S = \{P_1, P_2, ..., P_s\}$.

In the following we are dealing with the problem how many different points do contain the lines in T. In other words, to estimate the number of points necessary for building t lines sharing the same s points. We shall denote such a number by $m_s(t)$.

For two lines x_1 and x_2, it is always

$|\langle x_1 \rangle \cup \langle x_2 \rangle| = |x_1| + |x_2| - |\langle x_1 \rangle \cap \langle x_2 \rangle| = 2k - \lambda.$

The third line $< x_3 >$ can have with the former two at most $s + 2(\lambda - s)$ common points, admitting the possibility that the sets $\langle x_1 \rangle \cap \langle x_3 \rangle \setminus S$ and $\langle x_2 \rangle \cap \langle x_3 \rangle \setminus S$ are disjoint. Thus the third line contains at least $k - s - 2(\lambda - s) = k - 2\lambda + s$ new points.

Continuing in this way we see that $\langle x_i \rangle$ can share with the union of preceding lines at most $s + (i-1)(\lambda - s)$ common points — in the extreme case that all the sets $\langle x_i \rangle \cap \langle x_j \rangle \setminus S$, for $j < i$, are disjoint. Thus the i-th line x_i contains at least

$$(*) \quad n_i = k - s - (i-1)(\lambda - s) = k - (i-1)\lambda + (i-2)s$$

new points. Clearly, this holds only for the case that $n_i \geq 0$. Otherwise, the above extreme case cannot appear and we cannot conclude, arguing as above, that x_i and the further lines bring some new points.

Thus, we set $n_i = 0$ for $k - s - (i-1)(\lambda - s) \leq 0$, which is equivalent with

$$i - 1 \leq \frac{k-s}{\lambda - s},$$

that is

$$i \leq \lfloor \frac{k-s}{\lambda - s} + 1 \rfloor \equiv \mu_s.$$

The minimal number $m_s(t)$ of different points needed for building t lines sharing the same set of s points is, by previous argumentation, equal

$$m_s(t) = \sum_{i=1}^{t} n_i.$$

Now, for $t \leq \mu_s$ we have

$$\begin{aligned} m_s(t) &= \sum_{i=1}^{t}[k-(i-1)\lambda+(i-2)s] \\ &= tk - \sum_{i=2}^{t}(i-1)\lambda + \sum_{i=3}^{t}(i-2)s \\ &= tk - \binom{t}{2}\lambda + \binom{t-1}{2}s, \end{aligned}$$

and for $t > \mu_s$ it is $m_s(t) = m_s(\mu_s)$.

An additional condition on t is according the definition of symmetric block design the following one: $t \leq \lambda$, except if $s = 1$ when $t \leq k$.

The Theorem is proved.

Note: It remains to consider the case $s = \lambda$. Here $\langle x_i \rangle \cap \langle x_j \rangle = S$, for all i, j, and thus

$$m_\lambda(t) = |\bigcup_{i=1}^{t} \langle x_i \rangle|$$

$$= |S \bigsqcup \big(\bigcup_{i=1}^{t}(\langle x_i \rangle \setminus S)\big)|$$

$$= |S \sqcup \big(\bigsqcup_{i=1}^{t}(\langle x_i \rangle \setminus S)\big)| = \lambda + t(k - \lambda),$$

$m_\lambda(t)$ being the exact number of points needed.

Obviously, $\lambda + t(k - \lambda) \leq v = \frac{k(k-1)}{\lambda} + 1 \Rightarrow t \leq \frac{k+\lambda-1}{\lambda}$ and thus

$t \leq \lfloor \frac{k+\lambda-1}{\lambda} \rfloor = \lfloor \frac{k-1}{\lambda} + 1 \rfloor.$