Optimal Ellipsoid Approximations in Control Theory

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Linear diferential inclusion

- Let $\Omega \subset \mathbb{R}^{n \times n}$ be a given set of real square matrices.
- Trajectory $y(t) \in \mathbb{R}^n, t \in \mathbb{R}_+$ is called a state trajectory of uncertain time-varying linear system Ω iff

$$\frac{d}{dt} y(t) \in \Omega y(t), \ y(0) = y_0 \tag{1}$$

- Ω is describing the uncertainty frame for $A(t) \subseteq \Omega, t > 0$ with

$$\frac{d}{dt} y(t) = A(t) y(t). \tag{2}$$

- For any $A : \mathbb{R}_+ \to \Omega$, the solution of (2) is a trajectory of (1).

Linear time-invariant system

When $\Omega = \{A\}$, consider a differential equation for $y(t) : \mathbb{R} \to \mathbb{R}^n$:

$$\frac{dy}{dt}(t) = A \cdot y(t), \tag{3}$$

where $A \in \mathbb{R}^{n \times n}$ is a given matrix not depending on t.

Problem

Find conditions for solution's norm value $(||y(t)||)$[1] control: for every $t > 0$, $||y(t)|| < \infty$.

[1] A norm function: $||y(t)|| \geq 0$, $||y(t)|| = 0 \Leftrightarrow y(t) = 0$ and $||y(t_1) + y(t_2)|| \leq ||y(t_1)|| + ||y(t_2)||$.

Positive definite matrix

Symmetric matrix is a real square matrix equals with its transpose.

Definition ($P > 0$)

Symmetric matrix P is positive definite if for any $z \in \mathbb{R}^n$,

$$z \neq 0 \Rightarrow z^T P z = z^T (Pz) = (Pz)^T z > 0.$$

- Every eigenvalue[2] of a symmetric matrix is real and $P = U^T diag\left[\lambda_1, \ldots, \lambda_n\right] U$ with $U^T U = E$ [3]
- Every eigenvalue λ of a positive definite matrix is positive
  $$v^T P v = v^T (\lambda v) = \lambda v^T v > 0 \text{ after } v^T v > 0 \text{ for } v \neq 0.$$

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[2] a solution of $\det(\lambda E - P) = 0$ and there is a so-called eigenvector $v \neq 0$ such that $Pv = \lambda v$ and $||v|| = 1$

[3] E is the unit matrix: $X \cdot E = E \cdot X = X$ for any square matrix X

Generalized eigenvalues

Definition

Generalized eigenvalues are n (real) solutions λ that satisfy equality

$$\det(\lambda P - X) = 0$$

for a given positive matrix P and a symmetric matrix X.

There exists

$$\lambda_{\max}(X, P) = \max\{\lambda | \det(\lambda P - X) = 0\}.$$

Theorem

For two (real) symmetric matrices X and P, if $P > 0$ then

$$\begin{aligned} \lambda_{\max}(X, P) &= \inf\{\lambda \in \mathbb{R} | \lambda P - X > 0\} \\ &= \inf\{\lambda \in \mathbb{R} : X < \lambda P\} \\ &= \inf\{\lambda \in \mathbb{R} : z^T X z < z^T \lambda P z, z \in \mathbb{R}^n\} \end{aligned}$$

Proof.

If $\lambda \geq \lambda_{\max}$, then for any eigenvector v_i and its eigenvalue λ_i it slides that $v_i^T \lambda P v_i \geq v_i^T \lambda_{\max} P v_i \geq v_i^T \lambda_i P v_i = v_i^T X v_i$.
Since for every $z \in \mathbb{R}^n$ and every non-defective matrix, there exist real numbers γ_i such that $z = \sum_{i=1}^{n} \gamma_i v_i$, the $z^T x z \leq z^T \lambda_{\max} P z$ follows immediately, because generalized eigenvectors are perpendicular. □

Quadratic (Lyapunov)[4] function

Given any $P = P^T > 0$, let $V(z) = z^T Pz$. For $z = y(t)$ we have:

$$\frac{d}{dt}V(y(t)) = \left(\frac{d}{dt}y(t)\right)^T \cdot P \cdot y(t) + y(t)^T \cdot P \cdot \frac{d}{dt}y(t) =$$

If $y(t)$ satisfies (3), then

$$\begin{aligned} \frac{d}{dt}V(y(t)) &= (A \cdot y(t))^T Py(t) + y(t)^T PAy(t) \\ &= y(t)^T \left(A^T P + PA\right) y(t) \leq \\ &\leq \lambda_{\max}\left(A^T P + PA, P\right) \cdot V(y(t)) \end{aligned}$$

Splitting the beginning with the end:

$$\frac{d}{dt}V(y(t)) \leq \alpha \cdot V(y(t)). \quad (4)$$

[4] Aleksandr Vasiljevič Liapunov (1857–1918)

Rate estimation

Since $P > 0$, $V(y(t)) > 0$, and $dt > 0$:

$$\frac{dV(y(t))}{V(y(t))} \leq \alpha \cdot dt$$

$$\int_0^t \frac{dV(y(t))}{V(y(t))} \leq \int_0^t \alpha \cdot dt$$

$$\ln V(y(t)) - \ln V(y(0)) \leq \alpha t$$

$$\ln V(y(t)) \leq \alpha t + \ln V(y(0))$$

$$V(y(t)) \leq e^{\alpha t} V(y(0))$$

Proposition

If $\alpha := \lambda_{\max}\left(A^T P + PA, P\right)$, then rate is limited by $\frac{d}{dt} V(y(t)) \leq \alpha V(y(t)) \leq \alpha e^{\alpha t} V(0)$, regardless $y(t)$.

Estimation problem

Find a positive matrix P such that α is as minimal as possible:

$$\min_{P>0}\left(\lambda_{\max}\left(A^T P + PA, P\right)\right) \tag{5}$$

- homogeneous in P: $\mathrm{Tr}P = n$, where $\mathrm{Tr}P$ is a trace[5]
- impose: $P - bE > 0$, $0 < b < 1$

Initial feasible case:

$$\begin{aligned} P_0 &= E \\ \alpha = \lambda_0 &= \lambda_{\max}(A^T + A, E) = \lambda_{\max}(A^T + A) \end{aligned}$$

[5] the sum of diagonal elements

Initial limitation

Initial P is in the sphere

$$\mathcal{E}_0 = \left\{ P| \; ||P - E||^2 \le n(n-1), \mathrm{Tr} P = n \right\}, \tag{6}$$

containing a set of positive definite matrices with $\mathrm{Tr}P = n$, since

$$\begin{aligned} ||P - E||^2 &= \mathrm{Tr}(P - E)^T (P - E) = \mathrm{Tr}(P - E)^2 \\ &= \mathrm{Tr}(P^2 - 2P + E) \le n^2 - 2n + n. \end{aligned}$$

For any positive definite P:

$$\begin{aligned} \mathrm{Tr}(P \cdot P) &= \mathrm{Tr}(U diag[\lambda_1, \ldots, \lambda_n] U^T U diag[\lambda_1, \ldots, \lambda_n] U^T) \\ &= \mathrm{Tr}(diag[\lambda_1, \ldots, \lambda_n] diag[\lambda_1, \ldots, \lambda_n]) \\ &= \lambda_1^2 + \cdots + \lambda_n^2 \le (\lambda_1 + \cdots + \lambda_n)^2 = (\mathrm{Tr}P)^2. \end{aligned}$$

An Ellipsoid in $\{P : P = P^T, \mathrm{Tr} = n\}$

For a positive definite Q and fixed matrix P_0:

$$\mathcal{E} = \left\{ P : \mathrm{Tr}(P - P_0)^T Q^{-1} (P - P_0) \leq 1, \mathrm{Tr} P = n \right\},$$

with axis' squares given by positive eigenvalues.

Note that (6) is an ellipsoid:

$$\begin{aligned} \mathcal{E}_0 &= \{ P : \mathrm{Tr}(P - E)^T E (P - E) \leq n(n-1), \mathrm{Tr} P = n \} \\ &= \left\{ P : \mathrm{Tr}(P - E)^T \frac{1}{n(n-1)} E (P - E) \leq 1, \mathrm{Tr} P = n \right\} \\ Q &= n(n-1)E. \end{aligned} \tag{7}$$

Iteration for the infeasible case

Current ellipsoid

$$\mathcal{E}_k = \left\{ P : \ \mathrm{Tr}(P - P_k)^T Q_k^{-1}(P - P_k) \leq 1, \mathrm{Tr}P = n \right\}. \qquad (8)$$

- If $P_k - bE < 0$, then find eigenvector v for the minimum eigenvalue calculated for P_k, such that $||v|| = 1$.
- Wanted P for (5) is in the half-space:

  $$\{P : v^T P v \geq b, \mathrm{Tr}P = n\}, \qquad (9)$$

  since must be $v^T P v > v^T bEv = b \cdot ||v||^2 = b$.
- The next ellipsoid $\mathcal{E}_{k+1}$ is the minimum volume ellipsoid that contains the intersection between $\mathcal{E}_k$ and (9).

Iteration for the feasible case

- If $P_k - bE > 0$, then compute

$$\lambda_k = \lambda_{\max}\left(A^T P_k + P_k A, P_k\right),$$

  and its corresponding generalized eigenvector with $||v|| = 1$.
- Wanted P for (5) is in the half-space:

$$\left\{P : v^T\left(\lambda_k P - A^T P - PA\right) v \geq 0, \mathrm{Tr} P = n\right\}, \qquad (10)$$

  since from Theorem $\lambda_k P \geq A^T P + PA$.
- The next ellipsoid $\mathcal{E}_{k+1}$ is the minimum volume ellipsoid that contains the intersection between $\mathcal{E}_k$ and (10).

Current feasible case lower bound

- If $P_k - bE > 0$, then the lower bound in the current case is

$$\alpha = \lambda_k \geq \max_{P \in \mathcal{E}_k} \frac{v^T(A^T P + PA^T)v}{v^T P v}.$$

- It is a consequence of

$$\begin{aligned} \alpha P &\geq A^T P + PA^T \\ \alpha v^T P v &\geq v^T(A^T P + PA^T)v \Big| : v^T P v > 0 \\ \alpha &\geq \frac{v^T(A^T P + PA^T)v}{v^T P v} \end{aligned}$$

First iteration

Since (6), for $0 < b < 1$, $E - bE > 0$ the case is feasible and then we calculate λ_0 taking the maximum of solutions from

$$\det(\lambda E - A^T - A) = 0. \tag{11}$$

The half space is determined by the eigenvector v with $||v|| = 1$ that is calculated for λ_0:

$$\left\{P : v^T \left(\lambda_0 P - A^T P - PA\right) v \geq 0, \mathrm{Tr} P = n\right\}, \tag{12}$$

noting E is belonging to it.

Problem

Find $\mathcal{E}_1$ such that it contains intersection between $\mathcal{E}_0$ and a half space (12)

Lowner[6]-John[7] ellipsoid

- Ellipsoid $\mathcal{E}(M,a) = \left\{x \in \mathbb{R}^m : (x-a)^T M^{-1}(x-a) \leq 1\right\}$
- Hiperspace: $\left\{x \in \mathbb{R}^m : c^T(a-x) \geq 0\right\}$
- The minimum volume ellipsoid $\mathcal{E}'(M',a')$, containing the intersection between given two sets, is given with:

$$
\begin{aligned}
d &= \frac{1}{\sqrt{c^T M c}} M c \\
a' &= a - \frac{1}{m+1} d \\
M' &= \frac{m^2}{m^2-1}\left(M - \frac{2}{m+1} d^T d\right)
\end{aligned}
$$

[6] Karel Löwner (1893-1968)
[7] Fritz John (1910-1994)

Analogy

From (7) and (12) imply at first place dimension of $\{P : P = P^T, \mathrm{Tr}P = n\}$ and then it follows:

$$\begin{array}{rcl} m & \rightarrowtail & \dfrac{n(n+1)}{2} - 1 \\ a & \rightarrowtail & E \\ M & \rightarrowtail & n(n-1)E \\ c^T & \rightarrowtail & \lambda_0 E - A^T - A, \end{array}$$

since from (12) and (11) it slides:

$$\begin{aligned} & v^T(\lambda_0 P - A^T P - PA)v = \\ & = v^T(\lambda_0 P - A^T P - PA - \lambda_0 E + A^T + A)v \\ & = v^T(\lambda_0 (P - E) - A^T(P - E) - (P - E)A)v \end{aligned}$$

The first ellipsoid

According to shape (8), $\mathcal{E}_1$ is determined by:

$$
\begin{aligned}
d &= \frac{1}{\sqrt{\mathrm{Tr}(\lambda_0 E - A - A^T)^2}} \cdot (\lambda_0 E - A^T - A) \\
P_1 &= E - \frac{2}{n(n+1)} \frac{1}{\sqrt{\mathrm{Tr}(\lambda_0 E - A - A^T)^2}} \cdot (\lambda_0 E - A^T - A) \\
Q_1 &= \frac{n^2(n+1)^2 - 4n(n+1) + 4}{n^2(n+1)^2 - 4n(n+1)} \cdot \\
&\quad \cdot \left(E - \frac{4}{n(n+1)} \frac{1}{\mathrm{Tr}(\lambda_0 E - A - A^T)^2} \cdot (\lambda_0 E - A^T - A)^2 \right)
\end{aligned}
$$

Polytopic linear diferential inclusion

- When $\Omega = \mathrm{Co}\{A_1, \ldots, A_L\}$.
- Then consider $A(t) = \sum_{i=1}^{L} \theta_i(t) A_i$

  with $A_i \in \{A_1, \ldots, A_L\}$, $\theta_i(t) \geq 0$, and $\sum_{i=1}^{L} \theta_i(t) = 1$,
- Differential equation for the state trajectory:

$$\frac{dy}{dt}(t) = \left(\sum_{i=1}^{L} \theta_i(t) A_i \right) y(t), \qquad (13)$$

Problem

Find conditions for solution's norm value ($||y(t)||$) control.

Lyapunov function search

Given any $P = P^T > 0$, let $V(z) = z^T P z$. For $z = y(t)$ we have:

$$\frac{d}{dt} V(y(t)) = \left(\frac{d}{dt} y(t) \right)^T \cdot P \cdot y(t) + y(t)^T \cdot P \cdot \frac{d}{dt} y(t) =$$

If $y(t)$ satisfies (13), then

$$= \left(\left(\sum_{i=1}^{L} \theta_i(t) A_i \right) y(t) \right)^T P y(t) + y(t)^T P \left(\sum_{i=1}^{L} \theta_i(t) A_i \right) y(t)$$

$$= y(t)^T \left(\sum_{i=1}^{L} \theta_i(t) A_i^T \right) P y(t) + y(t)^T P \left(\sum_{i=1}^{L} \theta_i(t) A_i \right) y(t) =$$

Maximum generalized eigenvalue estimation

According to the previous:

$$
\begin{aligned}
&= y(t)^T \left(\sum_{i=1}^{L} \theta_i(t) A_i^T P + P \sum_{i=1}^{L} \theta_i(t) A_i \right) y(t) = \\
&= \sum_{i=1}^{L} \theta_i(t) y(t)^T \left(A_i^T P + P A_i \right) y(t) \leq \\
&\leq \max_i y(t)^T \left(A_i^T P + P A_i \right) y(t) \leq \\
&\leq \max_i \lambda_{\max} \left(A_i^T P + P A_i, P \right) \cdot V(y(t)).
\end{aligned}
$$

Splitting the beginning with the end:

$$
\frac{d}{dt} V(t) \leq \beta \cdot V(y(t)). \tag{14}
$$

Estimation problem

- If $\beta := \max_i \lambda_{\max}\left(A_i^T P + PA_i, P\right)$ then $\frac{d}{dt}V(t)) \leq \beta V(t) \leq \beta e^{\beta t} V(0)$ regardless of $y(t)$ and $\theta_i(t)$.
- Determine a symmetric matrix P such that:

$$\min_{P>0}\left(\max_i \lambda_{\max}\left(A_i^T P + PA_i, P\right)\right)$$

- Defining the block matrix $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, we have

$$\min_{\substack{\mathrm{Tr}P = n \\ P - bE > 0}} \lambda_{\max}\left(\bigoplus_{i=1}^{L}(A_i^T P + PA_i), \bigoplus_{i=1}^{L} P\right)$$

Initial solution

- Initial feasible solution is $P_0 = E$ with

$$\lambda_0 = \lambda_{\max}\left(\bigoplus_{i=1}^{L}(A_i^T + A_i), \bigoplus_{i=1}^{L} E\right)$$

- Initial ellipsoid

$$\mathcal{E}_0 = \left\{P | \; ||P - E||^2 \leq n(n-1), \mathrm{Tr}P = n\right\}.$$

- For the infeasible center of current ellipsoid

$$\mathcal{E}_k = \left\{P : \; \mathrm{Tr}(P - P_k)^T Q_k^{-1}(P - P_k) \leq 1, \mathrm{Tr}P = n\right\},$$

the half space is determined with the unit eigenvector v obtained for the minimum eigenvalue of P_k.

$$\left\{P : v^T P v \geq b, \; \mathrm{Tr}P = n\right\},$$

Feasible current case

- If $P_k > bE$, then

$$\beta \geq \min_{\substack{P \in \mathcal{E}_k \\ v^T \bigoplus_{i=1}^L P v \geq b}} \frac{v^T \bigoplus_{i=1}^L (A_i^T P + P A_i) v}{v^T \bigoplus_{i=1}^L P v}$$

- The next minimizer P is in the half-space:

$$\left\{ P : v^T \bigoplus_{i=1}^{L} \left(\lambda_k P - A_i^T P - P A_i \right) v \geq 0, \mathrm{Tr} P = n \right\},$$

where

$$\lambda_k = \lambda_{\max} \left(\bigoplus_{i=1}^{L} \left(A_i^T P_k + P_k A_i \right), \bigoplus_{i=1}^{L} P_k \right),$$

and v is the unit eigenvector associated with λ_k.

Literature

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Dear coleagues,

thank you for your patience